Pacific Journal of Mathematics

ON DOMINANT AND CODOMINANT DIMENSION OF QF -3 RINGS

DAVID A. HILL

Vol. 49, No. 1 May 1973

ON DOMINANT AND CODOMINANT DIMENSION OF QF-3 RINGS

DAVID A. HILL

In this paper the concept of codominant dimension is defined and studied for modules over a ring. When the ring R is artinian, a left R module M has codominant dimension at least n in case there exists a projective resolution

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

with P_i injective. It is proved that every left R-module has the above property if and only if R has dominant dimension at least n. The concept of codominant dimension is also used to study semi-perfect QF-3 rings.

Let R be an associative ring with an identity 1. Denote by $_{\Re}R$ (resp. R_{\Re}) the left (resp. right) R-module R. Using the terminology of [5], we have the following definitions:

- (1) R is left QF-3, if R has a faithful projective injective left ideal.
 - (2) R is left $QF 3^+$ if the injective hull E(R) is projective.
- (3) R is left QF 3' if $E(_{\Re}R)$ is torsionless, i.e., there exists a set A such that $E(R) \leq \prod_A R$.

In general $(1) \Rightarrow (3)$. For perfect rings the three conditions are equivalent for left and right QF - 3 rings. (See [5].)

The dominant dimension of a left (resp. right) R-module M, denoted by dom. dim $(_{\mathfrak{R}}M)$ (resp. dom. dim $(M_{\mathfrak{R}})$) is at least n, if there exists an exact sequence

$$0 \longrightarrow M \longrightarrow X_{\scriptscriptstyle 1} \longrightarrow \cdots \longrightarrow X_{\scriptscriptstyle n}$$

of left (resp. right) R-module where each X_i is torsionless and injective for $i = 1, \dots, n$. See [3] for details.

Note that this says when dom.dim $({}_{\mathfrak{R}}R) \geq 1$ and R is left-artinian that $E(Re_i)$ for $i=1,\cdots,n$ is projective where $\{e_i\}, i=1,\cdots,n$ is a complete set of orthogonal idempotents, and that each X_i is projective.

We define codominant dimension as follows:

Let M be a left R-module. The codom of M is at least n in case there exists an exact sequence

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

where P_i is torsionless and injective for $i = 1, \dots, n$.

Following the notation of [3], we say that if such an exact

sequence exists for $1 \le i \le n$, but no such sequence exists for $1 \le i \le n+1$, then codom. $\dim (_{\Re}M) = n$. If such a sequence exists for all n then codom. $\dim (_{\Re}M) = \infty$. If no such sequence exists codom. $\dim (_{\Re}M) = 0$.

An R-module U is defined to be a cogenerator if for any module M we can embed it in a product of copies of U. We have:

LEMMA. Let U, V be left injective cogenerators then the codom. dim (U) = codom. dim (V).

The proof follows easily from properties of injective cogenerators and shall omit it.

Let U be a left injective cogenerator. If the codom. dim (U)=n, we say that R has l. codom. dim $(_{\Re}R)=n$. In a similar manner one defines r. codom. dim (R_{\Re}) . Note that if $_{\Re}R$ is artinian, products of projectives are projective and direct sums of injectives are injective. Hence l. codom. dim $(_{\Re}R)=n$ is equivalent to the existence of a resolution

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow U \longrightarrow 0$$

where P_i is projective and injective and $U = E(S_1) \oplus \cdots \oplus E(S_n)$ where S_i : $i = 1, \dots, n$ is a copy of each simple left R-module.

In § 1 we characterize semi-perfect $QF-3^+$ rings in terms of their finitely generated projective, injectives.

In § 2 we show that l. dom. dim $(_{\Re}R)$ and l. codom. dim $(_{\Re}R)$ are the same for artinian rings. Hence, if R is artinian QF-3 then the l.-dom. dim (r-dom. dim) 1. codom. dim (r-codom. dim) are the same.

For notation we use J to donote the Jacobson radical, and $R^{(A)}(R^A)$ denotes a direct sum (resp. direct product) of A-copies of R. Also E(M) will be used to denote the injective hull of an R-module M and P(M) will denote the projective cover of M when M has a projective cover. For a left R-module M, we let $\mathcal{L}_{\mathbb{R}}(M) = \{x \in R \mid x \cdot M = 0\}$, and $\mathcal{L}_{\mathbb{R}}(I) = \{x \in M \mid I \cdot x = 0\}$ where $I \subseteq R$. We will use T(M) to denote M/J(M) where J(M) is the Jacobson radical of M.

