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The marginal subgroup for any outer commutator word has been characterized by R. F. Turner-Smith. This paper considers the marginal subgroup $E(G)$ of G for the Engel word $e_2(x, y) = [x, y, y]$ of length two. The principal result is that an element a of G is in $E(G)$ if and only if $[x, y, a][a, y, x]$ is a law in G. The method of proof relies upon properties of Engel elements established by W. Kappe.

Among other results are the following: (a) $E(G)/Z_2(G)$ is an elementary Abelian 3-group of central automorphisms on the commutator subgroup G'. (b) If $Z(G) \cap \gamma_3(G)$ has no elements of order 3 or if G' is Černikov complete, then $E(G) = Z_2(G)$. (c) If $[G:E(G)] = m$ is finite, then the verbal subgroup $e_2(G)$ is finite with order dividing a power of m.

1. Notation and assumed results. Let $\phi(x_1, \cdots, x_n)$ be any word in the variables $x_1, \cdots, x_n$. The verbal subgroup $\phi(G)$ is the subgroup of G generated by all elements of the form $\phi(a_1, \cdots, a_n)$ with $a_1, \cdots, a_n$ in G. We say ϕ is a law in G, or that G is in the variety determined by ϕ, if $\phi(G) = 1$.

The associated marginal subgroup $\phi^*(G)$ of G consists of all a in G such that $\phi(g_1, \cdots, ag_i, \cdots, g_n) = \phi(g_1, \cdots, g_i, \cdots, g_n)$ for every g_i in G, $1 \leqq i \leqq n$. We also refer to $\phi^*(G)$ as the ϕ-margin of G.

For x, y, a_i in G, define $[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$, $[a_1, \cdots, a_n] = [[a_1, \cdots, a_{n-1}], a_n]$, and $[x, (n + 1)y] = [[x, ny], y]$. Similarly, for subgroups H and K of G, $[H, K]$ denotes the subgroup generated by all elements of the form $[h, k]$, where $h \in H$, $k \in K$. We define $[H, (n + 1)K] = [[H, nK], K]$. If $H_1, \cdots, H_n$ are subgroups, then $[H_1, \cdots, H_n] = [[H_1, \cdots, H_{n-1}], H_n]$.

The word $\gamma_1 = d_0 = x$ is an outer commutator word of weight one. If $\theta = \theta(x_1, \cdots, x_m)$, $\lambda = \lambda(y_1, \cdots, y_n)$ are outer commutator words of weights m and n respectively, then $\phi = \phi(x_1, \cdots, x_{m+n}) = [\theta(x_1, \cdots, x_m), \lambda(x_{m+1}, \cdots, x_{m+n})]$ is an outer commutator word of weight $m + n$. We write $\phi = [\theta, \lambda]$. Particular examples are the derived (or solvable) words, defined by $d_n = [d_{n-1}, d_{n-1}]$, and the nilpotent (or lower central) words, defined by $\gamma_{n+1} = [\gamma_n, \gamma_1]$.

The following two theorems appear in [15]:

THEOREM 1.1. *For any group G and word ϕ,*

(a) *$\phi(G)$ is fully invariant in G and $\phi^*(G)$ is characteristic in G.*

(b) $\phi(\phi^*(G)) = 1$.

(c) *if* $K/\phi^*(G)$ *is the center of* $G/\phi^*(G)$, *then* $[K, \phi(G)] = 1$. *In particular,* $[\phi^*(G), \phi(G)] = 1$.

(d) *if* H *is a subgroup such that* $G = H\phi^*(G)$, *then* $\phi^*(H) = H \cap \phi^*(G)$ *and* $\phi(G) = \phi(H)$.

THEOREM 1.2. *Let* θ *and* λ *be two words in independent variables and* $\phi = [\theta, \lambda]$. *Then, in any group* G,

(a) $\phi(G) = [\theta(G), \lambda(G)]$.

