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Separable algebras have been studied recently by M. Auslander, D. Buchsbaum and Chase-Harrison-Rosenberg. The question of a Galois theory for linear topological rings opposite to the Krull type theory obtained in the above works was raised by H. Röhrl. In this paper, a Galois theory relating the complete subalgebras of restricted type of a complete algebra A to a set of subgroups of a discrete group G of automorphisms of A is developed.

The notion of a linear topological module has been discussed in [1], [5], [6], [7]; while the concepts pertaining to separables algebras are now available in the monograph [4] for the most part. We employ two results of [3] which we will state below. All rings considered will be commutative with 1.

DEFINITION 0.1 [3]. Two ring morphisms $A \underset{g}{\overset{f}{\rightrightarrows}} B$ are *strongly distinct* if, for each nonzero idempotent $e \in B$, there is $a \in A$ with $f(a)e \neq g(a)e$. Where B is connected, f and g are strongly distinct if and only if they are distinct.

THEOREM 0.2 [3]. *Let G be a finite group of automorphisms of the ring A having (pointwise) fixed ring k. The following statements are equivalent:*

(0) *A is a separable k-algebra [and the elements of G are pairwise strongly distinct].*

(1) *There are families of elements of A, $(x_i)_{i=1}^n$, $(y_i)_{i=1}^n$ with*

$$\sum_{i=1}^{n} x_i \sigma(y_i) = \delta_{1\sigma}$$

for each $\sigma \in G$, where $\delta_{1\sigma}$ is the Kronecker delta.

(2) *For each $\sigma \in G \backslash \{1\}$ and each maximal ideal $m < A$, there is $a \in A$ with $a - \sigma(a) \notin m$.*

(3) *For each connected k-algebra B and each pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ of k-algebra morphism, there is a unique $\sigma \in G$ with $\sigma g = f$.*

Proof. (0) $\to$ (1) $\to$ (2) $\to$ (0) is contained in [3], Theorem (1.3), and the implication (2) $\to$ (3) is Corollary (3.2) of [3]. We establish (3) $\to$ (2). Let $m < A$ be a maximal ideal and suppose $\sigma \in G \backslash \{1\}$. Then the

k-algebra A/m is connected, so the two k-algebra morphisms q, σq: $A \to A/m$ are distinct (q is the quotient map), otherwise $\sigma = 1$. Hence, there is $a \in A$ with $a - \sigma(a) \notin m$.

DEFINITION 0.3 [3]. When any of the equivalent conditions (0)-(3) of (0.2) hold for (A, G), we call (A, G) a Galois extension of k with group G.

Note that when A is connected and (A, G) is a Galois extension of k, (0.2)(3) shows that G the full group of k-algebra automorphisms of A.

DEFINITION 0.4 [3]. Let (A, G) be a Galois extension of k and let B be a subring of A. B will be called *G-strong* if the restrictions to B of any two elements of G are either equal or strongly distinct.

THEOREM 0.5 ([3] 2.3). *Let (A, G) be a Galois extension of k. Then there is Galois correspondence (g, r) between the set of separable k-subalgebras of A which are G-strong and the set of subgroups of G. If B is a separable G-strong k-subalgebra of A, then $g(B)\colon = \{\sigma \in G \mid \sigma(b) = b$ for all $b \in B\}$. Moreover, if $\sigma \in G$, $g(\sigma B) = \sigma g(B)\sigma^{-1}$. A subgroup H of G is normal in G if and only if $r(H)\colon = \{a \in A \mid \sigma(a) = a$ for all $\sigma \in H\}$ is a G-invariant subalgebra of A, in which case $(r(H), G/H)$ is a Galois extension of k with group G/H.*

We now pass to linear topological case.

DEFINITION 0.6. The ring A with a filter basis of ideals $\mathscr{U}(A)$ has a *linear topology* with $a \in A$ having a basis of neighborhoods the family $(a + U) U \in \mathscr{U}(A)$, and the pair $(A, \mathscr{U}(A))$ or briefly A will be called a *linear topological ring*. *A linear topological k-algebra* is a continuous ring morphism

$$(k, \mathscr{U}(k)) \xrightarrow{\rho} (A, \mathscr{U}(A)) .$$

1. Quasi-Galois extensions. Consider the following situation:

(0) $k \to A$ is a linear topological k-algebra.

(1) F is a final subset of $\mathscr{U}(A)$.

(2) $I \in F$ implies that A/I is a connected Galois extension of $k/k \cap I$ with Galois group G_I.

PROPOSITION 1.1. *There is a unique contravariant monic valued functor $G\colon F \to Gr$ (Gr is the category of groups) such that $G(I) = G_I$, and such that $I \leqq I'$ in F implies the commutativity of the diagram:*

$$\begin{array}{ccc} A/I & \xrightarrow{G(I',\, I)(\sigma)} & A/I \\ \downarrow & & \downarrow a^I_{I'} \\ A/I' & \xrightarrow{\sigma} & A/I' \end{array}$$

for each $\sigma \in G(I')$, *where* $a^I_{I'}$ *is the canonical quotient map.*

Proof. For each $\sigma \in G(I')$, there is by (0.2), (3), a unique $\sigma' \in G(I)$ such that $\sigma' a^I_{I'} = a^I_{I'}\sigma$. We define $G(I', I)(\sigma) := \sigma'$. The uniqueness available in (0.2), (3), guarantees that $G(I', I)$ is a group morphism, and the surjectivity of $a^I_{I'}$ entails the injectivity of $G(I', I)$.

DEFINITION 1.2. The triple (A, F, G) will be called *an extension of* k if:

(0) $k \rightarrow A$ is a linear topological k-algebra.

(1) F is a final subset of $U(A)$; so F is a filter basis.

(2) $G: F \rightarrow Gr$ is a contravariant monic valued functor such that

(i) $G(I)$ is a finite subgroup of the group of $k/k \cap I$-automorphisms of A/I;

(ii) for each $I \leqq I'$ in F and $\sigma \in G(I')$ the diagram of (1.1) is commutative.

If for each $I \in F$, $(A/I, G(I))$ is a Galois extension of $k/k \cap I$ with Galois group $G(I)$, we will call (A, F, G) a *quasi-Galois extension of* k *with group* G.

An immediate consequence of (1.1) is the

COROLLARY 1.3. *If* (A, F, G) *is a quasi-Galois extension of* k, *and if for each* $I \in F$, A/I *is connected, then the functor* G *is uniquely determined.*

Let (A, F, G) be an extension of K. We will define a group $\hat{G}$ of continuous k-automorphisms of $\hat{A}$

$$(\hat{A} = \varprojlim_{I \in \mathscr{U}(A)} A/I \quad \text{and} \quad \mathscr{U}(\hat{A}) = \{\ker(\hat{A} \xrightarrow{a_I} A/I) I \in \mathscr{U}(A)\})$$

and show that when (A, F, G) is a quasi-Galois extension of k, then there is a Galois correspondence (g, r) between a specific class of subgroups of $\hat{G}$ and a class of complete $\hat{k}$-subalgebras of $\hat{A}$. Each of these classes is characterized by the quality of their approximations, i.e., we require that their approximations satisfy a specific condition for each $I \in F$.