1. QF - 3 Rings. Recall that if $_{\Re}R$ is noetherian $rt \cdot QF - 3 \Leftrightarrow rt \cdot QF - 3^{+}$. (See [1] and [6].)

To begin with we shall prove that under those hypotheses

$$rt \cdot QF - 3^+ \iff rt \cdot QF - 3'$$
.

PROPOSITION 1.1. Let $_{\mathfrak{R}}R$ be noetherian. If $E(R_{\mathfrak{R}})$ is torsionless then $E(R_{\mathfrak{R}})$ is projective.

Proof. Given that $0 \to E \xrightarrow{\theta} R^A$ is monic, where A is an indexing set. We show that there exists a finite number of R_{α} 's, $\alpha \in A$ say $R_{\alpha_i}, \dots, R_{\alpha_m}$ such that $\pi \theta \mid_R = \tilde{\theta}$ where π is the projection $R^A \to \bigoplus \sum_{i=1}^m R_{\alpha_i}$ is monic. Let S be the set of all finite intersections of right ideals $\{K_{\alpha}\}_{\alpha \in A}$ where $K_{\alpha} = \ker (\pi_{\alpha} \circ \theta \mid_R)$. Note that $\bigcap_{i=1}^n K_{\alpha_i}$ induces a natural embedding of

$$0 \longrightarrow R / \bigcap_{i=1}^n K_{\alpha_i} \longrightarrow R^{(n)}$$
.

Thus $R/\bigcap_{i=1}^n K_{\alpha_i}$ is torsionless. Hence by [2, Thm. I, p. 350]

Now since $_{\Re}R$ noetherian, the set $\{\mathscr{L}_{\Re}(\bigcap_{i=1}^n K_{\alpha_i})\}$ has a maximal element $\mathscr{L}_{\Re}(\bigcap_{i=1}^m K_{\alpha_i})$ where $\bigcap_{i=1}^n K_{\alpha_i} \in S$. Thus $_{^{2}\Re}\mathscr{L}_{\Re}(\bigcap_{i=1}^m K_{\alpha_i}) = \bigcap_{i=1}^m K_{\alpha_i}$ is a minimal right ideal in S. But then $x \in \bigcap_{i=1}^m K_{\alpha_i} \Rightarrow x \in \bigcap_{\alpha \in A} K_{\alpha}$. Thus $\bigcap_{i=1}^m K_{\alpha_i} = 0$. This implies that $\widetilde{\theta}$ is monic. But then $\pi\theta$ is monic since $\ker(\pi\theta) \cap R \neq 0$ if $\ker(\pi\theta) \neq 0$. This shows E is projective.

We next show that $QF-3^+\Rightarrow QF-3$ for semi-perfect rings. First we need the following lemma.

Lemma 1.2. Let K be finitely generated. Suppose there exists an exact sequence

$$0 \longrightarrow K \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n$$

where $E(K) = E_1$, $E_{i+1} = E(E_i)$ for $1 \le i \le n-1$ and each E_i is projective. Then E_1, \dots, E_n are all finitely generated.

Proof. This follows easily from the proof of [4, Lemma 1].

PROPOSITION 1.3. Suppose R is semi-perfect. If R is left $QF-3^+$ then R is left QF-3.

Proof. By Lemma 1.2 E(R) is finitely generated. Since R is semi-perfect $E(R) \cong \bigoplus \sum_{i=1}^{n} Re_i$, where each e_i is an indecomposable idempotent.

Let Re_1, \dots, Re_k be a subset of Re_1, \dots, Re_n , where the set $\{Re_1, \dots, Re_k\}$ is a complete set of isomorphism classes of $\{Re_1, \dots, Re_n\}$. Then $U = Re_1 \oplus \dots \oplus Re_k$ is a minimal projective injective.

Now we come to the main theorem of this section.

Theorem 1.4. Let R be semi-perfect. The following are equivalent:

- (a) R is left $QF 3^+$.
- (b) $E(_{\Re}R)$ is finitely generated and every finitely generated left injective has an injective projective cover.
- (c) Every finitely generated left projective has a projective injective hull.

Proof. (b) \Rightarrow (a): Consider

$$P(E(R)) \longrightarrow E(R) \longrightarrow 0$$
.

Embed $R \xrightarrow{i_{\Re}} E(R)$ then by the projectivity of R there exists a map $\theta' \colon R \to P(E(R))$ such that θ' is monic.

