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## **SETS GENERATED BY RECTANGLES**

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For any family  $F$  of sets, let  $\mathcal{B}(F)$  denote the smallest  $\sigma$ -algebra containing  $F$ . Throughout this paper  $X$  denotes a set and  $\mathcal{R}$  the family of sets of the form  $A \times B$ , for  $A \subseteq X$  and  $B \subseteq X$ . It is of interest to find conditions under which the following holds:

- (1) Each subset of  $X \times X$  is a member of  $\mathcal{B}(\mathcal{R})$ .

The interesting case is when

$$\omega_1 < \text{Card } X \leq c,$$

since results for other cases are known.

It is shown in Theorem 9 that (1) is equivalent to

- (2) There is a countable ordinal  $\alpha$  such that each subset of  $X \times X$  can be generated from  $\mathcal{R}$  in  $\alpha$  Baire process steps.

It is also shown that the two-dimensional statements (1) and (2) are equivalent to the one-dimensional statement

- (3) There is a countable ordinal  $\alpha$  such that for each family  $H$  of subsets of  $X$  with  $\text{Card } H = \text{Card } X$ , there is a countable family  $G$  such that each member of  $H$  can be generated from  $G$  in  $\alpha$  steps.

It is shown in Theorem 5 that the continuum hypothesis (CH) is equivalent to certain statements about rectangles of the form (1) and (2) with  $\alpha = 2$ .

Rao [7, 8] and Kunen [2] have shown that

**THEOREM 1.** *If  $\text{Card } X \leq \omega_1$  (the first uncountable cardinal) then (1) is true and if  $\text{Card } X > c$  then (1) is false.*

The question of whether (1) is true (without the requirement  $\text{Card } X \leq \omega_1$ ) was raised by Johnson [1] and earlier by Erdős, Ulam, and others (see [8], p. 197). The arguments in Kunen's thesis actually showed that if  $\text{Card } X \leq \omega_1$  then

- (4) Each subset of  $X \times X$  can be generated from  $\mathcal{R}$  in 2 steps (i.e., each subset is a member of  $\mathcal{R}_{\sigma\delta}$ . See definitions in § 2.)

In Theorem 5 we generalize Theorem 1 and Kunen's result (4),

and give a new characterization of CH by showing it to be equivalent to certain statements about rectangles of the form (1) and (4).

If CH is assumed the  $\alpha$  appearing in statements (2) and (3) above is 2 (see Theorem 10). This raises the intriguing (but unanswered) question of whether  $\alpha$  must *always* be 2 if (1) holds and CH is false.

It would also be interesting to know whether statements (1), (2), and (3) are equivalent to statement (5) below. Clearly (3) implies (5).

If  $H$  is a family of subsets of  $X$  with  
 (5)  $\text{Card } H = \text{Card } X$ , then there is a countable  
 family  $G$  for which  $H \subseteq \mathcal{B}(G)$ .

The equivalence of (1) and (2) means for example, (assuming CH), that there is a countable family  $G$  from which all real Borel sets (or analytic sets, or projective sets) can be generated in *two* steps (i.e., Borel sets  $\subseteq G_{\omega_1}$ ). This is remarkable in view of the well known result [4, 8] that if  $G$  is a countable basis for the real topology, then the Borel sets cannot be generated from  $G$  in less than  $\omega_1$  steps.

As a generalization of this well known result we show in Theorem 12 that any countable family  $G$  which is closed to complementation and which generates the Borel sets (i.e., Borel sets  $\subseteq \mathcal{B}(G)$ ) must have order  $\omega_1$ . That is

$$\mathcal{B}(G) \not\subseteq G_\alpha$$

for any countable ordinal  $\alpha$ . Thus, even though  $G$  might generate the Borel sets in  $\alpha$  steps (or 2 steps if CH is assumed), the process, nevertheless, continues to produce new members of  $\mathcal{B}(G)$  until we reach  $G_{\omega_1}$ .