Since F is a filter basis, the family $(G(I))_{I \in F}$ of groups is cofiltered, and we can form the colimit $\hat{G} := \varinjlim G(I)$, the colimit being taken over $I \in F$. We denote by $g_I: G(I) \to \hat{G}$ the canonical colimit morphisms; they are injective, and for $I \leqq I'$ in F yield a commutative diagram:

$$\begin{array}{ccc} G(I') & \xrightarrow{G(I',\,I)} & G(I) \\ \downarrow{\scriptstyle g_{I'}} & & \downarrow{\scriptstyle g_I} \\ \hat{G} & = = = = & \hat{G}\,. \end{array}$$

Another useful description of $\hat{G}$ is obtained as follows. Fix $I' \in F$ and consider any $I \leqq I'$ in F. We then have a commutative diagram:

$$\begin{array}{ccc} A/I & \xrightarrow{G(I',\,I)(\sigma)} & A/I \\ \downarrow{\scriptstyle a^I_{I'}} & & \downarrow{\scriptstyle a^I_{I'}} \\ A/I' & \xrightarrow{\sigma} & A/I' \end{array}$$

for each $\sigma \in G(I')$. Evidently, the family of morphism $(G(I', I)(\sigma))_{I \leqq I'}$ is filtered and compatible with the quotient maps $a^I_{I'}$, so we can form the limit $\hat{\sigma}$ of this family, obtaining, for each $I \leqq I'$, the commutative diagram:

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\hat{\sigma}} & \hat{A} \\ \downarrow{\scriptstyle a_I} & & \downarrow{\scriptstyle a_I} \\ A/I & \xrightarrow{G(I',\,I)(\sigma)} & A/I\,. \end{array}$$

We let H denote the set of all such $\hat{\sigma}$ for $I' \in F$ and $\sigma \in G(I')$ arbitrary. The foregoing diagram shows that each $\hat{\sigma}$ is a continuous $\hat{k}$-automorphism of $\hat{A}$. If $\hat{\sigma}, \hat{\tau} \in H$, say $\sigma \in G(I')$ and $\tau \in G(\tau'')$, we define $\hat{\sigma}\hat{\tau} = \hat{\mu}$, where $\mu = G(I', I)(\sigma) \cdot G(I'', I)(\tau)$ and $I \leqq I', I''$. Since F is a filter basis, $\hat{\mu}$ does not depend on I, and so is well-defined; moreover, this multiplication makes H a group.

PROPOSITION 1.4. *The mapping $H \to \hat{G}$, given by $\hat{\sigma} \to g_I(\sigma)$, where $\sigma \in G(I)$, is a group isomorphism.*

Proof. Define $h_I: G(I) \to H$ by putting $h_I(\sigma) = \hat{\sigma}$. The h_I are then group morphisms compatible with the inclusions $G(I', I)$ for $I \leqq I'$; hence, there is a unique group morphism $h: \hat{G} \to H$ such that $g_I h = h_I$ for all $I \in F$. Next, define $g: H \to \hat{G}$ by putting $g(\hat{\sigma}) = g_I(\sigma)$ if $\sigma \in G(I)$. To see that g is well-defined, let $\hat{\sigma} = \hat{\tau}$, where $\sigma \in G(I')$ and $\tau \in G(I'')$, and choose $I \leqq I', I''$. Then

$$\begin{aligned} 1 = \hat{\sigma}(\hat{\tau})^{-1} &= [G(I', I)(\sigma)]^{\wedge} \cdot [G(I'', I)(\tau^{-1})]^{\wedge} \\ &= [G(I', I)(\sigma) G(I'', I)(\tau^{-1})]^{\wedge} . \end{aligned}$$

This shows that the diagram:

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\ 1\ } & \hat{A} \\ \Big\downarrow{\scriptstyle a_I} & & \Big\downarrow{\scriptstyle a_I} \\ A/I & \xrightarrow{\ \mu\ } & A/I \end{array}$$

is commutative, where $\mu = G(I', I)(\sigma)G(I'', I)(\tau^{-1})$. But a_I is surjective, so we conclude that $\mu = 1$, and so $G(I', I)(\sigma) = G(I'', I)(\tau)$, proving that $g_{I'}(\sigma) = g_I(G(I', I)(\sigma)) = g_I(G(I'', I)(\tau)) = g_{I''}(\tau)$ as required.

A similar argument shows that g is a group morphism. Finally, let $\sigma \in G(I)$, then $h(g(\hat{\sigma})) = h(g_I(\sigma)) = h_I(\sigma) = \hat{\sigma}$. On the other hand, each element x of $\hat{G}$ has the form $g_I(\sigma)$ for some $I \in F$, since F is a filter basis. It follows that $g(h(x)) = gh(g_I(\sigma)) = g(h_I(\sigma)) = g(\hat{\sigma}) = g_I(\sigma) = x$. Thus, we have the group identities $1 = gh$ and $1 = hg$ showing that g is a group isomorphism.

PROPOSITION 1.5. *If (A, F, G) is an extension of k such that for each $I \in F$, the fixed ring of $G(I)$ is $k/k \cap I$, then the fixed ring of $\hat{G}$ is $\hat{k}$.*

Proof. We have already observed that $G(I) \leqq \mathrm{Auto}_{k/k \cap I}(A/I)$ implies that the elements of $\hat{G}$ are $\hat{k}$-automorphisms of $\hat{A}$. Now suppose $\alpha \in \hat{A}$ belongs to the fixed ring of $\hat{G}$. Then we have commutative diagram:

$$\begin{array}{ccccccc} \hat{k} & \xrightarrow{\ u\ } & \hat{k}[\alpha] & \xrightarrow{\ v\ } & \hat{A} & \xrightarrow{\ \hat{\sigma}\ } & \hat{A} \\ \Big\downarrow{\scriptstyle k_I} & & & & \Big\downarrow{\scriptstyle a_I} & & \Big\downarrow{\scriptstyle a_I} \\ k/k \cap I & & \xrightarrow{\ \ \rho_I\ \ } & & A/I & \xrightarrow{\ \sigma\ } & A/I \end{array}$$

where ρ_I, u and v are the canonical inclusions and $uv = \hat{\rho}\colon \hat{k} \to \hat{A}$ is the limit of the morphisms ρ_I, and where $\sigma \in G(I)$. $\hat{k}[\alpha]$ has the topology induced by v, so all the morphisms are continuous. By hypothesis, $va_I\sigma = v\hat{\sigma}a_I = va_I$, so that va_I factors through the fixed ring of $G(I)$, namely $k/k \cap I$. Let the factorization be $va_I = w_I\rho_I$. For $I \leqq I'$ in F, we have $w_I k^I_{I'} \rho_{I'} = w_I \rho_I a^I_{I'} = va_I a^I_{I'} = va_{I'} = w_{I'}\rho_{I'}$ and since $\rho_{I'}$ is monic, $w_I k^I_{I'} = w_{I'}$. Thus, we obtain a family $(w_I) I \in F$ compatible with the morphisms $k^I_{I'}\colon k/k \cap I \to k/k \cap I'$. Passing to the limit, we obtain a commutative diagram

$$\begin{array}{ccc} \hat{k}[\alpha] & \xrightarrow{\;w\;} & \hat{k} \\ \| & & \downarrow{\scriptstyle \kappa_I} \\ \hat{k}[\alpha] & \xrightarrow{\;w_I\;} & k/k \cap I \end{array}$$

for each $I \in F$. w is continuous, and $va_I = w_I \rho_I = w k_I \rho_I = w(uv)a_I$ for each $I \in F$, so passing to the limit again, $v = (wu)v$. But v is monic, so we conclude that $1 = wu$ showing that u is surjective. Since u is already injective, u is an isomorphism and we conclude that $\alpha \in \hat{k}$ as desired.

