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SETS WHICH ARE TAME IN ARCS IN E^3

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Results of McMillan and Cannon may be combined to give an algebraic condition which is sufficient to show that an arc topologically embedded in E^3 is tame in E^3 . The main theorem of this paper gives an essentially algebraic condition involving an arc embedded in E^3 and a compact subset of that arc which is sufficient to show that the arc may be approximated arbitrarily closely without moving the subset, to obtain a tame arc.

1. Preliminaries. The usual Euclidean distance function will be denoted by d . An open neighborhood having radius r about a set S will be denoted by $N(S, r)$. An r -set will be a set having diameter less than r .

1.1. DEFINITION. Suppose that X is a compact subset of a finite complex K which is topologically embedded in E^3 . Then X is said to be tame in K iff given $r > 0$ there is a homeomorphism $h: K \rightarrow E^3$ such that

- (1) $d(x, h(x)) < r$ for each x in K ,
- (2) $h(x) = x$ for each x in X , and
- (3) $h(K)$ is tame.

1.2. DEFINITION. Suppose that X is a compact subset of an arc A which is topologically embedded in E^3 . Then X is said to be untangled iff for each $r > 0$, there is an $s > 0$ such that if J is a loop in $E^3 - X$ which bounds (homologically) on an s -set in $E^3 - X$, then J shrinks (homotopically) on an r -set in $E^3 - X$.

McMillan [3] has noted that an arc is untangled iff it has free local fundamental groups (1-FLG) at each of its points. He also proved that an arc which has 1-FLG at each point is tame if each of its subarcs pierces a disk. Cannon [2, Theorem 3.16] has shown that an arc which has 1-FLG at each point does pierce a disk. Hence, an arc which is untangled is tame.

1.3. NOTATION. For the remainder of this paper A will denote an arc topologically embedded in E^3 and X will denote a compact subset of A which is untangled. The arc A will be assumed to have a fixed order, compatible with, and inducing, the given topology on A .

1.4. DEFINITION. Let Y be a subset of A . An indexed collection C_1, \dots, C_n of disjoint connected subsets of E^3 is said to be ordered

with respect to Y iff, in the order on A , each point of $C_i \cap Y$ precedes each point of $C_{i+1} \cap Y$. ($i = 1, \dots, n - 1$.)

The lemma below gives a way to separate components of X by open sets in E^3 which are, roughly speaking, not much larger than the components.

1.5. SEPARATION LEMMA. Suppose that s is a positive number. Then there is a finite cover C_1, \dots, C_n of X by connected open subsets of E^3 with disjoint, polyhedral closures such that:

- (1) C_1, \dots, C_n is ordered with respect to X , and
- (2) For each i , there is a component X_i of X in C_i such that $C_i \subset N(X_i, s)$.

Proof. The lemma follows easily from the fact that X is a compact set.

2. Cellularity lemmas. In this section it is shown that if X is untangled, then each component K of X can be enclosed in a polyhedral ball which is "close" to K and which has boundary missing X . The proof falls naturally into two cases, depending on whether or not K is a nondegenerate component of X . The two cases are handled in 2.1 and 2.4 respectively. These results are referred to as cellularity lemmas.

2.1. CELLULARITY LEMMA FOR NONDEGENERATE COMPONENTS. Suppose that e is a positive number and that K is a nondegenerate component of X . Then there is a polyhedral ball B such that K is contained in B , $\text{Bd } B$ does not intersect X , and $B \subset N(K, e)$.

Proof. By the results of McMillan [3] and Cannon [2], K is a tame arc. Therefore, there is a 3-cell $C \subset N(K, e)$ which contains $K - \text{Bd } K$ in its interior, which does not intersect $A - K$, and which has boundary which is polyhedral modulo K .

In view of Dehn's lemma [5], it suffices to prove the fact stated below. Indeed, the fact may be used in conjunction with Dehn's lemma to alter $\text{Bd } C$ slightly near $K \cap \text{Bd } C$ and in $N(K, e)$ so as to miss X . The adjusted 2-sphere will then bound a ball B satisfying the requirements of the lemma.

Fact. Suppose that J is a simple closed curve in $\text{Bd } C$ which separates the endpoints of K from each other in $\text{Bd } C$, and suppose that J bounds a disk D in $\text{Bd } C$ of diameter less than some given positive number q . Then J bounds a singular disk D' in $E^3 - X$ of diameter less than q .