(b) *if* $U = C_G(\theta(G))$, $V = C_G(\lambda(G))$, $L/U = \lambda^*(G/U)$, *and* $M/V = \theta^*(G/V)$, *then* $\phi^*(G) = L \cap M$.

An immediate result of Theorem 1.2(b) is that $\gamma^*_{n+1}(G) = Z_n(G)$, the nth center of G. It is this theorem which makes possible a classification of marginal subgroups for all outer commutator words, since the variables in θ and λ are independent of each other (see [16, p. 328]).

An element x of G is called a left (right) Engel element of G if for every y in G there is a positive integer n such that $[y, nx] = 1$ ($[x, ny] = 1$). The Engel word of length n is $e_n(x, y) = [x, ny]$. We note that Theorem 1.2(b) can not be used to determine $e^*_n(G)$, since $e_{n-1}(x, y)$ and y are not independent.

For H a subgroup of G, $[G: H]$ is the index of H in G. If H is a proper (normal) subgroup of G, write $H < G (H \lhd G)$. If G is isomorphic to a subgroup of a group K, write $G \lesssim K$. $C_G(H)$ is the centralizer of H in G. For x in G, x^G denotes the subgroup generated by all conjugates of x in G.

2. The Engel margin. In this section "Engel word" will mean "Engel word of length two". We write $M(G) = d^*_2(G)$ and $E(G) = e^*_2(G)$ for the metabelian and Engel margins of G respectively.

Recall that $[Z_n(G), \gamma_m(G)] \subseteq Z_{n-m}(G)$ for all positive integers m and n.

LEMMA 2.1. *In any group* G,

(a) $d^*_n(G)/C_G(d_{n-1}(G)) = d^*_{n-1}(G/C_G(d_{n-1}(G)))$. *In particular,* $M(G) = \{a \in G \mid [[a, x], [y, z]]$ *is a law in* $G\}$.

(b) $Z_{n(n+1)/2}(G) \subseteq d^*_n(G)$. *In particular,* $Z_3(G) \subseteq M(G)$.

Proof. Part (a) follows from Theorem 1.2(b) with $\theta = \lambda = d_{n-1}$.

We prove (b) by induction on n. For $n = 1$, $Z_1(G) \subseteq d^*_1(G) = Z(G)$. For $n > 1$, let $\bar{G} = G/C_G(d_{n-1}(G))$. Then

$$\overline{d^*_n(G)} = d^*_{n-1}(\bar{G}) \supseteq Z_{n(n-1)/2}(\bar{G})$$

by part (a) and the induction hypothesis. Furthermore,

$$[Z_{n(n+1)/2}(G),\ n(n-1)/2(G)] \subseteq Z_{n(n+1)/2-n(n-1)/2}(G) = Z_n(G)$$

and $[Z_n(G), d_{n-1}(G)] \subseteq [Z_n(G), \gamma_n(G)] = 1$ so that

$$[Z_{n(n+1)/2}(G),\ n(n-1)/2(G)] \subseteq C_G(d_{n-1}(G)) .$$

Consequently,

$$\overline{Z_{n(n+1)/2}(G)} \subseteq Z_{n(n-1)/2}(\bar{G}) \subseteq \overline{d_n^*(G)}$$

and $Z_{n(n+1)/2}(G) \subseteq d_n^*(G)C_G(d_{n-1}(G)) = d_n^*(G)$, as desired.

We define $E_1(G) = \{a \in G \mid [ax, y, y] = [x, y, y] \text{ for all } x, y \in G\}$ and $L(G) = \{a \in G \mid [a, x, x] \text{ is a law in } G\}$ to be the subgroup of right Engel elements of length two. It is not difficult to show that $E(G) \subseteq E_1(G)$ and $E_1(G)$ is a characteristic subgroup of G.

The following properties of $L(G)$ were established by W. Kappe in [6]:

LEMMA 2.2. *In any group G, where $a \in L(G)$, $g, h, \in G$,*

(a) *$L(G)$ is a characteristic subgroup of G.*

(b) $[a, g, h] = [a, h, g]^{-1}$.