THEOREM 1.6. *Let (A, F, G) be an extension of k such that for each $I \in F$, the fixed ring of $G(I)$ is $k/k \cap I$. Then the following statements are equivalent.*

(0) *(A, F, G) is a quasi-Galois extension of k.*

(1) *For each $\hat{\sigma} \in \hat{G} \backslash 1$ and each open, maximal ideal $m < \hat{A}$, there is $x \in \hat{A}$ with $x - \hat{\sigma}(x) \notin m$.*

In addition, if $I \in F$ implies that A/I is connected, (0) *and* (1) *are equivalent to a third condition.*

(2) *A is a quasi-separable k-algebra, i.e., $I \in F$ implies A/I is a separable $k/k \cap I$ — algebra.*

Proof. Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & \hat{A} \\ \| & & \downarrow{\scriptstyle a_I} \\ A & \xrightarrow{\;\alpha_I\;} & A/I \end{array}$$

where i is the canonical limit morphism, and α_I and a_I are the quotient maps. Let $m < \hat{A}$ be an open, maximal ideal and let $\hat{\sigma} \in \hat{G} \backslash 1$. We may suppose $I \in F$ is such that $m \geqq \ker(a_I)$ and $\hat{\sigma} = g_I(\sigma)$. Since $i^{-1}(m)$ is an open, maximal ideal of A, $\alpha_I(i^{-1}(m))$ is a maximal ideal of A/I, and $\sigma \in G \backslash 1$ shows that there is $a \in A/I$ such that $a - \sigma(a) \notin \alpha_I(i^{-1}(m))$, assuming (0), by (0.2). Suppose $y \in A$ is such that $\alpha_I(y) = a$, then $i(y) - \hat{\sigma} i(y) \notin m$; otherwise, $a_I i(y) - a_I \hat{\sigma} i(y) = \alpha_I(y) - \sigma \alpha_I(y) \in a_I(m) = \alpha_I(i^{-1}(m))$ contrary to our choice of $\alpha_I(y) = a$. Thus, $i(y) - \hat{\sigma} i(y) \notin m$ as desired.

Now suppose m is a maximal ideal of A/I and let $\sigma \in G(I) \backslash 1$. Then $a_I^{-1}(m)$ is an open, maximal ideal of $\hat{A}$, and $g_I(\sigma) = \hat{\sigma} \in \hat{G} \backslash 1$. We obtain, therefore, $x \in \hat{A}$ with $x - \hat{\sigma}(x) \notin a_I^{-1}(m)$. It follows that $a_I(x) - a_I \hat{\sigma}(x) = a_I(x) - \sigma a_I(x) \notin m$ showing that A/I is a Galois extension of $k/k \cap I$ with Galois group $G(I)$ by (0.2).

If, in addition, $I \in F$ implies that A/I is connected, and (0) holds, then by definition A is a quasi-separable k-algebra. The converse implication follows from (0.2).

COROLLARY 1.7. *Suppose (A, F, G) is an extension of k such that for each $I \in F$, the fixed ring of $G(I)$ is $k/k \cap I$. If the condition $(*)$ below holds, then (A, F, G) is a quasi-Galois extension of k.*

$(*)$ For each $\hat{k}$-algebra B and each pair of continuous $\hat{k}$-algebra morphisms $f, g: \hat{A} \rightarrow B$, there is a unique $\hat{\sigma} \in \hat{G}$ such that $\hat{g} = \hat{\sigma} f$.

Proof. Let $\hat{\sigma} \in \hat{G} \backslash 1$ and let $m < \hat{A}$ be an open, maximal ideal. If $a - \hat{\sigma}(a) \in m$ for all $a \in A$, then the two $\hat{k}$-algebra morphisms $q: \hat{A} \rightarrow \hat{A}/m$ and $\hat{\sigma} q$ agree on $\hat{A}$, so by $(*)$ we must have that $\hat{\sigma} = 1$ which is a contradiction. We conclude that there is $a \in \hat{A}$ with $a - \hat{\sigma}(a) \notin m$, and so by (1.6) (A, F, G) is a quasi-Galois extension of k.

DEFINITION 1.8. Let (A, F, G) be an extension of k. For each subgroup H of $\hat{G}$ let $r(H)$ denote the pointwise fixed ring of H and let $H_I := g_I^{-1}(H)$. For each $\hat{k}$-subalgebra B of $\hat{A}$ let $g(B)$ denote the subgroup of $\hat{G}$ fixing B elementwise.

For $I \leqq I'$ in F we then have a commutative diagram:

$$\begin{array}{ccc}
H & \xrightarrow{\quad h \quad} & G \\
\uparrow{\scriptstyle J_I} & & \uparrow{\scriptstyle g_I} \\
H_I & \xrightarrow{\quad h_I \quad} & G(I) \\
\uparrow{\scriptstyle J^I_{I'}} & & \uparrow{\scriptstyle G(I', I)} \\
H_I & \xrightarrow{\quad h_{I'} \quad} & G(I')
\end{array}$$

where h, h_I, and $h_{I'}$ are the canonical inclusions, and J_I and $J^I_{I'}$ are the monomorphisms induced by g_I and $G(I', I)$ respectively.

PROPOSITION 1.9. *The colimit of the family $(H_I, J^I_{I'})$ is H with the colimit morphisms being the J_I.*

Proof. We have just observed the compatibility of the family of morphisms J_I with the mappings $J^I_{I'}$ for $I \leqq I'$ in F, and it remains to establish their universality. Let $x_I: H_I \rightarrow X$ be any family of group morphisms compatible with the mappings $J^I_{I'}$ $(I \leqq I'$ in $F)$. Define $x: H \rightarrow X$ by putting $x(\hat{\sigma}) := x_I(\sigma)$, if $g_I(\sigma) = \hat{\sigma}$. If $g_{I'}(\sigma') = \hat{\sigma}$ also, choose $I'' \leqq I, I'$ so that $J^{I''}_I(\sigma) = J^{I''}_{I'}(\sigma')$. Then $x_I(\sigma) = x_{I''}(J^{I''}_I(\sigma)) = x_{I''}(J^{I''}_{I'}(\sigma')) = x_{I'}(\sigma')$ shows that x is a group morphism, and the equality $J_I x = x_I$ for $I \in F$ shows that x is uniquely determined. Hence, $J_I: H_I \rightarrow H$ is a colimit for $(H_I, J^I_{I'})$.