Consider the following diagram:

$$0 \longrightarrow R \xrightarrow{i_{\Re}} E(R)$$

$$\theta' \Big| \bigvee_{\theta''}$$

$$P(E(R)) .$$

Here $\theta''(r) = \theta'(r)$ for all $r \in R$. Also θ'' is monic. The injectivity of E(R) forces E(R) to be a direct summand of P(E(R)), hence projective.

- (a) \Leftrightarrow (c): Consider $R^{(n)}$, $R^{(n)} \leq E(R)^{(n)}$. Thus $E(P) \leq E(R)^n$, where $P \oplus P' = R^{(n)}$, as a direct summand. Hence E(P) is projective. The converse is trivial.
 - (a) \Rightarrow (b): By Lemma 1.2 E(R) is finitely generated.

Consider $P(E) \xrightarrow{\theta} E \to 0$ where P(E) is finitely generated injective. Let $R^{(n)} \xrightarrow{\rho} E \to 0$. Combining the above maps we have the following diagrams:

So we have ho' epic and $ho' \circ i_{\Re}^{(n)} =
ho$. Further we have

$$P(E) \xrightarrow{\rho'' / \bigoplus_{\theta'}} E \xrightarrow{\rho'} 0$$

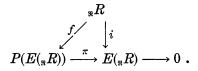
Noting that ρ'' is epic and P(E) is projective, P(E) is a direct summand of $E(R)^{(n)}$. Hence injective.

A ring is perfect in case every module has a projective cover. We show that $QF - 3^+$ rings can be characterized in terms of the

projective cover of $E(_{\pi}R)$.

THEOREM 1.5. Let R be perfect. Then every indecomposable summand of $P(E(_{\Re}R))$ is injective if and only if R is left $QF - 3^+$.

Proof. ⇒ Consider the following diagram:



Here i is a monomorphism and π is epic. Since R is projective there exists on f such that $\pi f = i$. Clearly f is monic. Since R is perfect $P(E(_{\Re}R)) \cong \sum_{\alpha \in A} Re_{\alpha}$, where e_{α} are primitive idempotents of R. Now $\operatorname{Im}(f)$ is contained in $\sum_{\alpha=1}^{n} Re_{\alpha}$, for n a positive integer, since $_{\Re}R$ is cyclic. Thus using the hypothesis, $E(_{\Re}R)$ is projective and R is left $QF - 3^+$. \Leftarrow This is trivial.

2. Codominant dimension of rings. We begin with a lemma which holds the key to the main results of this section.

LEMMA 2.1. Let R be a ring. The following conditions are equivalent.

(1) For every projective left R-module P, there exists an exact sequence

$$0 \longrightarrow P \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n$$

where E_i , $1 \leq i \leq n$, are injective and projective.

(2) For every injective left R-module Q, there exists an exact sequence

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow Q \longrightarrow 0$$

where P_i , $1 \leq i \leq n$, are injective and projective.

Proof. $(1) \Rightarrow (2)$. For n=1 a modification for the proof of Theorem 1.4 will suffice. We assume the lemma is true for the nth case and prove the n+1 case. So consider the following exact sequences.

$$(1) 0 \longrightarrow P_{n+1} \xrightarrow{J_1} E_1 \xrightarrow{J_2} E_2 \longrightarrow \cdots \xrightarrow{J_{n+1}} E_{n+1}$$

$$(2) P_{n+1} \xrightarrow{\theta_1} P_n \xrightarrow{i_n} \cdots \longrightarrow P_1 \xrightarrow{i_1} Q \longrightarrow 0.$$

Here Q is an arbitrary injective module and

$$P_1, \cdots, P_n, E_1, \cdots, E_{n+1}$$

are both projective and injective and P_{n+1} is projective.

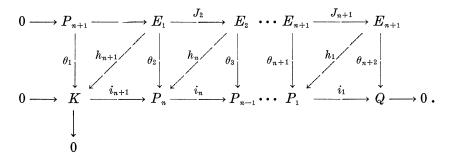
Also E_k is the injective hull of $\operatorname{Cok}(J_k)$.

Denote by K the image of θ_1 . Using the injectivity of P_n , there is a map θ_2 : $E_1 \to P_n$ such $\theta_2 J_1 = i_{n+1} \theta_1$ where i_{n+1} is the embedding of K into P_n . The injectivity of P_{n-1} and the exact sequence $0 \to E_1/P_{n+1} \to E_2$ induce a map θ_3 : $E_2 \to P_{n-1}$ which one can easily check has the property $\theta_3 J_2 = i_n \theta_2$.