(c) $[a, [g, h]] = [a, g, h]^2$.

(d) $[a, g, [h, g]] = 1$.

(e) $a^4 \in Z_3(G)$.

THEOREM 2.3. *In any group G,*

(a) $Z_2(G) \subseteq E(G) \subseteq L(G)$.

(b) $E_1(G) = \{a \in G \mid [a, x] \in C_G(x^G)$ *for all* $x \in G\} = L(G)$.

(c) $[a, x] \in C_G(x^G) \cap C_G(a)$ *for all* $a \in E_1(G), x \in G$. *Furthermore,* $[a, x]^{rs} = [a^r, x^s]$ *for all integers r and s.*

(d) a^G *and* $x^{L(G)}$ *are Abelian for all a in $L(G)$, x in G.*

(e) $E_1(G) \subseteq C_G((x^G)') \lhd G$ *for all x in G.*

Proof. Part (a) follows immediately from the definitions.

(b) Let $a \in E_1(G)$. Then $[ay, x, x] = [y, x, x]$ for all x, y in G. This is equivalent to saying that $1 = [[ay, x][y, x]^{-1}, x] = [[a, x]^y \times [y, x][y, x]^{-1}, x] = [[a, x]^y, x]$ for all x, y in G. Since x and y are independent, we may conclude that a is in $E_1(G)$ if and only if $1 = [a, x, x^y]$ for all x, y in G or, equivalently, $[a, x] \in C_G(x^G)$ for all x.

That $E_1(G) \subseteq L(G)$ follows from $[a, x, x^y] = 1$ by letting $y = 1$. Finally, let $a \in L(G)$. We have for x, y in G that

$$[a, x, x^y] = [a, x, x[x, y]] = [a, x, [x, y]][a, x, x]^{[x,y]} .$$

From the definition of $L(G)$ we must have that $[a, x, x] = 1$. By Lemma 2.2(d) we also have that $[a, x, [x, y]] = 1$. Hence $[a, x, x^y] = 1$ and $a \in E_1(G)$.

(c) Since a is a right Engel element, we have that $[a, x]$ is in $C_G(a)$ by [6, Lemma 2.1]. Part (b) says that $[a, x] \in C_G(x^G)$ for all x in G. The remainder of part (c) follows from [13, Theorem 3.4.4].

(d) From part (c) we see that $a^x = a[a, x] \in C_G(a)$, since a and $[a, x]$ are in $C_G(a)$. This implies that a^G is Abelian.

The proof that $x^{L(G)}$ is Abelian follows similarly from the observation that $x^a = x[x, a]$, $[x, a] \in C_G(x^G) \subseteq C_G(x)$.

(e) By part (c) we may conclude that $[a, x^y] \in C_G((x^y)^G) = C_G(x^G)$ for all a in $E_1(G)$, x, y in G.

Let $a \in E_1(G)$. By Lemma 2.2(c), we have $[a, [x^w, x^z]] = [[a, x^w], x^z]^2 = 1$. This implies that $a \in C_G((x^G)')$.

THEOREM 2.4. *In any group G, $E(G) = \{a \in G \mid [x, a, y][x, y, a] = 1$ for all x, y in $G\}$.*

Proof. Set $E_2(G) = \{a \in G \mid [x, ay, ay] = [x, y, y]$ for all x, y in $G\}$. We see then that $E(G) = E_1(G) \cap E_2(G)$. Let S be the set described on the right in the statement of the theorem. Suppose $a \in S$, $x \in G$. Then $1 = [x, a, x][x, x, a] = [x, a, x]$. This implies that $a \in E_1(G) = L(G)$. Since also $E(G) \subseteq E_1(G)$, it suffices to show that $E(G) \cap E_1(G) = E_1(G) \cap E_2(G) = E_1(G) \cap S$. Then, for x, y in G, $a \in E_1(G) \cap E_2(G)$ if and only if

$$\begin{aligned}
[x, y, y] &= [x, ay, ay] \\
&= [x, ay, y][x, ay, a]^y \\
&= [[x, y][x, a]^y, y][[x, y][x, a]^y, a]^y \\
&= [x, y, y]^{[x,a]^y}[[x, a]^y, y][x, y, a]^{[x,a]^y y}[[x, a]^y, a]^y .
\end{aligned}$$