Next, let H be a subgroup of G, and obtain the diagram:

$$\begin{array}{ccccc}
r(H) & \xrightarrow{\alpha} & \hat{A} & \xrightarrow{\sigma} & \hat{A} \\
\downarrow{\scriptstyle r_I} & & \downarrow{\scriptstyle a_I} & & \downarrow{\scriptstyle a_I} \\
r(H_I) & \xrightarrow{\alpha_I} & A/I & \xrightarrow{\sigma} & A/I \\
\downarrow{\scriptstyle r^I_{I'}} & & \downarrow{\scriptstyle a^I_{I'}} & & \downarrow{\scriptstyle a^I_{I'}} \\
r(H_{I'}) & \xrightarrow{\alpha_{I'}} & A/I' & \xrightarrow{\sigma'} & A/I'
\end{array}$$

which is commutative, where α, α_I, $\alpha_{I'}$ are inclusions providing their respective domains with the induced topology. For each $\sigma \in H_I$, $\alpha a_I \sigma = \alpha \hat{\sigma} a_I = \alpha a_I$, so that a_I factors through $r(H_I)$, defining r_I. Then $\alpha a_I = r_I \alpha_I$ for all $I \in F$. Similarly, if $I \leqq I'$ in F, and $\sigma' \in G(I')$ and $\sigma = G(I', I)(\sigma')$, then $\sigma_I a^I_{I'} \sigma^1 = \alpha_I a^I_{I'}$, so that $\alpha^I_{I'}$ factors through $r(H_{I'})$, defining $r^I_{I'}$. Then $r^I_{I'} \alpha_{I'} = \alpha_I a^I_{I'}$. Still using the above diagram, we obtain from the equality $r_{I'} \alpha_{I'} = r_I r^I_{I'} \alpha_{I'}$ the relation $r_{I'} = r_I r^I_{I'}$ since $\alpha_{I'}$ is monic. This shows that the mapping $r_I \colon r(H) \rightarrow r(H_I)$ are compatible with the mapping $(r^I_{I'}) I \leqq I'$ in F.

PROPOSITION 1.10. *The mappings $r_I \colon r(H) \rightarrow r(H_I)$ form a limit for the family $(r(H_I), r^I_{I'})$.*

Proof. Let $x_I \colon X \rightarrow r(H_I)$ be any family of continuous ring morphisms compatible with the $r^I_{I'}$. Composing this family coordinatewise with the family $(\alpha_I) I \in F$, we obtain a family $(x_I \alpha_I) I \in F$ compatible with the canonical quotient maps $a^I_{I'}$. Hence, there is a unique continuous mapping $x \colon X \rightarrow \hat{A}$ such that $x a_I = x_I \alpha_I$ for each $I \in F$. Now let $\hat{\sigma} \in H$, say $\hat{\sigma} = g_{I'}(\sigma)$ for some $I' \in F$. For all $I \leqq I'$ in F, $x \hat{\sigma} a_I = x a_I G(I', I)(\sigma) = x_I \alpha_I G(I', I)(\sigma) = x_I \alpha_I = x a_I$ since $G(I', I)(\sigma) \in H_I$. This being true for all small $I \in F$, passing to the limit, we have $x \hat{\sigma} = x$, showing that x must factor through $r(H)$. Let $x = y\alpha$ for some $y \colon X \rightarrow r(H)$. y is then unique, since α is monic, and $y r_I \alpha_I = y \alpha a_I = x_I \alpha_I$ implies that $y r_I = x_I$ since α_I is monic. This completes the proof.

REMARK. The topology induced by α on $r(H)$ coincides with the limit topology for $\ker(r_I) = \ker(r_I \alpha_I) = \ker(\alpha a_I)$. For the remainder of this section we assume (A, F, G) is a quasi-Galois extension of k.

For each subgroup H of $\hat{G}$ we are led to a commutative diagram:

$$\begin{array}{ccccccc}
r(H) & = & r(H) & \xrightarrow{\alpha} & \hat{A} & \xrightarrow{\hat{\sigma}} & \hat{A} \\
\downarrow{\scriptstyle e_I} & & \downarrow{\scriptstyle r_I} & & \downarrow{\scriptstyle a_I} & & \downarrow{\scriptstyle a_I} \\
r(H)_I & \xrightarrow{m_I} & r(H_I) & \xrightarrow{\alpha_I} & A/I & \xrightarrow{\sigma_I} & A/I
\end{array}$$

where $r(H)$ is the image of αa_I and $r(H)_I \leqq r(H_I)$, since $\sigma \in H_I$ implies $e_I\alpha'_I\sigma = e_I\alpha'_I$, where $e_I\alpha'_I$ is the canonical factorization of a_I through $r(H)_I$. Since e_I is surjective, $\alpha'_I\sigma = \alpha'_I$ shows that $r(H)_I \leqq r(H_I)$, say $m_I\colon r(H)_I \to r(H_I)$ so that $\alpha'_I = m_I\alpha_I$. Since α_I is monic and $e_I m_I \alpha_I = r_I\alpha_I$, $e_I m_I = r_I$, so the first square is commutative.

It follows immediately from the definitions that $H \leqq gr(H)$ for each subgroup H of $\hat{G}$.

LEMMA 1.11. *Suppose $H \leqq \hat{G}$ satisfies the condition $I \in F \to H_I = g[r(H)_I]$, where g is appropriately defined. Then $gr(H) = H$.*

Proof. Of course, by $g[r(H)_I]$ we mean the set

$$\{\sigma \in G(I) \mid x \in r(H)_I \longrightarrow \sigma(x) = x\} .$$

Let $\hat{\sigma} \in gr(H)$ and suppose $g_I(\sigma) = \hat{\sigma}$. Then the equality $m_I\alpha_I\sigma = m_I\alpha_I$ shows that $\sigma \in g[r(H)_I] = H_I$ by hypothesis; hence $\hat{\sigma} = g_I(\sigma) \in H$.

DEFINITION 1.12. Call a $\hat{k}$-subalgebra B of $\hat{A}$ *G-strong* if for each $I \in F$, B_I is a $G(I)$-strong subalgebra of A/I.

LEMMA 1.13. *Let $H \leqq \hat{G}$. The following statements are equivalent:*

1.14. (0) *$I \in F \to r(H)_I = r(H_I)$, i.e., r_I is surjective.*
(1) *$I \in F \to H_I = g[r(H)_I]$ and $r(H)$ is a G-strong separable $\hat{k}$-subalgebra of $\hat{A}$.*

Proof. Suppose (0), then since (A, F, G) is a quasi-Galois extension of k, $r(H)_I = r(H_I)$ shows that $r(H_I)$ is a $G(I)$-strong separable $k/k \cap I$-subalgebra of A/I for $I \in F$. $r(H)$ is a closed $\hat{k}$-subalgebra of the complete separated ring $\hat{A}$, i.e., is complete. Finally, $H_I = gr(H_I) = g[r(H)_I]$ by (0) and (0.5). Conversely, if (1) holds, then

$$r(H_I) = rg[r(H)_I] = r(H)_I$$

since $r(H)$ is a G-strong quasi-separable $\hat{k}$-subalgebra of $\hat{A}$ and $rg = 1$ by (0.5).

COROLLARY 1.15. *If $H \leqq \hat{G}$ satisfies* (1.14), *$gr(H) = H$.*

Now let B be a complete $\hat{k}$-subalgebra of $\hat{A}$ and put $H = g(B)$. We obtain the following supplement to the last diagram

$$\begin{array}{ccc} B & \xrightarrow{\beta} & r(H) \\ \downarrow{\scriptstyle b_I} & & \downarrow{\scriptstyle e_I} \\ B_I & \xrightarrow{\beta_I} & r(H)_I \end{array}$$

for each $I \in F$. For evidently $B \leqq rg(B) = r(H)$.

