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Given a compact, connected, k-dimensional, oriented Lie group G or a faithful orthogonal representation T of such a G there arises an element of the kth framed cobordism group Ω_k^{fr}. The study of these elements is begun, and some algebraic properties of the situation are discussed. The remaining problem is to relate such properties of the elements in Ω_k^{fr} as order or Adams d and e invariants to Lie theory.

If G is a k-dimensional Lie group its tangent bundle may be trivialized by choosing a linear isomorphism of its Lie algebra $\mathscr{L}(G)$ with Euclidean space R^k, and using right multiplication to give an isomorphism of the tangent space at any point with the tangent space at the identity which is, of course, the Lie algebra. If G is compact and oriented every trivialization of the tangent bundle gives rise to a trivialization of the stable normal bundle (see the discussion of tangential and normal structures on p. 23 of [2]) and hence to an element of the kth framed cobordism group Ω_k^{fr}. If two choices of linear isomorphisms of $\mathscr{L}(G)$ with R^k differ by an element of $GL_k(R)$ of positive determinant it is easily seen that the corresponding tangential trivializations are homotopic through trivializations and hence determine the same element of Ω_k^{fr}. Thus, a compact, oriented k-dimensional Lie group G gives rise to a well-defined element $[G] \in \Omega_k^{fr}$.

Now assume in addition that G is connected and let $T: G \to \mathrm{SO}(n)$ be a faithful representation of G. T embeds G in Euclidean n^2-space. If G is k-dimensional then $k \leqq n(n-1)/2 < n^2/2$, since $\dim G \leqq \dim \mathrm{SO}(n)$, so that $\operatorname{codim} G > k$ and the normal bundle of G in this embedding is already stable. We shall always assume a fixed orientation in any Euclidean space we discuss; in particular view Euclidean n^2-space as $M(n)$, the space of $n \times n$ real matrices and choose an orthonormal basis e_{ij}, $1 \leqq i, j \leqq n$, where e_{ij} is the matrix with one in the ijth position and zeroes elsewhere. Orient $M(n)$ by putting the e_{ij} in lexicographic order, so that the ordered basis is $e_{11}, e_{12}, \cdots, e_{1n}, e_{21}, \cdots, e_{2n}, \cdots, e_{n1}, \cdots, e_{nn}$. Make the convention that $M(n)$ is always oriented this way, and also assume that the matrix of any linear transformation from $M(n)$ to itself is always written with respect to this ordered basis.

Returning to the faithful representation T, choose an orthonormal basis $\tau_1, \cdots, \tau_k$ for τ_I the tangent k-plane to $T(G)$ at the identity I.

Then choose an orthonormal basis $\nu_1, \cdots, \nu_{n^2-k}$ for $\tau_I^\perp$ in such a way that the basis $\tau_1, \cdots, \tau_k, \nu_1, \cdots, \nu_{n^2-k}$ determines the fixed orientation of $M(n)$.

Since G (identified with $T(G)$) is contained in $\mathrm{SO}(n)$, right multiplication of elements of $M(n)$ by an element $g \in G$ defines an element ρ_g in $\mathrm{SO}(n^2)$. Since $\rho_{g|G}$ is a diffeomorphism of G and $d_I\rho_g$, the differential of ρ_g (viewed as a map $\rho_g: M(n) \to M(n)$) is a constant element of $\mathrm{SO}(n^2)$, $d_I\rho_g$ maps the tangent (resp. normal) space at I to the tangent (resp. normal) space at g. Define $\nu_i(g) = d_I\rho_g(\nu_i)$, and observe that this defines a framing of the stable normal bundle of G, $\nu(G)$, and thus an element $[T] \in \Omega^k_{fr}$. Observe also that $\tau_j(g) = d_I\rho_g(\tau_j)$ defines a framing of $\tau(G)$ which coincides with the framing discussed above in connection with the element $[G]$.