In like manner we can define θ_k : $E_{k-1} \to P_{n+2-k}$ such that

$$heta_k J_{k-1} = i_{n+3-k} heta_{k-1}$$
 , $k=2,\cdots,n+2$.

This information is summed up in the following diagram:



Having constructed θ_{n+2} , the projectivity of E_{n+1} induces a map h_1 : $E_{n+1} \rightarrow P_1$ such $i_1h_1 = \theta_{n+2}$. Now consider the map $h_1J_{n+1} - \theta_{n+1}$: $E_n \rightarrow P_1$. We have $i_1(h_1J_{n+1} - \theta_{n+1}) = \theta_{n+2}J_{n+1} - i_1\theta_{n+1} = 0$. So Im $(h_1J_{n+1} - \theta_{n+1}) \leq \ker(i_1)$.

Now consider the following diagram:

$$E_n igg| h_1 J_{n+1} - heta_{n+1} \ P_2 \longrightarrow \operatorname{Im} \ (i_2) \longrightarrow 0 \ .$$

We can construct h_2 using the projectivity of E_n . By a similar argument we can show that $\operatorname{Im}(h_2J_n-\theta_n) \leq \ker(i_2)$. By a recursive argument we can construct $h_kJ_{n+2-k}-\theta_{n+2-k}$ for $k=1,\cdots,n$ in like manner. In particular we have $h_nJ_2-\theta_2$: $E_1\to P_n$ where $\operatorname{Im}(h_nJ_2-\theta_2)\leq K$. We need only show equality to complete the proof. Let $k\in K$. Then there exists an $x\in P_{n+1}$ such that $\theta_1(x)=k$. Thus $(h_nJ_2-\theta_2)(J_1(-x))=\theta_2J_1(x)=\theta_1(x)=k$. Thus $h_nJ_2-\theta_2$ maps on to K. The proof $(2)\to (1)$ is similar. This completes the proof.

Noting that for left artinian rings products of projectives are projective, and direct sums of injectives are injective one can easily show that dom. dim $(R) \ge n$ implies dom. dim. $(P) \ge n$ for all projective P.

Likewise letting $I = \bigoplus \sum E_{\alpha}(S_{\alpha})$ be the minimal injective cogenerator of R, we find that codom. $\dim(I) \geq n$ implies codom. $\dim(Q) \geq n$ for all injectives Q. Thus we have:

Theorem 2.2. Let R be left artinian then the following are equivalent:

- (1) The $\inf \{m \in Z \mid \text{dom. dim } (P) = m \text{ for all } P \text{ projectives} \} = n$.
- (2) The inf $\{m \in Z \mid \text{dom. dim } (Q) = m \text{ for all } Q \text{ injectives}\} = n$.
- (3) 1. dom. dim $(_{\Re}R) = n$.
- (4) l. codom. dim $(_{\mathfrak{R}}R) = n$.

If no such n exists we say $l. dom. dim(R) = \infty$

Proof. (3) \Rightarrow (1), (4) \Rightarrow (2) by our previous discussion. (1) \Rightarrow (3): There exists a projective module P such dom. dim (P) = n.

Now $P \cong \bigoplus \sum_{\alpha} Re_{\alpha}$, $\{e_{\alpha}\}$ primitive idempotents such that for some e_{β} dom. dim $(Re_{\beta}) < n+1$ where $e_{\beta} \in \{e_{\alpha}\}$. Since $Re_{\beta} < R$, n+1 >dom. dim $(R) \geq n$. This yields the desired result. $(2) \Rightarrow (4)$ is similar. $(1) \Rightarrow (2)$: By Lemma 2.1 inf $\{m \in Z \mid \text{codom. dim } (Q) = m\} \geq n$. If inf of the above set is strictly greater than n, another application of the lemma forces inf $\{m \in Z \mid m = \text{dom. dim } (P), P \text{ projective}\} > n$ which is impossible. $(2) \Rightarrow (1)$ is similar.

Let R be left artinian and both left and right QF-3. Then by [4, Thm. 10] 1. dom. dim $(_{\Re}R)=r$. dom. dim (R_{\Re}) . Thus in view of 2.2 we have:

PROPOSITION 2.3. Let $_{\Re}R$ be artinian and QF-3. Then 1. domdim $(_{\Re}R) = r$. domdin $(R_{\Re}) = l$. codomdin $(_{\Re}R) = r$. codomdim $(R_{\Re}) = n$.