We would like to point out in conjunction with our characterization of CH that Kunen [2] has proved that if Martin's Axiom A holds (see [6]) and  $\text{Card } X \leq c$  then (4) holds. He also proved that if  $\omega_1 < \text{Card } X \leq c$  then (1) is independent of ZFC (Zermelo-Frankel Axioms + the Axiom of Choice) together with the negation of CH.

**2. Notation and definitions.** If  $G$  is any family of sets, let  $G_0$  be the family  $G$ , and for each ordinal  $\alpha$ ,  $\alpha > 0$ , let  $G_\alpha$  be the family of all countable unions (intersections) of sets in  $\bigcup_{\gamma < \alpha} G_\gamma$ , if  $\alpha$  is odd (even). Limit ordinals will be considered even. (Compare Kuratowski [3].) Thus we have

$$G_0 = G, G_1 = G_\sigma, G_2 = G_{\sigma\sigma}, G_3 = G_{\sigma\sigma\sigma}, \dots, G_\alpha, \dots$$

Also  $G_\alpha \subseteq G_{\alpha+1}$  for each ordinal  $\alpha$  and  $G_{\omega_1} = G_{\omega_1+1}$ , where  $\omega_1$  is the first uncountable ordinal. If  $\alpha > 0$ , then the family  $G_\alpha$  is closed under countable unions (intersections) if  $\alpha$  is odd (even).

We define the *order of G* to be the first ordinal  $\alpha, \alpha > 0$ , such that  $G_{\alpha+1} = G_\alpha$ .

For each  $A \subseteq X$  (or  $A \subseteq X \times X$ ), let  $A'$  be the complement of  $A$  with respect to  $X$  (or  $X \times X$ ), and for each family  $G$  of subsets of  $X$  (or  $X \times X$ ) let  $\mathcal{C}(G)$  be the family of complements of  $G$ . Note that if  $\mathcal{C}(G) \subseteq G$ , or even if  $\mathcal{C}(G) \subseteq G_{\omega_1}$ , then the family  $G_{\omega_1}$  is the family  $\mathcal{B}(G)$ , the  $\sigma$ -algebra generated by  $G$ . Thus, since

$$(A \times B)' = A \times B' \cup A' \times X \in \mathcal{R}_1,$$

it follows that  $\mathcal{R}_{\omega_1} = \mathcal{B}(\mathcal{R})$ .

If  $G$  is a family of subsets of  $X$ , let  $VG = \{A \times B: A \subseteq X, B \in G\}$ , and let  $HG = \{A \times B: A \in G, B \subseteq X\}$ .

If  $Z \subseteq X \times X$  and  $x \in X$ , let  $Z_x$  denote the vertical section of  $Z$  at  $x, Z_x = \{y: (x, y) \in Z\}$ .

**3. Results.** The following lemma is easily proved by transfinite induction.

**LEMMA 2.** *If  $1 \leq \alpha < \omega_1$  and  $A \in G_\alpha$ , then there is a set  $B$  in  $G_1$  such that  $A \subseteq B$ .*

**THEOREM 3.** *If  $G$  is a countable family of subsets of  $X, Z \subseteq X \times X$ , and  $0 < \alpha < \omega_1$ , then  $Z \in (VG)_\alpha$  if and only if  $Z_x \in G_\alpha$  for each  $x \in$  domain  $Z$ .*

*Proof.* By considering the natural projections of the sets involved on the second coordinate axis, it is easily seen that

$$\text{if } Z \in (VG)_\alpha, \text{ then } Z_x \in G_\alpha \text{ for each } x \in \text{domain } Z.$$

Now suppose that  $Z_x \in G_\alpha$  for each  $x \in$  domain  $Z$ , and let  $G = \{\theta_1, \theta_2, \theta_3, \dots\}$ . We complete the proof by transfinite induction on  $\alpha$ .

*Case 1.*  $\alpha = 1$ .