The fact is proved as follows. Let $r = q - \text{diam } D$. Choose $s > 0$ so small that loops which bound on s -sets in $E^3 - X$ shrink on r -sets in $E^3 - X$. Pick a 3-cell T of diameter less than s such that $\text{Bd } T$ separates the endpoints of K in E^3 and $(\text{Bd } T) \cap (\text{Bd } C)$ is a simple closed curve in $\text{Int } D$. Let E denote the disk $C \cap \text{Bd } T$. Use the separation lemma 1.5 to cover components of X which intersect $(\text{Bd } T) - E$ by a finite collection of disjoint open sets whose polyhedral closures miss $E \cup K$. Let W be the union of these sets and assume that $\text{Bd } W$ is in general position with respect to $\text{Bd } T$. Because $\text{Cl } W$ does not intersect $K \cup E$, $\text{Bd } E$ bounds homologically on the s -set $\text{Bd } (T - W) - \text{Int } E$ in $E^3 - X$ and therefore bounds a singular r -disk F in $E^3 - X$. The set $(D - K) \cup F$, which lies in $E^3 - X$, contains a singular disk D' of diameter less than q which is bounded by J . This establishes the fact and completes the proof of the lemma.

If, in the proof of 2.1, C is first partitioned by means of disjoint spanning disks D_1 and D_2 into three 3-cells—a central 3-cell C_3 , whose intersection with K is an arc $K_3 \subset \text{Int } K$, and end cells C_1 and C_2 whose intersections with K are the closed components K_1 and K_2 of $K - K_3$ —and $\text{Bd } C$ is adjusted only very near $(\text{Bd } C_1 - D_1) \cap K$ and $(\text{Bd } C_2 - D_2) \cap K$ in constructing $\text{Bd } B$, then the following is evident.

2.2. ADDENDUM. The ball B in 2.1 may be chosen in such a manner that it can be partitioned by disjoint spanning disks D_1 and D_2 into 3-cells B_1, B_2 , and B_3 ($B_i \cap B_3 = D_i$ for $i = 1, 2$) satisfying

(1) $B_3 \cap A$ is a subarc of $\text{Int } K$ which spans the cell B_3 from D_1 to D_2 ,

(2) the diameter of B_i is less than ϵ ($i = 1, 2$), and

(3) $B_i \cap A$ lies in an ϵ -arc on A ($i = 1, 2$) with one endpoint of this ϵ -arc missing X (unless B_i contains an endpoint of A).

The next lemma is well-known.

2.3. LEMMA. Suppose that J is a simple closed curve in E^3 which bounds an orientable surface T of diameter less than r , and suppose that L is an arc of diameter less than s which misses J . Then J bounds a surface of diameter less than $r + s$ in $E^3 - L$ and this surface may be chosen in an arbitrarily small neighborhood of $T \cup L$.

2.4. CELLULARITY LEMMA FOR DEGENERATE COMPONENTS. Suppose that s is a positive number and that $\{p\}$ is a degenerate component of X . Then there is a polyhedral 3-cell B such that p lies in B , $\text{Bd } B \cap X = \emptyset$, and $B \subset N(p, s)$.

Proof. As a first approximation to the desired 3-cell, let $B' = \text{Cl } (N(p, r))$ where $0 < r < s/2$ and r is so small that B' intersects no

component of X with diameter as large as $s/4$. Choose $q > 0$ so small that loops which bound on q -sets in $E^3 - X$ shrink on $r/2$ -sets in $E^3 - X$.

Choose a collection D_1, \dots, D_n of small disjoint polyhedral disks on $\text{Bd } B'$ with boundaries missing X and a collection S_1, \dots, S_m of small polyhedral spheres in $E^3 - X$ such that $X \cap \text{Bd } B'$ is contained in $D_1 \cup \dots \cup D_n \cup (\text{Int } S_1 \cap \text{Bd } B') \cup \dots \cup (\text{Int } S_m \cap \text{Bd } B')$. Specifically, select for each degenerate component $\{x\}$ of X which lies on $\text{Bd } B'$ a disk $D_x \subset \text{Bd } B'$ with $\text{Bd } D_x \cap X = \emptyset$ and so small that $\text{diam } D_x < q/2$ and $D_x \cap A$ lies on a subarc of A with diameter less than $q/2$ which has endpoints in $A - X$. Select a sphere for each nondegenerate component of X which intersects $\text{Bd } B'$ using the cellularity lemma for nondegenerate components, 2.1, with $s/4$ as the value of e in that lemma. A finite collection of these disks and spheres satisfies the conditions above except for the requirement that the disks be disjoint. Since their boundaries miss X , the disks may be made disjoint by simply putting their boundaries in general position and cutting them apart.