LEMMA 1.16. *Suppose B is a complete $\hat{k}$-subalgebra of $\hat{A}$ satisfying the condition.*

1.17. $I \in F \rightarrow B_I = r[g(B)_I]$.
Then B is a G-strong quasi-separable $\hat{k}$-subalgebra of $\hat{A}$, $rg(B) = B$, and $g(B)$ satisfies Condition 1.14.

Proof. Since $B_I = r[g(B)_I]$ is the fixed ring of a subgroup of $G(I)$, it follows from (0.5) that B_I is a $G(I)$-strong separable $k/k \cap I$-subalgebra of A/I, proving our first assertion. Next, we have the equalities:

$$B = \varprojlim_I B_I = \varprojlim_I (r[g(B)_I]) = r(\varinjlim_I [g(B)_I]) = rg(B)$$

by (1.9) and (1.10). Using this fact, we obtain $[rg(B)]_I = B_I = r[g(B)_I]$ showing that (1.14) holds for $g(B)$.

REMARK. If $H \leqq \hat{G}$ satisfies Condition 1.14, then $r(H)$ satisfies Condition 1.17 for $r(H)_I = r(H_I) = r[(gr(H))_I]$ since $H = gr(H)$.

THEOREM 1.18. *Let (A, F, G) be a quasi-Galois extension of k. Then the pair of maps (g, r) is a Galois correspondence between the set of all complete $\hat{k}$-subalgebras of $\hat{A}$ satisfying Condition* 1.17 *and the set of all subgroups of $\hat{G}$ satisfying Condition* 1.14.

Proof. We need only observe that $gr = 1$ and $rg = 1$ are valid equations when restricted to the sets mentioned in the statement of the theorem.

PROPOSITION 1.19. *Suppose H is normal subgroup of $\hat{G}$ satisfying Condition* 1.14. *Then for each $I \in F$, H_I is a normal subgroup of $G(I)$.*

Proof. Form the diagram:

$$\begin{array}{ccccccc}
r(H) & \longrightarrow & r(H) & \xrightarrow{\alpha} & \hat{A} & \xrightarrow{\hat{\sigma}} & \hat{A} \\
\downarrow{\scriptstyle e_I} & & \downarrow{\scriptstyle r_I} & & \downarrow{\scriptstyle a_I} & & \downarrow{\scriptstyle a_I} \\
r(H)_I & \xrightarrow{m_I} & r(H)_I & \xrightarrow{\alpha_I} & A/I & \xrightarrow{\sigma} & A/I .
\end{array}$$

Our hypotheses on H show that r_I is surjective. Now let $\sigma \in G(I)$ and $h \in H_I$. Then $r_I\alpha_I\sigma^{-1}h\sigma = \alpha(\sigma^{-1})\hat{\ }\hat{h}\hat{\sigma}a_I = \alpha a_I = r_I\alpha_I$, since

$$(\sigma^{-1})\hat{\ }\hat{h}\hat{\sigma} \in H .$$

However, r_I is surjective, so $\alpha_I\sigma^{-1}h\sigma = \alpha_I$, and we conclude that $\sigma^{-1}h\sigma \in H_I$ since $gr(H_I) = H_I$. Hence, H_I is a normal subgroup of $G(I)$.

Consider the following diagram of groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{\;h\;} & G & \xrightarrow{\;g\;} & G/H & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle t} & & \\ 0 & \longrightarrow & H' & \xrightarrow{\;h'\;} & G' & \xrightarrow{\;g'\;} & G'/H' & \longrightarrow & 0 \end{array}$$

where the rows are exact, r and s are monomorphisms, while t is the unique group morphism making the right square commutative.

LEMMA 1.20. *If (H', r, h') is a pullback for h and s, then t is a monomorphism.*

Proof. Let $t(x') = 1$, then $g'(y') = x'$ for some $y' \in G'$, and so $gs(y') = 1$. Hence $h(z) = s(y')$ for some $z \in H$. But since H' is a pullback, there is $z' \in H'$ such that $r(z') = z$ and $h'(z') = y'$. Therefore, $1 = g'h'(z') = g'(y') = x'$, and we conclude that t is a monomorphism.

Now suppose H is a normal subgroup of $\hat{G}$ satisfying condition (1.14). For each $I \leqq I'$ in F we are led to a commutative diagram of groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_I & \xrightarrow{\;h_I\;} & G(I) & \xrightarrow{\;q\;} & G(I)/H_I & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle J^I_{I'}} & & \uparrow{\scriptstyle G(I', I)} & & \uparrow{\scriptstyle G/H(I', I)} & & \\ 0 & \longrightarrow & H_I' & \xrightarrow{\;h_{I'}\;} & G(I') & \xrightarrow{\;q_{I'}\;} & G(I')/H_{I'} & \longrightarrow & 0 \end{array}$$

where q_I and $q_{I'}$ are the canonical quotient maps, and $G/H(I', I)$ is the map produced by the remainder making the whole diagram commutative with exact rows. Since $J^I_{I'}$ and $G(I', I)$ are monic, while H_I, is a pullback, it follows from our foregoing *Lemma* that $G/H(I', I)$ is also a monomorphism.

Thus, we obtain a contravariant monic valued functor $G/H\colon F \longrightarrow G$ such that $I \in F$ implies that $G/H(I) = G(I)/H_I$ is the Galois group of $r(H_I)$ over $k/k \cap I$ by (0.5). Finally, the diagram

$$\begin{array}{ccc} r(H_I) & \xrightarrow{\;G/H(I', I)(\bar{\sigma})\;} & r(H_I) \\ \downarrow{\scriptstyle r^I_{I'}} & & \downarrow{\scriptstyle r^I_{I'}} \\ r(H_I) & \xrightarrow{\;\;\bar{\sigma}\;\;} & r(H_{I'}) \end{array}$$

is commutative for each $\bar{\sigma} \in G/H(I')$. For if $\bar{\sigma} = q_{I'}(\sigma)$, then $G/H(I', I)(\bar{\sigma}) = q_I(G(I', I)(\sigma))$ and the corresponding diagram

$$\begin{array}{ccc} A/I & \xrightarrow{G(I',\,I)(\sigma)} & A/I \\ \Big\downarrow{\scriptstyle a^I_{I'}} & & \Big\downarrow{\scriptstyle a^I_{I'}} \\ A/I' & \xrightarrow{\quad\sigma\quad} & A/I' \end{array}$$

is commutative.

This establishes the corollary below.

COROLLARY 1.21. *Let A be a separated and complete linear topological k-algebra. Suppose (A, F, G) is a quasi-Galois extension of k, and suppose H is a normal subgroup of $\hat{G}$ satisfying condition* (1.14). *Then there is a final subset F' of F such that $(r(H), F' \cap r(H), G/H)$ is a quasi-Galois extension of k, where*

$$F' \cap r(H) = \{I' \cap r(H) \mid I' \in F'\} .$$

Proof. Define F' to be the smallest subset of F such that for each intersection $r(H) \cap I$ with $I \in F$, there is $I' \in F'$ with $r(H) \cap I' = r(H) \cap I$. Because $r(H)$ has the induced topology, F' is final in $\mathscr{U}(r(H))$ and our foregoing constructions show that $(r(H), F' \cap r(H), G/H)$ is a quasi-Galois extension of k.