By assumption, $[a, x] \in C_G(x^G)$. Since $C_G(x^G) \lhd G$, we also have that $[a, x]^y \in C_G(x^G)$. Consequently, conjugation by $[x, a]^y$ is irrelevant in the last statement above because all the commutators are in x^G. Therefore, the above is equivalent to

$$[x, y, y] = [x, y, y][[x, a]^y, y][x, y, a]^y[[x, a]^y, a]^y$$

or

$$1 = [x, a, y][x, y, a][[x, a]^y, a]$$

for all $x, y \in G$, $a \in E(G)$.

Now a and $[x, a]^y$ are elements of a^G. By Theorem 2.3(d), a^G is Abelian. This implies that $[[x, a]^y, a] = 1$. Therefore, $E(G)$ is contained in the set S.

We have already shown that S is a subset of $E_1(G) = L(G)$. Consequently, all the above arguments are reversible and we may conclude that $S = E(G)$.

LEMMA 2.5. (a) $E(G) \cap C_G(G') = Z_2(G)$.
(b) $[x, a, y] = [a, y, x]$ *for all* x, y *in* G, a *in* $L(G)$.

Proof. (a) We need only verify that $E(G) \cap C_G(G') \subseteq Z_2(G)$ by Theorem 2.3(a) and the remark before Lemma 2.1. Let $a \in E(G) \cap C_G(G')$. By Theorem 2.4, $1 = [x, a, y][x, y, a]$ for all x, y in G. But $a \in C_G(G')$ implies that $[x, y, a] = 1$ and thus that $[x, a, y] = 1$ for all x, y in G. Hence $a \in Z_2(G)$.

(b) $[a, y, x] = [a, x, y]^{-1}$ by Lemma 2.2(b), $= [[x, a]^{-1}, y]^{-1} = (([x, a, y]^{-1})^{-1})^{[a,x]} = [x, a, y]$ since $[a, x] \in C_G(x^G)$ by Theorem 2.3(c).

From Theorem 2.4 and Lemma 2.5(b) we have our characterization of $E(G)$:

THEOREM 2.6. *For any group* G, $E(G) = \{a \in G \mid [x, y, a][a, y, x]$ *is a law in* $G\}$.

COROLLARY 2.7. *For any* $a \in E(G)$, $[a, G, G]^3 = [a^3, G, G] = 1$.

Proof. Let $x, y \in G$. By Theorem 2.6, $[x, y, a][a, y, x] = 1$. Then $[x, y, a] = [a, [x, y]]^{-1} = ([a, x, y]^2)^{-1}$ by Lemma 2.2(c), $= [a, y, x]^2$ by Lemma 2.2(b). Hence $1 = [x, y, a][a, y, x] = [a, y, x]^2[a, y, x] = [a, y, x]^3$.

By Theorem 2.3(d) we have that a^G is Abelian. Hence $[a, x, y]^3 = 1$ for all $x, y \in G$ implies $[a, G, G]$ has exponent dividing three, and $[a, x, y]^3 = [a^3, x, y] = 1$.

COROLLARY 2.8. *For any group* G, $E(G) \subseteq Z_3(G) \subseteq M(G)$.

Proof. Let $a \in E(G)$. By Lemma 2.2(e) we have that $a^4 \in Z_3(G)$. Since also $a^3 \in Z_2(G) \subseteq Z_3(G)$ by Corollary 2.7, it follows that $a \in Z_3(G)$.