Application of Lemma 2.3 shows that, for each i , $\text{Bd } D_i$ bounds on a q -set in $E^3 - X$, and hence shrinks on an $r/2$ -set in $E^3 - X$. Because of this, if $\text{Int } D_i$ is thrown away and $\text{Bd } D_i$ shrunk, a singular 2-sphere R_i may be produced which intersects X in "fewer" places than did $\text{Bd } B'$ and which is essential in $E^3 - \{p\}$ since it is homotopic to $\text{Bd } B'$ in $E^3 - N(p, r/2)$. If any of the spheres S_i contains p , the proof is finished since each S_i has diameter less than s . Otherwise, for each i in turn, choose a point x_i in $\text{Int } S_i - R_{i-1}$ and project "radially" the part of R_{i-1} in $\text{Int } S_i$ into S_i to form a singular sphere R_i . The final result is a singular sphere R_m in $E^3 - X$, essential in $E^3 - \{p\}$, and lying in $N(p, s)$. Using a strong form of the sphere theorem (which is implicit in the original proof in [5] and is stated explicitly in [6]), there is a nonsingular, polyhedral sphere with the same properties. Taking B to be the closure of the interior of this sphere completes the proof.

3. The following theorem is the main result of this paper. It says that if a compact subset of an arc is untangled, then it is tame in that arc.

THEOREM 3.1. *Suppose that X is a compact subset of an arc A which is topologically embedded in E^3 , and that for each positive number r , there is a positive number s such that if J is a loop in $E^3 - X$ which bounds on an s -set missing X , then J shrinks on an r -set missing X . Then for each positive number q there is a homeomorphism $f: A \rightarrow E^3$ such that*

$$(1) \quad f(x) = x \text{ for each } x \text{ in } X,$$

- (2) $d(x, f(x)) < q$ for each x in A , and
 (3) $f(A)$ is tame.

Proof. The idea of the proof is as follows. Construct a homeomorphism $h: E^3 \rightarrow E^3$ such that the restriction of h to X takes X in an order preserving fashion into the x -axis. Define $f(A) = h^{-1}(I)$ where I is a suitably chosen subinterval of the x -axis. Care must be taken in the construction of h in order that A and $f(A)$ be close homeomorphically. The details of this process are described below.

It may be assumed that A is locally polyhedral modulo X . Use the cellularity lemma for nondegenerate components, 2.1, to construct a collection B'_1, \dots, B'_n of disjoint polyhedral 3-cells with boundaries in $E^3 - X$, one for each component of X with diameter as large as $q/12$. Choose ϵ in that lemma and its addendum, 2.2, less than $q/6$, and so small that the collection of cells is ordered with respect to X (although it need not cover X).

The set $X' = X - \bigcup B'_i$ is a compact subset of A with no component having diameter as large as $q/12$; X' is untangled. "Small" 3-cells covering X' are now constructed. By an appropriate choice of s in the separation lemma, 1.5, a cover C_1, \dots, C_m of X' may be obtained, the closures of whose elements miss $\bigcup B'_i$, have diameters less than $q/6$, and are ordered with respect to X . A suitable choice of s also guarantees that each C_i intersects at most one of the $q/6$ -arcs associated (by the Addendum 2.2) with any B'_j , and that $C_i \cap A$ lies in a $q/6$ -arc on A .

Each C_i may be changed to a ball, using the two cellularity lemmas to cover each component of X inside C_i by a polyhedral ball, picking a finite subcover of these, and following the methods of Bing [1] to cut apart and reconnect these balls to form a single ball containing $X \cap C_i$ in its interior. Retain the notation C_i for such a polyhedral ball.

Modify each B'_j ($j = 1, \dots, n$) as follows. The set $B'_j \cap Cl(A - B'_j)$ is contained in (at most) two $q/6$ -arcs A_1 and A_2 on A , and the intersection of each of these with B'_j lies in a $q/6$ -cell (by the addendum). If A_i intersects $X - B'_j$, then $A_i \cap (X - B'_j)$ is contained in some of the C_i 's. Connect these C_i 's to the $q/6$ -cells using disjoint $q/6$ -cells, so that the result is two $5q/6$ -cells, each of which intersects one of A_1 or A_2 and no other point of $A - B'_j$.

Let $B_1, \dots, B_n, B_{n+1}, \dots, B_k$ be the modified B'_j 's ($j = 1, \dots, n$) plus the C_j 's not used in the modifications. Assume that the indices are arranged so that this collection is ordered with respect to X .

Figure 1 illustrates the situation at this point of the proof.