For each  $n$ , let  $A_n = \{x \in \text{domain } Z: \theta_n \subseteq Z_x\}$ , and let  $Z_n = A_n \times \theta_n$ . Then  $Z_n \in VG$ , for each  $n$ , and

$$Z = \bigcup_{n=1}^{\infty} Z_n \in (VG)_1.$$

Now suppose  $1 < \alpha < \omega_1$ , and that the theorem holds for every  $\gamma, 0 < \gamma < \alpha$ .

*Case 2.*  $\alpha$  is even.

Let  $\{\gamma_n\}_{n=1}^\infty$  be a sequence of odd ordinals less than  $\alpha$  such that each odd ordinal less than  $\alpha$  appears infinitely often in  $\{\gamma_i\}_{n=1}^\infty$ . For each  $x \in \text{domain } Z$ , let

$$D_1(x), D_2(x), D_3(x), \dots$$

be a sequence such that  $D_i(x) \in G_{\gamma_i}$  for each  $i$ , and

$$Z_x = \bigcap_{i=1}^\infty D_i(x).$$

This can be done in view of Lemma 2. For each  $i$ , let

$$Z^i = \bigcup_{x \in \text{domain } Z} \{x\} \times D_i(x).$$

First note that  $Z = \bigcap_{i=1}^\infty Z^i$ . Also each nonempty section  $(Z^i)_x$  of  $Z^i$  is equal to  $D_i(x) \in G_{\gamma_i}$ . Hence, by the induction hypothesis,  $Z^i \in (VG)_{\gamma_i}$ , for each  $i$ , and therefore

$$Z = \bigcap_{i=1}^\infty Z^i \in (VG)_\alpha,$$

by the definition of the family  $(VG)_\alpha$ .

*Case 3.*  $\alpha$  is odd and greater than 1.

For each  $x \in \text{domain } Z$ , let  $\{D_i(x)\}_{i=1}^\infty$  be a sequence of members of  $G_{\alpha-1}$  for which  $Z_x = \bigcup_{i=1}^\infty D_i(x)$ , and let  $Z^i = \bigcup_{x \in \text{domain } (Z)} \{x\} \times D_i(x)$ , for each  $i$ .

Again it follows that  $Z^i \in G_{\alpha-1}$ , for each  $i$ , and

$$Z = \bigcup_{i=1}^\infty Z^i \in (VG)_\alpha.$$

**COROLLARY 4.** *If  $Z \subseteq X \times X$  is the graph of a function then  $Z \in \mathcal{R}_2 \subseteq \mathcal{B}(\mathcal{R})$ .*

*Proof.* Let  $G$  be a countable basis for the real topology and note that, for each  $x \in X$ ,  $Z_x$  is a singleton and hence  $Z_x \in G_2$ . Thus by Theorem 3,  $Z \in (VG)_2 \subseteq \mathcal{R}_2 \subseteq \mathcal{B}(\mathcal{R})$ . Also see [7].

**THEOREM 5.** *Let  $X$  be the real numbers and let  $G$  be a countable base for the usual topology on  $X$ . The following three statements are equivalent:*

(1) CH holds

(2) if  $Z \subseteq X \times X$ , then  $Z = A \cap B$ , where  $A \in (VG)_2$  and  $B \in (HG)_2$

and

(3) if  $E \subseteq X \times X$ , then  $E = C \cup D$ , where  $C \in \mathcal{B}(VG)$  and  $D \in \mathcal{B}(HG)$ .

*Proof.* First, assume CH and suppose  $Z \subseteq X \times X$ . As is well known [7], the complement of  $Z$  is the union of two sets  $H$  and  $K$  such that each vertical section of  $H$  is countable and each horizontal section of  $K$  is countable.

Let  $A$  be the complement of  $H$  and let  $B$  be the complement of  $K$ . Then each vertical section of  $A$  is a  $G_2$  set and by Theorem 3,  $A \in (VG)_2$ . Similarly,  $B \in (HG)_2$ . Of course,  $Z = A \cap B$ .