Acknowledgement. The author wishes to thank the referee for his proof to Theorem 1.5 which is simpler than the author's original version.

REFERENCES

- 1. J. P. Jans, Projective injective modules, Pacific J. Math., 9 (1959), 1103-1108.
- 2. T. Kato, Duality of cyclic modules, Tohoku Math. J., 14 (1967), 349-356.
- 3. _____, Rings of dominant dimension ≥ 1, Proc. Japan Acad., 44 (1968), 579-584.
- 4. B. J. Muller, Dominant dimension of semi-primary rings, J. reine angew. Math., 232 (1968), 173-179.
- 5. H. Tachikawa, On left QF-3 rings, Pacific J. Math., 31 (1970), 255-268.
- 6. ——, Lectures on QF-3 and QF-1 Rings, Carleton Mathematical Lecture Notes No. 1, July, 1972.

Received February 8, 1972 and in revised form Junuary 3, 1973.

UNIVERSITY OF WESTERN AUSTRALIA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California

Los Angeles, California 90024

R. A. BEAUMONT

University of Washington Seattle, Washington 98105 J. Dugundji*

Department of Mathematics University of Southern California Los Angeles, California 90007

D. GILBARG AND J. MILGRAM

Stanford University Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. Wolf

K. Yoshida

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA STANFORD UNIVERSITY UNIVERSITY OF TOKYO UNIVERSITY OF UTAH WASHINGTON STATE UNIVERSITY UNIVERSITY OF WASHINGTON

AMERICAN MATHEMATICAL SOCIETY NAVAL WEAPONS CENTER

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 49, No. 1

May, 1973

A. Digard, Free lattice-ordered modules	1
Richard Bolstein and Warren R. Wogen, Subnormal operators in strictly cyclic operator algebras	7
Herbert Busemann and Donald E. Glassco, II, <i>Irreducible sums of simple multivectors</i>	13
W. Wistar (William) Comfort and Victor Harold Saks, <i>Countably compact groups</i>	33
and finest totally bounded topologies	33
Mary Rodriguez Embry, Maximal invariant subspaces of strictly cyclic operator algebras	45
Ralph S. Freese and James Bryant Nation, Congruence lattices of semilattices	51
Ervin Fried and George Grätzer, A nonassociative extension of the class of	31
distributive lattices	59
John R. Giles and Donald Otto Koehler, <i>On numerical ranges of elements of locally</i>	
m-convex algebras	79
David A. Hill, On dominant and codominant dimension of QF – 3 rings	93
John Sollion Hsia and Robert Paul Johnson, <i>Round and Pfister forms over</i> $R(t)$	101
I. Martin (Irving) Isaacs, Equally partitioned groups	109
Athanassios G. Kartsatos and Edward Barry Saff, <i>Hyperpolynomial approximation</i>	
of solutions of nonlinear integro-differential equations	117
Shin'ichi Kinoshita, On elementary ideals of θ -curves in the 3-sphere and 2-links in	
the 4-sphere	127
Ronald Brian Kirk, Convergence of Baire measures	135
R. J. Knill, The Seifert and Van Kampen theorem via regular covering spaces	149
Amos A. Kovacs, <i>Homomorphisms of matrix rings into matrix rings</i>	161
Young K. Kwon, <i>HD-minimal but no HD-minimal</i>	171
Makoto Maejima, On the renewal function when some of the me <mark>an renewal lifetimes</mark>	
are infinite	177
Juan José Martínez, Cohomological dimension of discrete modules over profinite	
groups	185
W. K. Nicholson, Semiperfect rings with abelian group of units	191
Louis Jackson Ratliff, Jr., Three theorems on imbedded prime divisors of principal	
ideals	199
Billy E. Rhoades and Albert Wilansky, <i>Some commutants in B</i> (α) which are almost	
matrices	211
John Philip Riley Jr., Cross-sections of decompositions	219
Keith Duncan Stroyan, A characterization of the Mackey uniform (L^{∞}, L^{1}) for	
finite measures	223
Edward G. Thurber, The Scholz-Brauer problem on addition chains	229
Joze Vrabec, Submanifolds of acyclic 3-manifolds	243
Philip William Walker, Adjoint boundary value problems for compactified singular	0.55
differential operators	265
Roger P. Ware, When are Witt rings group rings	279
James D. Wine, <i>Paracompactifications using filter bases</i>	285