We recall a theorem of F. W. Levi (see [12]): If e_2 is a law in a group G, then G is nilpotent of class at most three and $\gamma_3(G)$ has exponent dividing three. This, together with Theorem 1.1(b), yields the first statement in the following:

THEOREM 2.9. *$E(G)$ is nilpotent of class no greater then three and metabelian, and $\gamma_3(E(G))$ has exponent dividing three. If $C_G(G') \subseteq E(G)$, then $M(G) = Z_3(G)$.*

Proof. Suppose $C_G(G') \subseteq E(G)$. By Lemma 2.5(a) this implies that $C_G(G') = Z_2(G)$. From Lemma 2.1(a), $M(G)/C_G(G') = Z(G/C_G(G'))$. Hence $M(G) = Z_3(G)$.

THEOREM 2.10. *Let G be a group, $M = M(G)$, $E_1 = E_1(G) = L(G)$. Then*

(a) $[G', M, E_1] = [G', E_1, M] = [M, G, G'] = 1$.

(b) $[G, M', E_1] = [M', E_1, G] = [G', M'] = 1$. *In particular*, $[M', E_1] \subseteq Z(G)$.

Proof. (a) By Lemma 2.1(a), $[M, G] \subseteq C_G(G') \cap G' = Z(G')$ so that $1 = [M, G, G']$. Now let $a \in E_1$, $m \in M$, $x \in G'$. By Lemma 2.2(c), $[a, [m, x]] = [a, m, x]^2 = 1$. This implies $[G', M, E_1] = 1$. Consequently $[G', E_1, M] = 1$ by [13, Theorem 3.4.8(i)].

(b) As in the proof of part (a), we have $M' \subseteq Z(G')$ so that $1 = [G', M']$. Let $a \in E_1$, $x \in M'$, $g \in G$. Then $[a, [g, x]] = [a, g, x]^2 = 1$. Hence $[M', G, E_1] = 1$ and, as above, $[M', E_1, G] = 1$.

3. Central automorphisms on G'. It follows from Theorem 2.10(a) that $[M(G), G'] \subseteq Z(G')$. This implies that $M(G)/C_G(G')$ acts as an Abelian group of central automorphisms on G'. Then

$$(E_1(G) \cap M(G))/(E_1(G) \cap C_G(G')) \lesssim M(G)/C_G(G')$$

is also such a group. Denote the corresponding group of automorphisms on G' by $\mathfrak{A}_2$. Furthermore,

$$E(G)/Z_2(G) = (E(G) \cap M(G))/(E(G) \cap C_G(G')) \lesssim \mathfrak{A}_2$$

by Lemma 2.5(a) and Corollary 2.8. Let $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ denote the corresponding group of automorphisms. From Corollary 2.7 we see that $E(G)/Z_2(G)$ has exponent 3. Hence $\mathfrak{A}_1$ is an elementary Abelian 3-group of central automorphisms on G'.

THEOREM 3.1. (a) *If the exponent* $\operatorname{Exp}(Z(G')) = n$ *is finite, then* $\operatorname{Exp}(\mathfrak{A}_2)$ *divides* n.

(b) *If G' is a p-group, $\mathfrak{A} \subseteq \mathfrak{A}_2$ is periodic, then $\mathfrak{A}$ is a p-group.*

(c) *Assume G' is polycyclic; that is, G' has a finite ascending normal series with cyclic factors. Then $E(G)/Z_2(G)$ is finite.*

Proof. (a) Suppose $Z(G')$ has exponent n. Then, for $x \in G'$, $a \in \mathfrak{A}_2$, $1 = [x, a]^n = [x, a^n]$ by Theorem 2.3(c). Consequently, $a^n = 1$ and $\mathfrak{A}_2$ has exponent dividing n.

(b) Now assume $\mathfrak{A}$ is periodic. By Theorem 2.10(a) we may conclude that $[G', M(G), E_1(G)] = [G', \mathfrak{A}, \mathfrak{A}] = 1$. Thus $\mathfrak{A}$ stabilizes the normal series $1 \lhd [G', \mathfrak{A}] \lhd G'$ of G'. By [1, Corollary 5.3.3] we have that $\mathfrak{A}$ is a p-group.

