Pacific Journal of Mathematics

A COUNTER EXAMPLE TO THE BLUM HANSON THEOREM IN GENERAL SPACES

MUSTAFA AGAH AKCOGLU, JOHN PHILIP HUNEKE AND HERMANN ROST

Vol. 50, No. 2 October 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 50, No. 2, 1974

A COUNTER EXAMPLE TO THE BLUM HANSON THEOREM IN GENERAL SPACES

M. A. AKCOGLU, J. P. HUNEKE AND H. ROST

Let T and S be two bounded linear operators on a Banach space B. One studies the question whether weak convergence of the powers T^n to S implies convergence of the Cesaro averages $1/n \sum_{k=1}^{n} T^{i(k)}$ to S for all subsequences $0 \leqq i(1) < i(2) < \cdots$ of the integers. It is well known that this implication holds if B is the L^2 of a finite measure space and T is induced by a measure preserving transformation of that space (this is the Blum Hanson theorem) or, more generally, if B is a Hilbert space and T of norm at most 1, or if B is a L^1 space and T a positive operator of norm at most 1. In the present paper the conjecture that the above implication holds in general Banach spaces for all T with $\|T\| \leqq 1$ is disproved by constructing a counterexample in a Banach space of the type $B = \mathscr{C}(X)$, X a compact Hausdorff space.

Specifically, let B and B^* be a Banach space and its adjoint, respectively, and let $T: B \to B$ and $S: B \to B$ be two linear and bounded operators. Consider the following two statements:

(i) T^n converges weakly to S; i.e., for each $f \in B$ and $F \in B^*$, $\lim_{n\to\infty} F(T^n f) = F(Sf)$.

(ii) Let $i(n)$ be a sequence of integers so that $0 \leqq i(n) < i(n+1)$ for each $n \geqq 1$. Then, $1/n \sum_{k=1}^{n} T^{i(k)}$ converges strongly to S; i.e., if $f \in B$, then $\lim \| 1/n \sum_{k=1}^{n} T^{i(k)} f - Sf \| = 0$.

It is easy to see that (ii) always implies (i). The Blum Hanson theorem [2] states that if B is the L_2 space of a finite measure space and if T is induced by a measure preserving transformation of this measure space, then (i) also implies (ii). Later it was shown that the equivalence of (i) and (ii) is true if T is a contraction (i.e., if $\|T\| \leqq 1$) and B is a Hilbert space [1], [3] or the L_1 space of a σ-finite measure space [1]. It is then natural to ask if these two conditions (i) and (ii) are always equivalent. In this note, we give a counterexample to show that in general (i) does not imply (ii), even if T is a contraction.

2. Reducing the question to a topological one. Let X be a compact Hausdorff space and let $\mathscr{C} = \mathscr{C}(X)$ be the Banach space of all real valued continuous functions on X, with the usual, supremum norm. If $\tau: X \to X$ is a continuous transformation, then there is an induced linear contraction $T: \mathscr{C} \to \mathscr{C}$, defined as $(Tf)(x) = f(\tau x)$ for each $f \in \mathscr{C}$ and $x \in X$. Note that $T^n f$ converges weakly in $\mathscr{C}$ if and only if $f(\tau^n x)$ converges for each $x \in X$, as a sequence of real numbers.

Hence, if there is a point $x_0 \in X$ so that $\lim_{n\to\infty} \tau^n x = x_0$ for every $x \in X$, then T^n converges weakly to $S: \mathscr{C} \to \mathscr{C}$, defined as $(Sf)(x) = f(x_0)$ for each $x \in X$ and $f \in \mathscr{C}$.

Now assume that τ is such a transformation and also that there is a compact $K \subset X$, not containing the point x_0 and satisfying the following condition:

(A) For each integer $N \geqq 0$ there is a point $x = x(N)$ in X so that K contains more than N terms of the sequence $\tau^n x$, $n = 0, 1, 2, \cdots$.

Before we give an example for such an X and τ in the next section, here we note that in this case (i) does not imply (ii). Let $f \in \mathscr{C}$ be a nonnegative function so that $f(x_0) = 0$ and $f(x) \geqq 1$ for all $x \in K$. Hence, $T^n f$ converges weakly to zero. Now define a sequence $i(n)$ of integers as follows. Let $i(1) = 0$. For each $r \geqq 1$, if the first 2^{r-1} terms are determined then the next 2^{r-1} terms [i.e., the terms $i(2^{r-1}+1), \cdots, i(2^r)$] are chosen as follows. With the notations of Condition (A), let $x_r = x(i(2^{r-1}) + 2^{r-1})$ and let the following conditions be satisfied: $\tau^i(2^{r-1}+s)_{x_r} \in K$ for each $s = 1, 2, \cdots, 2^{r-1}$ and $i(2^{r-1}) < i(2^{r-1}+1) < i(2^r)$. Then,

$$\frac{1}{2^r}\sum_{k=1}^{2^r} f(\tau^{i(k)}x_r) \geqq \frac{1}{2}$$

for each $r \geqq 1$. Hence,

$$\frac{1}{n}\sum_{k=1}^{n} T^{i(k)}f$$

does not converge strongly to zero.

3. The topological example.

We are now going to give an example of a compact Hausdorff space X and a continuous transformation $\tau: X \to X$ so that all the assumptions of the second section are satisfied.