2. **Examples.** In this section we will show how to construct a number of examples of the foregoing material. Two lemmata are useful in this direction.

LEMMA 2.1. *Let X and $Y = (Y_i)_{i \in I}$ be distinct indeterminants over the ring A. Let $f \in A[X]$ be a monic polynomial, and suppose $I \leqq (A[X]/(f))[Y]$ is an ideal. Let I' be the ideal generated by the image of I in $A[X, Y]$ under the canonical inclusion $A[X]/(f) \subset A[X, Y]$. Then we have $(A[X]/(f))[Y]/I \cong A[X, Y]/(fA[X, Y] + I')$.*

Proof. We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & fA[X] \otimes_A A[Y] & \longrightarrow & A[X] \otimes_A A[Y] & \longrightarrow & \dfrac{A[X]}{(f)} \otimes_A A[Y] & \longrightarrow & 0 \\ & & \Big\| \wr & & \Big\| \wr & & \Big\| \wr & & \\ 0 & \longrightarrow & fA[X, Y] & \longrightarrow & A[X, Y] & \longrightarrow & \dfrac{A[X]}{(f)}[Y] & \longrightarrow & 0 \end{array}$$

with exact rows. Hence, $\ker(\alpha) = fA[X, Y]$. If β is the quotient mapping $(A[X]/(f))[Y] \to (A[X]/(f))[Y]/I$ and $\beta\alpha(P) = 0$, then $\alpha(P) \in I$, so there is $Q \in I'$ such that $\alpha(P) \in I' + fA[X, Y]$. Evidently, this latter ideal is contained in $\ker(\alpha\beta)$, completing the proof.

LEMMA 2.2. *Suppose* $I \leqq k[X_1, \cdots, X_n] \subsetneq k[X]$, $X = (X_i)_{i \geqq 1}$. *Then* $k[X]/([X] \cdot I + k[X] \cdot \langle X_{n+1}, X_{n+2}, \cdots \rangle) \cong k[X_1, \cdots, X_n]/I$.

Proof. Let $k[X] \xrightarrow{\Phi} k[X_1, \cdots, X_n] \xrightarrow{\psi} k[X_1, \cdots, X_n]/I$ be the composition of the evaluation at the point $(X_1, X_2, \cdots, X_n, X_n, 0, 0, \cdots)$ followed by the canonical quotient morphism ψ. Clearly, $k[X] \cdot I + k[X] \cdot \langle X_{n+1}, \cdots \rangle$ is contained in the kernel of the surjection $\Phi\psi$; if $\psi(\Phi(f)) = 0$, then $f = (f - \Phi(f)) + \Phi(f) \in k[X]$ shows that

$$f \in k[X]I + k[X] \cdot \langle X_{n+1}, \cdots \rangle .$$

1. *Example of a quasi-Galois extension.* Suppose A_0 is a complete Noetherian local ring with residual field k_0. Let $k_0 < k_1 < \cdots$ be a tower of finite Galois field extensions of k_0 with corresponding Galois groups $G(k_i/k_0)$.

Since k_1 is a finite Galois extension of k_0, we can find a monic polynomial $f_1 \in A_0[X_1]$ such that $k_0[X_i]/(\bar{f}_1) \cong k_1$, where $\bar{f}_1$ is the reduction of f_1 modulo $j(A_0)$, the Jacobson radical of A_0. Following [8] p. 63 we see that $A_1 = A_0[X_1]/(f_1)$ is a complete Noetherian local ring which is an A_0-algebra of finite type; moreover, A_1 is a Galois extension of A_0 with Galois group isomorphic to $G(k_1/k_0)$ in the sense of [3].

Since k_2 is a finite Galois extension of k_1, we repeat the above construction obtaining a monic polynomial $f_2 \in A_1[X_2]$ such that $A_2 := A_1[X_2]/(f_2)$ is a Galois extension of A_1 with Galois group $G(k_2/k_1)$.

We have the ring inclusions $A_0 \leqq A_0[X_1]/(f_1) \leqq (A_0[X_1]/(f_1))[X_2]/(f_2)$. Since f_1 is monic, we can view $f_2 \in A_0[X_1, X_2]$ and apply Lemma 2.2 to obtain the isomorphism:

$$\frac{A_0[X_1]}{(f_1)}[X_2]/(f_2) \cong \frac{A_0[X_1, X_2]}{f_1A_0[X_1, X_2] + f_2A_0[X_1, X_2]} = \frac{A_0[X_1, X_2]}{\langle f_1, f_0 \rangle} .$$

Iterating the above, we obtain $A_{n+1} \cong A_0[X_1, \cdots, X_{n+1}]/ < f_1, \cdots, f_{n+1}\rangle$ and have that A_{n+1} is a finite Galois extension of A_n with Galois group $G(k_{n+1}/k_n)$; A_{n+1} is also a finite Galois extension of A_0 with Galois group $G(k_{n+1}/d_0)$.

Now define ideals $I_n \leqq B := A_0[X_1, X_2, \cdots]$ as follows:

$$I_n := B\langle f_1, \cdots, f_n\rangle + B \cdot \langle X_{n+1}, X_{n+2}, \cdots \rangle \quad \text{for} \quad n \geqq 1 .$$

LEMMA 2.3. (1) $I_n \geqq I_{n+1}$.
(2) $I_n \cap A_0 = (0)$.
(3) $B/I_n \cong A_n$.

Proof. (1): Since $f_{n+1} \in A_0[X_1, \cdots, X_{n+1}] \subset B$, it follows that

$Bf_{n+1} \subset I_n$ so that $I_n \geqq I_{n+1}$.

(2): Is clear.

(3): Follows from Lemma (2.2).

Let $U(B)$ have as filter basis the family $(I_n)_{n\geqq 1}$. Then for $I_n \geqq I_{n+1}$, we have a commutative diagram

$$\begin{array}{ccccc} A_0 & \longrightarrow & A_{n+1} \cong B/I_{n+1} & : & G(k_{n+1}/k_0) \\ \| & & \downarrow & & \uparrow \\ A_0 & \longrightarrow & A_n \cong B/I_n & : & G(k_n/k_0) \end{array}$$

where A_i is a Galois extension of A_0 with group $G(k_i/k_0)(i = n, n+1)$. By (1.1) there is a group morphism $G(k_n/k_0) \to G(k_{n+1}/k_0)$ which is injective and satisfies the commutativity criterion of (1.1).

Letting $F = (I_n)_{n\geqq 1}$ and $G\colon F \to Gr$ be such that $G(I_n) = G(k_n/k_0)$ we obtain a quasi-Galois extension (B, F, G) of A_0.

2. Another quasi-Galois extention. Let $K_0 < K_1 < \cdots$ be a tower of Galois field extensions (all finite), K_{n+1} is a finite Galois extension of K_n, so $K_{n+1} \cong K_n[X_{n+1}]/(f_{n+1})$ for a monic polynomial f_{n+1}, and repeating the technique of 1, we get for $A = K_0[X_1, X_2, \cdots]$ and $F = (I_n)_{n\geqq 1}$, I_n appropriately defined, that $A/I_n \cong K_n$ so that finally (A, F, G) is a quasi-Galois extension of K_0 with $G(I_n) = G(K_n/K_0)$.