Since  $A \in (VG)_2 \subseteq \mathcal{R}_2$  and  $B \in (HG)_2 \subseteq \mathcal{R}_2$  and  $\mathcal{R}_2$  is closed under finite intersections,  $Z \in \mathcal{R}_2$ . Thus, if CH holds, then the order of  $\mathcal{R}$  is  $\leq 2$ . Since the graph of the identity function,  $f(x) = x$ , is not in  $\mathcal{R}_1$ , it follows that the order of  $\mathcal{R}$  is 2.

Now, suppose statement 2 holds and  $E \subseteq X \times X$ . Then, the complement of  $E$  can be expressed as the intersection of sets  $A$  and  $B$  with  $A \in (VG)_2$  and  $B \in (HG)_2$ . It follows that  $A' \in (VG)_3 \subseteq \mathcal{B}(VG)$  and  $B' \in (HG)_3 \subseteq \mathcal{B}(HG)$ . Thus,  $E$  is the union of two sets  $C$  and  $D$ , where  $C \in \mathcal{B}(VG)$  and  $D \in \mathcal{B}(HG)$ .

Finally, assume statement 3 holds. Let  $T$  be a totally imperfect subset of  $X$  of cardinality  $c$ . The existence of such a set can be proven without assuming CH [3, p. 514]. Let  $E = T \times T$  and let  $E = C \cup D$ , with  $C \in \mathcal{B}(VG)$  and  $D \in \mathcal{B}(HG)$ . Then each vertical section of  $C$  is a subset of  $T$  which is a Borel set. Since an uncountable Borel set contains a perfect set and  $T$  contains no perfect set, we have that each vertical section of  $C$  is countable. Similarly, each horizontal section of  $D$  is countable. But, as is well known [10] this implies CH.

This completes the proof of Theorem 5.

The following two lemmas are well known.

LEMMA 6. *If  $F$  is a family of sets,  $\alpha$  is a countable ordinal, and  $A \in F_\alpha$ , then there is a countable subfamily  $J$  of  $F$  for which  $A \in J_\alpha$ .*

LEMMA 7. *If  $F$  is a family of sets,  $\mathcal{C}(F) \subseteq F$ , and  $A \in \mathcal{B}(F)$  then there is a countable subfamily  $J$  of  $F$  and a countable ordinal  $\alpha$  for which  $A \in J_\alpha$ .*

THEOREM 8. (a) *The following two statements are equivalent:*

(i) *For each subset  $Z$  of  $X \times X$  there is a countable ordinal  $\alpha$  such that  $Z \in \mathcal{R}_\alpha$ .*

(ii) *If  $H$  is a family of subsets of  $X$  and  $\text{Card } H = \text{Card } X$ , then there is a countable family  $G$  of subsets of  $X$  and a countable ordinal  $\alpha$  for which  $H \subseteq G_\alpha$ .*

(b) *If  $\alpha$  is a countable ordinal, the following two statements are equivalent:*

(i) *Each subset of  $X \times X$  is a member of  $\mathcal{R}_\alpha$ .*

(ii) *If  $H$  is a family of subsets of  $X$  and  $\text{Card } H = \text{Card } X$  then there is a countable family  $G$  of subsets of  $X$  for which  $H \subseteq G_\alpha$ .*

*Proof.* The proof of part (b) is similar to that of part (a) which is given below.

First suppose (i) holds, and suppose that  $H$  satisfies the hypotheses of (ii). Define the subset  $Z \subseteq X \times X$  by letting each member of  $H$  be a vertical section of  $Z$ . More precisely, let  $f$  be a 1-1 function from  $X$  to  $H$  and let

$$Z = \bigcup_{x \in X} \{x\} \times f(x).$$

By (i) there is a countable ordinal  $\alpha$  such that  $Z \in \mathcal{R}_\alpha$  and hence by Lemma 6, there is a countable subfamily  $J$  of  $\mathcal{R}$  for which  $Z \in J_\alpha$ . Let

$$G = \{B : A \times B \in J\},$$

note that  $Z \in (VG)_\alpha$  and use Theorem 3 to conclude that  $H \subseteq G_\alpha$ .