Let R be the real line with the usual topology and let $C = [0, 1) = \{x \mid 0 \leqq x < 1\}$ be the unit interval with its circle topology. Let $\varphi: C \to C$ be a homeomorphism that is linear in $[0, 1/2)$ and in $[1/2, 1)$ and satisfies $\varphi 0 = 0$, $\varphi 1/2 = 3/4$. Note that if $0 < x < 1$ then $\varphi^n x \to 1$ in R. Hence, $\varphi^n x \to 0$ in C, for each $x \in C$. Also, let $\alpha: C \to C$ be a continuous function that is linear in $[1/4, 1/2)$, vanishes identically on $[1/2, 1)$ and is equal to $-A/\log x$ at every $x \in (0, 1/4)$. Here, A is a positive constant so that $\max_{x \in C} \alpha x = +A/\log 4$ is less than $1/4$.

Now let $X = C^2$ be the two-dimensional torus with its usual topology. The points of X are denoted as (x, y), where $x, y \in C$. Let a mapping $\tau: X \to X$ be defined as $\tau(x, y) = ([\varphi x + \alpha y] \bmod 1, \varphi y)$. It is then clear that τ is continuous.

LEMMA. *If* $(x, y) \in X$, *then* $\lim_{n\to\infty} \tau^n(x, y) = (0, 0)$.

Proof. Let $\tau^n(x, y) = (x_n, y_n)$. Hence, $x_0 = x$, $y_0 = y$ and if $n \geqq 1$, then $y_n = \varphi^n y$, $x_n = \xi_n \bmod 1$, where $\xi_n = \varphi x_{n-1} + \alpha y_{n-1}$. If $y_0 = 0$, then $y_n = 0$ for all $n \geqq 0$ and $x_n = \varphi x_{n-1} = \varphi^n x_0 \to 0$ in C. If $y_0 > 0$, then there is an integer $m \geqq 0$ so that $y_n = \varphi^n y_0 > 1/2$ for all $n \geqq m$. This means that $\alpha y_n = 0$ and $x_n = \varphi^{n-m} x_m$ for all $n \geqq m$. Hence, $(x_n, y_n) = (\varphi^{n-m} x_m, \varphi^n y_0)$ converges to $(0, 0)$ for all $(x_0, y_0) \in X$.

LEMMA. *If* $K = \{(x, y) \mid (x, y) \in X, 1/8 \leqq x \leqq 7/8\}$, *then* K *satisfies Condition* A *of* § 2.

Proof. With the notations of the previous proof, let

$$\delta_n = \delta_n(x, y) = \xi_n - x_{n-1} = \varphi x_{n-1} - x_{n-1} + \alpha y_{n-1} .$$

Then,

$$x_n = \left[x_0 + \sum_{k=1}^{n} \delta_k\right] \bmod 1 .$$

Now note that if

$$\sum_{k=n_1}^{n_2} \delta_k \geqq 1$$

then there is an integer n, so that $n_1 \leqq n \leqq n_2$ and that $(x_n, y_n) \in K$. In fact, for each $k \geqq 1$, $0 \leqq \delta_k \leqq \max_{x\in C} (\varphi x - x) + \max_{y \in C} \alpha y \leqq 1/2$, and hence, if

$$\sum_{k=n_1}^{n_2} \delta_k \geqq 1$$

then,

$$\left[x_0 + \sum_{k=1}^{n} \delta_k\right] \bmod 1$$

is between 1/8 and 7/8 for some n, $n_1 \leqq n \leqq n_2$. Therefore, to prove the present lemma, it is enough to show that given any number N, there is a point $(x, y) \in X$ so that

$$\sum_{n=1}^{\infty} \delta_n(x, y) \geqq N .$$

Let $0 < y < 1/4$ be given and let $M = M(y)$ be the largest integer in the set $\{n \mid n \geqq 0, \varphi^n y < 1/4\}$. Let $z = \varphi^M y$. Hence, $z < 1/4$, but $\varphi z = (3/2)z \geqq 1/4$. Therefore, if $0 \leqq n \leqq M$, then $y_n = \varphi^n y = (3/2)^n y = (3/2)^{n-M} z \geqq (3/2)^{n-M} 1/6$, and $\alpha y_n = -A/\log y_n \geqq A/((M-n)\log 3/2 + \log 6)$. This means that

$$\sum_{k=1}^{M} \delta_n \geqq \sum_{n=1}^{M} \alpha y_n \geqq A \sum_{n=0}^{M-1} \frac{1}{n \log 3/2 + \log 6}\,.$$

But it is clear that there are points $y \in (0, 1/4)$ for which $M = M(y)$ is arbitrarily large, hence, for which $\sum_{n=1}^{\infty} \delta_n$ is also arbitrarily large.

References

1. M. A. Akcoglu and L. Sucheston, *On operator convergence in Hilbert space and in Lebesgue space*, to appear.
2. J. R. Blum and D. L. Hanson, *On the mean ergodic theorem for subsequences*, Bull. Amer. Math. Soc., **66** (1960), 308–311.
3. L. K. Jones and V. Kuftinek, *A note on the Blum Hanson theorem*, Proc. Amer. Math. Soc., **30** (1971), 202–203.

Received October 12, 1972 and in revised form June 25, 1973. Research of the first author was in part supported by NRC Grant 3974.

UNIVERSITY OF TORONTO
THE OHIO STATE UNIVERSITY
AND
JOHANN WOLFGANG GOETHE UNIVERSITY, FRANKFURT, GERMANY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$60.00 a year (6 Vols., 12 issues). Special rate: \$30.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Pacific Journal of Mathematics
Vol. 50, No. 2 October, 1974