Remark. In 1 each term B/I_n is a local ring, while in 2 each term A/I_n is an integral domain. These are two general classes of connected rings. Later we will give an example of a quasi-Galois extension where the approximating terms are not connected, i.e., have proper idempotents.

3. *Quasi-Galois extensions in rings of continuous functions.* This example is fairly complicated, so I first state the results. Let $(X_i)_{i\in I}$ be a cofiltered family of topological spaces such that $i \leqq j$ in I implies $x_{ij}\colon X_i \to X_j$ is an inclusion for which the identity

$$x_{ij}^{-i}(\mathrm{Top}\,(X_i)) = \mathrm{Top}\,(X_i)$$

holds. Let $X = \varinjlim_I X_i$, and let $x_i\colon X_i \to X$ be the colimit morphisms. Then the x_i are injective.

Next, let $C\colon \mathrm{Top} \to RIN$ be the functor assigning to each topological space X, the ring of continuous real valued functions with domain X, where Top denotes the category of topological spaces.

Lemma 2.4. $C(X) \cong \underset{\longleftrightarrow}{\lim}_I C(X_i)$ via $f \to (x_i f)_{i\in I}$.

Now suppose $(G_i)_{i\in I}$ is a cofiltered family of groups such that

$i \leqq j$ implies $g_{ij}: G_i \to G_j$ is the monomorphism, and let $G = \varinjlim_I G_i$ with $g_i: G_i \to G$ being the canonical colimit morphisms. The g_i are injective. We will suppose that G_i acts continuously on X_i, $G_i: X_i \to X_i$, in such a way that for $i \leqq j$ in I we have a commutative diagram for all $\sigma \in G_i$:

$$\begin{array}{ccc} X_i & \xrightarrow{x_{ij}} & X_j \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle g_{ij}(\sigma)} \\ X_i & \xrightarrow[x_{ij}]{} & X_j\,. \end{array}$$

LEMMA 2.5. *G acts continuously on X, and if $g \in G$, there is $I \in I$ for which $g_i(\sigma) = g$ and the diagram below is commutative*:

$$\begin{array}{ccc} X_i & \xrightarrow{x_i} & X \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle g = g_i(\sigma)} \\ X_i & \xrightarrow{x_i} & X \end{array} .$$

Due to the foregoing assumptions we obtain commutative diagrams:

$$\begin{array}{ccc} X_i & \xrightarrow{x_{ij}} & X_j \\ \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle q_j} \\ X_i/G_i & \xrightarrow[x_{ij}]{} & X_j/G_j \end{array} \quad \text{and} \quad \begin{array}{ccc} C(X_j/G_j) & \longrightarrow & C(X_j) \\ \downarrow & & \downarrow \\ C(X_i/G_i) & \longrightarrow & C(X_i) \end{array}$$

for $i \leqq j$ in I, where X_i/G_i is the space of G_i-orbits of X_i with the quotient topology, while q_i is the canonical quotient morphism. A more general result than (2.4) is the following:

LEMMA 2.6. *$C(X/G) \cong \varprojlim_I C(X_i/G_i)$ via $f \to (f_i)_{i \in I}$, where $q_i f_i = x_i q f$ and $q: X \to X/G$ is the quotient map.*

Finally, suppose the following conditions are fulfilled.

(a) Each X_i is compact.

(b) $G_i: X_i \to X_i$ is a finite group without fixed points.

(c) Both $C(X) \to C(X_i)$ and $C(X/G) \to C(X_i/G_i)$ are surjective.

Then:

(0) $\ker [C(X) \to C(X_i)] \cap C(X/G) = \ker [C(X/G) \to C(X_i/G_i)]$.

(1) $(C(X), F, H)$ is a quasi-Galois extension of $C(X/G)$, where $F = (\ker [C(X) \to C(X_i)])_{i \in I}$ and $H(\ker [C(X) \to C(X_i)]) = G_i$.

Proof. Draw the diagram:

$$\begin{array}{ccccc}
X_i & \xrightarrow{x_\iota} & X & \xrightarrow{f} & R \\
\big\downarrow{\scriptstyle q_i} & & \big\downarrow{\scriptstyle q} & & \big\| \\
X_\iota/G_i & \xrightarrow{\bar{x}_n} & X/G & \xrightarrow{\bar{f}} & R
\end{array}$$

and assume $x_\iota f = 0$ and $q\bar{f} = f$. Then $q_\iota \bar{x}_i \bar{f} = 0$ implies $\bar{x}_\iota \bar{f} = 0$ which implies $\bar{f} \in \ker [C(X/G) \to C(X_\iota/G_\iota)]$. Conversely, $\bar{x}_i \bar{f} = 0$ implies $x_i q \bar{f} = 0$ and $q\bar{f}: = f \in C(X/G) \cap \ker [C(X) \to C(X_\iota)]$ which completes the proof of (0).

For (1), it follows that for each $i \in I$ the diagram

$$\begin{array}{ccc}
C(X/G) & \longrightarrow & C(X) \\
\big\downarrow & & \big\downarrow \\
C(X_i/G_\iota) & \longrightarrow & C(X_\iota)
\end{array}$$

is commutative. $H(\ker [C(X) \to C(X_\iota)]) = G_\iota$ acts on $C(X_\iota)$ by the formula $\sigma f(x) = f(\sigma(x))$ for all $x \in X_i$ and $\sigma \in G_i$. Since X_i is compact and G_ι acts without fixed points, it follows from (0.2), (2), that $C(X_\iota) \to C(X_\iota)$ is a Galois extension with group G_ι. Moreover, we have for $i \leqq j$ in I, a commutative diagram

$$\begin{array}{ccccc}
C(X_\jmath) & \xrightarrow{C(g_{ij}(\sigma))} & C(X_j)\colon G_j & & C(g_{ij}(\sigma)) \\
\big\downarrow & & \big\downarrow \quad \big\uparrow{\scriptstyle G_{ij} =: H(i \leqq j)} & & \\
C(X_i) & \xrightarrow{C(\sigma)} & C(X_i)\colon G_i & & C(\sigma)
\end{array}$$

since the corresponding diagram omitting the C's is commutative.

Letting $U(C(X))$ have as filter basis the family $F = (\ker [C(X) \to C(X_\iota)])_{i \in I}$ we see that $(C(X), H, F)$ is a quasi-Galois extension of $C(X/G)$.

As example of such a situation as described above, let, for each $n \geqq 1$, X_n be the topological coproduct of 3^n copies of $[0, 1]$, and let G_n the cyclic group of order 3^n acting on X_n by permuting the summands. G_n acts continuously and has no fixed points, while X_n is compact. We have $\varinjlim_{n \geqq 1} G_n = Z(3^\infty)$ and $\varinjlim_{n \geqq 1} X_n$ is simply the coproduct of a countable number of copies of $[0, 1]$, where we interpret always $X_n \leqq X_{n+1}$ and $G_n \leqq G_{n+1}$. It is clear that the diagrams following (2.4) and (2.5) are commutative, and that the conditions (a)-(c) are fulfilled in this case.