The purpose in constructing B_1, \dots, B_n so carefully is to make it possible to change A homeomorphically by moving only points of

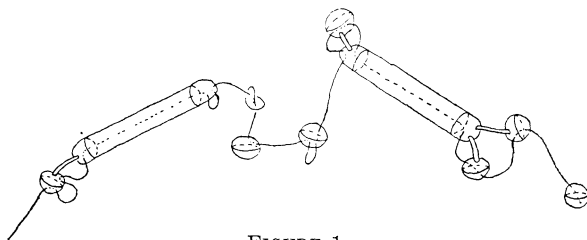


FIGURE 1

A which are near small components of X and near the endpoints of large components of X . Define a homeomorphism f' on A to move some subarcs of the $q/6$ -arcs associated with B_1, \dots, B_n into the $5q/6$ -cells at the "ends" of these "large" balls, and to move some subarcs of the $q/6$ -arcs associated with B_{n+1}, \dots, B_k into these balls; do this so that $f'(A)$ intersects each $\text{Bd } B_j$ in at most two points (and only one point if B_j contains an endpoint of A).

The first approximation h_1 to the homeomorphism h taking X into the x -axis is defined to take $f'(A) - \bigcup B_i$ into the x -axis, to take each B_i to a small neighborhood of an arc on the x -axis, and to take each component of X with diameter as large as $q/12$ into the x -axis, everything with order preserved. Subsequent approximations to h will be the identity outside the images under h_1 of the $5q/6$ -cells associated with B_1, \dots, B_n and of B_{n+1}, \dots, B_k , and this will ensure that conclusion (2) of the theorem is true.

Approximations to h are now obtained sequentially. At the second stage of the construction, a new collection of balls is chosen, inside the first, closer to X , and separating the components of X with diameters as large as $q/24$. A homeomorphism h_2 of E^3 onto itself is obtained which is the identity on each of the h_1 -images of the components of X which have diameter as large as $q/12$. This homeomorphism maps h_1 -images of the new balls to neighborhoods of arcs on the x -axis, and $h_2 h_1$ also sends components of X with diameter as large as $q/24$ into the x -axis, preserving order of cells.

Continue in this manner, choosing the balls so small that the sequence $h_1, h_2 h_1, h_3 h_2 h_1, \dots$ of homeomorphisms converges to a homeomorphism h . Let I be the smallest subarc of the x -axis which contains $h(f'(A)) \cap (x\text{-axis})$. The arc $h^{-1}(I)$ differs from $f'(A)$ only in the end cells of B_1, \dots, B_n and in B_{n+1}, \dots, B_k and so a homeomorphism $f: A \rightarrow E^3$ may be defined so that $f(A) = h^{-1}(I)$ and $d(x, f(x)) < q$ for each x in A . The homeomorphism f may also be chosen to fix points of X since the restriction of h to X is a homeomorphism of X into the x -axis. This completes the proof of the theorem.

There is also a relative version of 3.1:

THEOREM 3.2. *Suppose that X is a compact subset of an arc A*

which is topologically embedded in E^3 and that X is untangled. Suppose that $g: A \rightarrow [0, \infty)$ is a continuous function so that $g^{-1}(0) \cap X = \emptyset$. Then there is a homeomorphism $f: A \rightarrow E^3$ such that

- (1) $f(x) = x$ for each x in X ,
- (2) $d(x, f(x)) < g(x)$ for each x in A , and
- (3) $f(A)$ is locally tame modulo the set $g^{-1}(0)$.

Proof. The proof is almost exactly the same as for 3.1, except for beginning with an approximation to A which is locally polyhedral modulo $X \cup g^{-1}(0)$ and which is homeomorphically within g of A . This new arc is then modified on subarcs close to X in the same way as in the proof of 3.1.

4. Theorem 3.1 may be combined with a characterization of subsets of arcs due to R. L. Moore [4, Theorem 135] to yield a characterization of subsets of tame arcs in E^3 . Moore's theorem is proved for a space satisfying his axioms 0-5 and E^3 does not satisfy axiom 4. However the proof is not difficult in this case.

THEOREM (R. L. Moore) 4.1. *In order that the compact point set M in E^3 be a subset of an arc it is necessary and sufficient that every closed and connected subset of M be either a degenerate point set or an arc t such that no point of t , except for its endpoints is a limit point of $M - t$.*

4.2. CHARACTERIZATION OF SUBSETS OF TAME ARCS IN E^3 . Suppose that X is a compact subset of E^3 . Then X is a subset of a tame arc in E^3 if and only if each component of X is a point or an arc t such that no point of t , except possibly for an endpoint, is a limit point of $X - t$, and for each positive number r , there is a positive number s such that each loop which bounds on a s -set in $E^3 - X$ shrinks on an r -set in $E^3 - X$.

Proof. Sufficiency follows from 3.1 and 4.1. Necessity is obvious.

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