(c) Smirnov [14] has shown that a solvable group of automorphisms of a polycyclic group is polycyclic. Since then $\mathfrak{A}_1$ is finitely generated, it must be finite.

THEOREM 3.2. *If $\mathfrak{A}_2 \neq 1$ is not torsionfree, then G' has a proper subgroup of finite index and $Z(G')$ is not torsionfree.*

Proof. For $1 \neq \alpha \in \mathfrak{A}_2$, the homomorphism from G' into $Z(G')$ defined by $f_\alpha(x) = [x, \alpha]$ for each x in G' is nontrivial. We choose $a \in E_1(G) \cap M(G) \backslash E_1(G) \cap C_G(G')$ such that $[x, \alpha] = [x, a]$ for all x in G'. If α has finite order, then there is an integer n such that $a^n \in C_G(G')$. Thus $1 = [x, \alpha]^n = [x, a^n]$ and $G'/\mathrm{Ker} f_\alpha \subsetneq Z(G')$ is a nontrivial direct sum of cyclic groups each of order bounded by n. In particular, there are subgroups H and C of G' such that $G'/\mathrm{Ker} f_\alpha = H/\mathrm{Ker} f_\alpha + C/\mathrm{Ker} f_\alpha$ and $C/\mathrm{Ker} f_\alpha$ is nontrivial and finite. Consequently $H < G'$ and $G'/H \cong C/\mathrm{Ker} f_\alpha$ is finite.

Let $1 \neq \alpha \in \mathfrak{A}_2$, $o(\alpha) = n < \infty$. Then there is an $x \in G'$ such that $1 \neq [x, \alpha] \in Z(G')$. But $[x, \alpha]^n = [x, \alpha^n] = 1$ so that the order of $[x, \alpha]$ divides n.

COROLLARY 3.3. *If $E(G) > Z_2(G)$, then G' has a proper subgroup of finite index.*

Proof. If $E(G) > Z_2(G)$, then $\mathfrak{A}_1$ is a nontrivial torsion subgroup of $\mathfrak{A}_2$. Hence $\mathfrak{A}_2 \neq 1$ is not torsionfree and the theorem applies.

It is known that no complete, or even Černikov complete, group can have a proper subgroup of finite index (see [7, p. 234]). From this fact we derive part of the following:

COROLLARY 3.4. *If G' is Černikov complete, or if $Z(G) \cap \gamma_3(G)$ has no elements of order three, then $E(G) = Z_2(G)$.*

Proof. We shall show that $\mathfrak{A}_1 = 1$. By Corollary 2.8, $E(G) \subseteq Z_3(G)$. Hence $[G', E(G)] = [G', \mathfrak{A}_1] \subseteq Z(G) \cap \gamma_3(G)$.

Let $a \in \mathfrak{A}_1$, $x \in G'$. Then, by Corollary 2.7 and Theorem 2.3(c), $1 = [x, a^3] = [x, a]^3$. By hypothesis, this implies that $1 = [x, a]$. Consequently $a = 1$.

EXAMPLE 3.5. We now construct a group G such that $Z_2(G) < E(G) < Z_3(G)$.

Let $H = \langle a_1, a_2, a_3 \colon x^3 \rangle$. Levi and van der Waerden [8] have shown that H has nilpotence class exactly three and is in the variety determined by e_2. Hence $E(H) = H = Z_3(H) > Z_2(H)$. Let K be any group of nilpotence class at least three having no elements of order three (see for example [12, p. 198]). By Corollary 3.4, $E(K) = Z_2(K) < Z_3(K) \subseteq K$. Letting $G = H \times K$, we see that $E(G) = E(H) \times E(K) = H \times Z_2(K)$. Hence $Z_2(G) < E(G) < Z_3(G)$.

REMARK 3.6. Define $N_A(G) = \bigcap \{N_G(H) \mid H$ maximal Abelian subgroup of $G\}$ to be the A-Norm of G. Kappe [6] has shown that $a \in N_A(G)$ if and only if $[g, h] = 1$ for g, h in G implies that $[a, g, h] = 1$. From Theorem 2.6 it follows immediately that $E(G) \subseteq N_A(G) \subseteq E_1(G)$.