We will now prove assertions (2.4), (2.5), and (2.6).

LEMMA 2.4. $C(X) \cong \varprojlim_I C(X_\iota)$.

Proof. For each $i \leqq j$ in I, we have by definition a commutative diagram:

$$\begin{array}{ccc} X_i & \xrightarrow{x_{ij}} & X_j \\ \Vert & & \downarrow{\scriptstyle x_j} \\ X_i & \xrightarrow{x_i} & X\,. \end{array}$$

If $(f_i)_{i \in I} \in \varprojlim C(X_i)$, then for $i \leqq j$ we have a diagram

$$\begin{array}{ccc} X_i & \xrightarrow{x_{ij}} & X_j \\ \Vert & & \downarrow{\scriptstyle f_j} \\ X_i & \xrightarrow{f_i} & R \end{array}$$

so there is a unique $f\colon X \rightarrow R$ such that $x_i f = f_i$ for $i \in I$. This shows that $f \rightarrow (x_i f)_{i \in F}$ is bijective, and the uniqueness guarantees that this mapping is a ring morphism.

Lemma 2.5. *G acts continuously on X.*

Proof. G is formed by taking colimits of diagrams like:

$$\begin{array}{ccc} X_i & \xrightarrow{x_{ij}} & X_j \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle g_{ij}(\sigma)} \\ X_i & \longrightarrow & X_j \end{array}$$

where $j \geqq i$ for all $\sigma \in G(i)$. This leads to commutative diagrams:

$$\begin{array}{ccc} X_j & \xrightarrow{x_j} & X \\ \downarrow{\scriptstyle g_{ij}(\sigma)} & & \downarrow{\scriptstyle g} \\ X_j & \xrightarrow{x_j} & X \end{array}$$

where $g = \varinjlim_{j \geqq i} g_{ij}(\sigma)$. It follows immediately that $x_j^{-1} g^{-1}(0) \in \mathrm{Top}\,(X_j)$ for all $j \geqq i$ and all $0 \in \mathrm{Top}\,(X)$; moreover, if $k \in I$, let $j \geqq i, k$, then $x_k^{-1} g^{-1}(0) = x_{kj}^{-1} x_j^{-1} g^{-1}(0) \in \mathrm{Top}\,(X_k) = X_{kj}^{-1}(\mathrm{Top}\,(X_j))$ by definition of $\mathrm{Top}\,(X_k)$.

Hence, g is continuous.

Lemma 2.6. *$C(X/G) \cong \varprojlim_I C(X_i/G_i)$ via $f \rightarrow (\bar{x}_i f)_{i \in I}$.*

Proof. Let $y_i\colon X_i/G_i \rightarrow Y$ be such that $\bar{x}_{ij} y_j = y_i$ for $i \leqq j$ in I. Then composing $q_i\colon X_i \rightarrow X_i/G_i$ with y_i yields a family $(q_i y_i)_{i \in I}$ compatible with the $x_{ij}\colon X_i \rightarrow X_j$ for $i \leqq j$. Hence, there is a unique $y\colon X \rightarrow Y$

such that $x_i y = q_i y_i$ for $i \in I$ by (2.4). Next, let $g \in G$, say $g = g_i(\sigma)$ for $\sigma \in G(i)$. We then have the equations: $x_j g y = g_{ij}(\sigma) x_j y = x_j y$ since y is constant on G_j-orbits of X_j, i.e., $x_j y = q_j y_j$. Passing to the colimit over $j \geqq i$, we get $gy = y$ showing that y is constant on G-orbits of X. Hence, there is a unique $\bar{y}: X/G \to Y$ such that $y = q\bar{y}$. Since q_i is surjective and $q_i y_i = x_i y = x_i q \bar{y} = q_i \bar{x} \bar{y}$, we conclude that $y_i = \bar{x}_i \bar{y}$ for all $i \in I$. Thus, the mapping $f \to (\bar{x}_i f)_{i \in I}$ is bijective and as before the uniqueness assures that it is a ring morphism.

4. *A non-connected quasi-Galois extension.* Let (A, F, G) be a quasi-Galois extension of k and let $n \geqq 2$. Put $A^n = A\pi \cdots \pi A$ (n factors) and $F^{(n)} = \{I^n \mid I \in F\}$. The diagonal map $\varDelta: k \to A^n$ makes A a k-algebra, and $I \in F$ implies $A^n/I^n \cong (A/I)^n$. Moreover, $I \leqq I'$ in F induces $(a^I_{I'})$: $(A/I)^n \to (A/I')^n$ which is surjective. It follows from [2] (Chapter IX §7, Prop. 7.3) by induction that $(A/I)^n$ is a separable k_I-algebra via the diagonal map $\varDelta_I: k_I \to (A/I)^n$, where $k_I = k/k \cap I$.

Next, let $G^n(I) = G(I)\pi \cdots \pi G(I)$ (n factors) and let $H(I)$ denote the diagonal subgroup of $G^n(I)$, that is the image of the diagonal map $\varDelta: G(I) \to G^n(I)$. $G^n(I)$ acts componentwise on $(A/I)^n$. Let H be any subgroup of the symmetric group of n letters which moves all the letters to all positions, e.g., the cyclic group of order n. We think of H as acting on each $(A/I)^n$ as a permutation of the factors. Finally, let $K(I)$ be the normal product of H with $H(I)$, so that each element of $K(I)$ may be put in the form $\pi\varDelta(\sigma)$ with $\pi \in H$ and $\sigma \in H(I)$.

LEMMA 2.7. (a) *$K(I)$ acts on $(A/I)^n$ with fixed ring $\varDelta_I(k/k \cap I)$ for $I \in F$.*

(b) *$(A/I)^n$ is a Galois extension of $k/k \cap I$ with group $K(I)$ for $I \in F$.*

Proof. It is clear how $K(I)$ acts on $(A/I)^n$ using the representation of elements of $K(I)$ in the form $\pi\varDelta(\sigma)$. If $(a_1, \cdots, a_n)$ is fixed by $K(I)$, then because $K(I)$ moves each component to every other component, and each component lies in $k/k \cap I \cdot 1$, we must have that the element $(a_1, \cdots, a_n) \in \varDelta_I(k/k \cap I)$, proving (a).

Next, let (x_i), (y_i) be two families of elements of A/I such that $\sum_i x_i \sigma(y_i) = \delta_{1\sigma}$ for all $\sigma \in G(I)$. Such exist by (0.2), (1). Then we have $\sum_i \varDelta_I(x_i)\pi\varDelta(\sigma)(\varDelta_I(y_i)) = \varDelta_I(\sum_i x_i \sigma(y_i)) = \varDelta_I(\delta_{1\sigma}) = \delta_{1\varDelta(\sigma)} = \delta_{1\pi\varDelta(\sigma)}$; hence, (b) holds using (0.2), (1), again.

There is an evident group morphism $K(I') \to K(I)$ extending $G(I') \to G(I)$ which is monic. We denote the so generated functor by $K: F^{(n)} \to G$, and obtain a quasi-Galois extension $(A^n, F^{(n)}, K)$ of k such that $(A/I)^n$ is not connected.

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