Now suppose (ii) holds, and that  $Z \subseteq X \times X$ . Let  $H$  be the family of vertical sections of  $Z$ , and use (ii) to secure a countable family  $G$  and a countable ordinal  $\alpha$  for which  $H \subseteq G_\alpha$ . Thus  $Z_x \in G_\alpha$  for each  $x \in \text{domain } Z$  and by Theorem 3

$$Z \in (VG)_\alpha \subseteq \mathcal{R}_\alpha.$$

**THEOREM 9.** *The following four statements are equivalent:*

- (i) *Each subset of  $X \times X$  is a member of  $\mathcal{B}(\mathcal{R})$ .*
- (ii) *If  $H$  is a family of subsets of  $X$  and  $\text{Card } H = \text{Card } X$  then there is a countable family  $G$  and a countable ordinal  $\alpha$  for which  $H \subseteq G_\alpha$ .*
- (iii) *There is a countable ordinal  $\alpha$  such that, for each family  $H$  of subsets of  $X$  with  $\text{Card } H = \text{Card } X$ , there is a countable family  $G$  for which  $H \subseteq G_\alpha$ .*
- (iv) *There is a countable ordinal  $\alpha \geq 2$  such that each subset of  $X \times X$  is a member of  $\mathcal{R}_\alpha$ .*

*Proof.* Statements (i) and (ii) are equivalent by Lemma 7 and Theorem 8a. Clearly (iii) implies (ii) and (iv) implies (i). Also by Theorem 8b it follows that (iii) implies (iv).  $\alpha$  cannot be equal to 1 in (iv) because by (i) the identity function  $f(x) = x$  is not in  $\mathcal{R}_1$ .

We complete the proof by showing that (ii) implies (iii). Since this result is immediate if  $X$  is countable we will assume that  $\text{Card } X \geq \omega_1$ .

Suppose that (ii) holds and that (iii) does not. Then for each  $\alpha < \omega_1$ , there is a family  $H(\alpha)$  of subsets of  $X$  for which  $\text{Card } H(\alpha) =$

Card  $X$  and

(1) for each countable  $G$ ,  $H(\alpha) \not\subseteq G_\alpha$ .

Let  $H' = \bigcup_{\alpha < \omega_1} H(\alpha)$ . Thus  $\text{Card } H' = \text{Card } X$  and hence by (ii) there is a countable family  $G'$  and a countable ordinal  $\alpha'$  for which  $H' \subseteq G'_{\alpha'}$ . But then  $H(\alpha') \subseteq H' \subseteq G'_{\alpha'}$  in contradiction of (1).

Therefore (ii) implies (iii).

In part (ii) above the family  $G$  can be chosen so that  $G_{\omega_1}$  is closed to complementation (i.e., is a  $\sigma$ -algebra).

In view of condition (ii) of Theorem 9, it is interesting to note that R. Mansfield has shown that if  $G$  is a countable family of Lebesgue measurable sets, then  $B(G)$  does not contain all analytic sets [5].

As was mentioned in the introduction it would be interesting to know whether the formula " $H \subseteq G_\alpha$ " in Theorem 9 could be replaced by  $H \subseteq \mathcal{B}(G)$ . We do not know the answer to this question.

**THEOREM 10.** *If CH holds,  $\text{Card } X = c$ ,  $H$  is a family of subsets of  $X$ , and  $\text{Card } H = c$ , then there is a countable family  $G$  for which  $H \subseteq G_2$ .*

*Proof.* By Theorem 5 each subset  $Z$  of  $X \times X$  is a member of  $\mathcal{R}_2$ . The desired conclusion now follows from Theorem 8b.