4. **Finiteness conditions.** We shall say that a word ϕ satisfies the Schur-Baer property if $[G: \phi^*(G)] = m$ finite implies $\phi(G)$ finite with order which divides a power of m for all groups G.

Schur showed that γ_2 satisfies the Schur-Baer property; Baer extended this result to any outer commutator word ϕ (see [15]).

Recall that a group G is residually finite if for every x in G, $x \neq 1$, there is a normal subgroup N_x of G such that $x \notin N_x$ and G/N_x is finite. A group is locally residually finite if every finitely generated subgroup is residually finite.

We shall need the following theorem. For a proof (due to P. Hall), see [15, Theorem 2].

THEOREM 4.1. *If ϕ generates a locally residually finite variety, then ϕ satisfies the Schur-Baer property.*

THEOREM 4.2. *If $\phi \in \{e_2, e_3\}$, then ϕ satisfies the Schur-Baer property.*

Proof. Suppose $\phi = e_2$. A group in the variety generated by ϕ is nilpotent by Levi's Theorem. A finitely generated nilpotent group is residually finite by P. Hall [4]. Therefore, a finitely generated group in the variety generated by ϕ is residually finite and Theorem 4.1 applies.

Let $\phi = e_3$. Heineken [5] has shown that a group in the variety generated by ϕ is locally nilpotent. Hence a finitely generated group in this variety is also residually finite and the theorem follows as above.

Recall that a group is an SN^* group if it possesses an ascending normal series with Abelian factors (see [7]). Also, the unique maximum locally nilpotent normal subgroup of a group is called its Hirsch-Plotkin radical (see [12]).

We note that in P. Hall's proof of Theorem 4.1 that we may extend the result somewhat if we put some restrictions on G itself. That is, if $\phi^*(G)$ is locally residually finite for all G in some quotient- and subgroup-closed class Σ, then ϕ satisfies the Schur-Baer property for all G in Σ.

THEOREM 4.3. *If G satisfies the maximum or the minimum condition, or if G is an SN^* group, then e_n satisfies the Schur-Baer property for G.*

Proof. Suppose G satisfies the maximum condition. Then, by [12, Theorem VI. 8. j], we have that the set of left Engel elements (of all lengths) is the Hirsch-Plotkin radical R. Since then $e_n^*(G) \subseteq R$ is locally nilpotent, it is locally residually finite. By the preceding remark, we have that e_n satisfies the Schur-Baer property for G.

Vilyacer [18] has shown that an Engel group satisfying the minimum condition is locally nilpotent. Plotkin [11] has proved that an Engel group which is also an SN^* group is locally nilpotent. Hence the remainder of the theorem follows as above.

The validity of the Schur-Baer property in general is one of several conjectures which have been proposed for the group functions ϕ and ϕ^* (see [9] and [16]). Modified solutions of two of these come from the following lemma.

LEMMA 4.4. *Suppose G is in a class of groups in which the Schur-Baer property is satisfied locally for ϕ. If G is locally residually finite and ϕ is finite-valued on G, then $\phi(G)$ is finite.*

Proof. This follows from the arguments used in the proofs of Proposition 1 and its two corollaries in [17].

We note in particular in these proofs that there is a finitely generated subgroup H of G such that $\phi(H) = \phi(G)$. It follows that $H/\phi^*(H)$ is finite. Since H and ϕ satisfy the Schur-Baer property, $\phi(H) = \phi(G)$ is finite.

The following two theorems are immediate from these observations.

THEOREM 4.5. *If $\phi \in \{e_2, e_3\}$, G is locally residually finite, and ϕ is finite-valued on G, then $\phi(G)$ is finite.*

THEOREM 4.6. *If $\phi \in \{e_2, e_3\}$, ϕ is finite-valued on G, and G is finitely generated and residually finite, then $G/\phi^*(G)$ is finite.*

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