**4. Generating Borel sets.** Let  $R$  be the set of reals, and let  $H$  be the family of all Borel subsets of  $R$ . This family has cardinality  $c$ . Suppose  $G$  is a countable family of subsets of  $R$  such that  $H \subseteq G_{\omega_1}$  and  $G_{\omega_1}$  is closed to complementation. The next two theorems show that, even if the family  $G$  generates all the Borel sets at an early stage, the order of  $G$  is  $\omega_1$ . This is a generalization of the well known result [4, 9] that if  $G$  is a countable basis for the real topology then  $G$  has order  $\omega_1$ . Our proof which is a usual "diagonal" type argument, parallels somewhat Lebesgue's proof of that result [3, p. 368].

Let  $G = \{V_1, V_2, V_3, \dots\}$ , let  $N$  be the set of irrational numbers between 0 and 1 and let  $K$  be the family  $\{\theta_1, \theta_2, \theta_3, \dots\}$  of all intersections of the members of  $G$  with  $N$ ,

$$\theta_i = V_i \cap N.$$

It will be shown that the order of  $K$  is  $\omega_1$ . It then follows that the order of  $G$  is  $\omega_1$ .

For each  $z \in N$ , let  $(z_1, z_2, z_3, \dots)$  be the sequence of integers appearing in the continued fraction expansion of  $z$ . This defines a



reversible transformation from  $N$  onto the set of all sequences of positive integers. Let

$$\begin{aligned}
 z^1 &= (z_1, z_3, z_5, \dots) && \text{(odd indices)} \\
 z^2 &= (z_2, z_6, z_{10}, \dots) \\
 z^3 &= (z_4, z_{12}, z_{20}, \dots) \\
 &\vdots \\
 (*) \quad z^n &= (z_{2^{n-1}}, z_{3 \cdot 2^{n-1}}, z_{5 \cdot 2^{n-1}}, \dots) \\
 &\vdots
 \end{aligned}$$

This defines a homeomorphism between  $N$  and  $N^{\aleph_0}$  (see Kuratowski [3], p. 369). Also note that if  $f$  is a continuous function from  $N$  into  $N$ , then the functions  $f_n$  from  $N$  into the space of positive integers are continuous, where

$$f(z) = (f_1(z), f_2(z), f_3(z), \dots), \quad \text{or} \quad (f_n(z) = f(z)_n).$$

Recall that  $K = \{\theta_1, \theta_2, \theta_3, \dots\}$ . The family  $K_\alpha$  which appears in Theorem 11 is defined in § 2.

**THEOREM 11.** *For each countable ordinal  $\alpha$ ,  $\alpha > 0$ , there is a function  $U_\alpha$  from  $N$  onto  $K_\alpha$  such that if  $f$  is a continuous function from  $N$  into  $N$ , then the set*

$$A_f = \{z: z \in U_\alpha(f(z))\}$$

is in  $\mathcal{B}(K)$ .

*Proof.* Let  $U_1(z) = \bigcup_{n=1}^{\infty} \theta_{z_n}$ , for each  $z \in N$ . Clearly  $U_1$  maps  $N$  onto  $K_1$ .

Let  $f$  be a continuous function from  $N$  onto  $N$ .

We have

$$\begin{aligned}
 A_f &= \{z: z \in U_1(f(z))\} \\
 &= \left\{ z: z \in \bigcup_{n=1}^{\infty} \theta_{f_n(z)} \right\} \\
 &= \bigcup_{n=1}^{\infty} \{z: z \in \theta_{f_n(z)}\}.
 \end{aligned}$$

For each  $n$ ,

$$\{z: z \in \theta_{f_n(z)}\} = \bigcup_{i=1}^{\infty} \{J_{n_i} \cap \theta_i\}$$

where  $J_{n_i} = \{z: f_n(z) = i\}$ . Since each  $f_n$  is continuous it follows that each  $J_{n_i}$  is open and therefore the set  $A_f$  belongs to  $G_{\omega_1}$ .

Suppose  $1 < \alpha < \omega_1$  and suppose that the function  $U_\gamma$  has been defined for each ordinal  $\gamma$  with  $1 \leq \gamma < \alpha$ . (Induction hypothesis.)

If  $\alpha$  is odd, let

$$U_\alpha(z) = \bigcup_{n=1}^{\infty} U_{\alpha-1}(z^n), \quad \text{for } z \in N.$$

Clearly  $U_\alpha$  maps  $N$  onto  $K_\alpha$ .

If  $\alpha$  is even, let  $\{\gamma_n\}_{n=1}^{\infty}$  be a sequence of odd ordinals less than  $\alpha$  such that each odd ordinal less than  $\alpha$  appears infinitely often in  $\{\gamma_i\}_{i=1}^{\infty}$  and let

$$U_\alpha(z) = \bigcap_{n=1}^{\infty} U_{\gamma_n}(z^n), \quad \text{for } z \in N.$$

If  $A \in K_\alpha$  ( $\alpha$  is still even), then

$$A = \bigcap_{n=1}^{\infty} D_n,$$

where  $D_n \in K_{\gamma_n}$ , for each  $n$ . For each  $n$ , let  $y_n$  be a point of  $N$  such that

$$D_n = U_{\gamma_n}(y_n).$$

And let  $z$  be the point mapped by the transformation described by (\*) to the point  $(y_1, y_2, y_3, \dots)$  of  $N^{\aleph_0}$ . Thus

$$U_\alpha(z) = A$$

and  $U_\alpha$  maps  $N$  onto  $K_\alpha$ .

This completes the definition of the functions  $U_\alpha$ . Now let  $f$  be a continuous function from  $N$  into  $N$ . It will be shown that if  $\alpha$  is even the set

$$A_f = \{z: z \in U_\alpha(f(z))\}$$

is in  $G_{\omega_1}$ . The argument for the case  $\alpha$  is odd is similar.

We have

$$\begin{aligned} A_f &= \left\{ z: z \in \bigcap_{n=1}^{\infty} U_{\gamma_n}((f(z))^n) \right\} \\ &= \bigcap_{n=1}^{\infty} \{z: z \in U_{\gamma_n}((f(z))^n)\}. \end{aligned}$$

But, for each  $n$ , the function  $z \rightarrow (f(z))^n$ , being the composition of two continuous functions, is a continuous function from  $N$  to  $N$ .

Thus by the induction hypothesis, the sets  $\{z: z \in U_{\gamma_n}((f(z))^n)\}$  are in the family  $G_{\omega_1}$ . Therefore  $A_f \in G_{\omega_1}$ .

**THEOREM 12.** *If  $G$  is a countable family of subsets of real numbers with  $\mathcal{S}(G) \subseteq G$ , and each Borel set is a member of  $\mathcal{B}(G)$  then  $G$  has order  $\omega_1$ .*

*Proof.* Let  $\alpha$  be any countable ordinal, and let

$$I_\alpha = \{z: z \notin U_\alpha(z)\}.$$

Suppose  $I_\alpha \in K_\alpha$ , and let  $U_\alpha(z) = I_\alpha$ . If  $z \in I_\alpha$  then  $z \in U_\alpha(z)$ . But this contradicts the definition of  $I_\alpha$ . If  $z \notin I_\alpha$ , then  $z \in U_\alpha(z) = I_\alpha$ ,  $z \in I_\alpha$ . This contradiction shows that  $I_\alpha \notin K_\alpha$ .

Since  $\mathcal{B}(G) = G_{\omega_1}$  (because  $\mathcal{C}(G) \cong G$ ), and  $I_\alpha = \{z: z \in U_\alpha(z)\} \in G_{\omega_1}$  by Theorem 11, it follows that  $I_\alpha \in G_{\omega_1} - G_\alpha$ . Thus  $G_\alpha \neq G_{\omega_1}$ , and hence  $G$  has order  $\omega_1$  [3, p. 371].

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