The object of this note is to begin the study of the elements $[G]$ and $[T]$. It is first shown that the element $[T]$ does not depend on the choice of framing at I, and the question of the dependence of $[T]$ on the equivalence class of T as representation is then discussed. The algebraic properties of the map $T \mapsto [T]$ are considered. The following statements are proved by example:

(A) If S and T are two faithful representations of G, $[S + T] \neq [S] + [T]$ in general.

(B) If $T + m$ is the representation obtained by adding m trivial representations to T, then in general $[T + m] \neq [T]$.

Some positive statements may be made, however,

PROPOSITION 1. *If* $T_1: G_1 \to \mathrm{SO}(n_1)$ *and* $T_2: G_2 \to \mathrm{SO}(n_2)$ *are two faithful representations of two compact, connected, oriented Lie groups of dimensions* k_1 *and* k_2 *respectively, and* $T_1 \oplus T_2: G_1 \times G_2 \to \mathrm{SO}(n_1 + n_2)$ *is their exterior direct sum, then* $[T_1 \oplus T_2] = (-1)^e[T_1 + n_2][T_2 + n_1]$, *where* $e = k_2(n_1 + k_1 + n_1n_2)$ *and the product on the right is taken in the framed cobordism ring* Ω^{fr}_*.

PROPOSITION 2. *Let* $T: G \to \mathrm{SO}(n)$ *be a faithful representation of a compact, connected* k*-dimensional Lie group.*

(a) *The sequence* $[T], [T + 1], \cdots, [T + m], \cdots$ *of elements of* Ω^{fr}_k *is cyclic in* m.

(b) *The element* $[G]$ *occurs in the sequence of* (a).

(c) *If* r, s *are positive integers then* $[rT + s]$ *occurs in the sequence of* (a); *in fact,* $[rT + s] = [T + (r^2 - 1)n + s]$.

The problems that remain open may be summarized as: How do the Lie-theoretic properties of G and T affect the elements $[G]$ and $[T]$? We list two such problems: What conditions on G and T allow one to conclude that $[G] = 0$ or $[T] = 0$? Express the numbers that

occur-such numbers as the orders of $[G]$ and $[T]$, the primes that occur in those orders, the Adams d and e invariants of $[G]$ and $[T]$, the length of the cycle of Proposition (2c) in terms of Lie theory.

Before proceeding to the proofs of the statements above, it is useful to make the matrix form of $d\rho_g$ explicit. Since ρ_g is a linear map of $M(n)$ to itself, $d\rho_g$ is constant and $d\rho_g = \rho_g$. If $x = (x_{ij}) \in M(n)$ $g = (g_{kl}) \in T(G)$, then $\rho_g(x) = (y_{st})$, where $y_{st} = \sum_q x_{sq} g_{qt}$. Setting $x = e_{ij}$ and looking at the formula proves Lemma 1.

LEMMA 1. *As a matrix in* $\mathrm{SO}(n^2)$, $d\rho_g$ *is obtained by stringing* n *copies of* $g^t = g^{-1}$ *down the diagonal.*

LEMMA 2. *In* Ω_k^{fr}, $[T]$ *is independent of choice of basis* $\tau_1, \cdots, \tau_k$ *(resp.* $\nu_1, \cdots, \nu_{n^2-k}$*) of tangent (resp. normal) space at the identity.*

Proof. Let $\tau_1', \cdots, \tau_k', \nu_1', \cdots, \nu_{n^2-k}'$ be another choice of orthonormal, properly oriented bases of τ_I and $\tau_I^{\perp} = \nu_I$ respectively. Define $L \in \mathrm{SO}(n^2)$ by $L(\tau_i) = \tau_i'$ and $L(\nu_j) = \nu_j'$. L is in that subgroup H of $\mathrm{SO}(n^2)$ which keeps τ_I and ν_I invariant, so that H is isomorphic to $\mathrm{SO}(k) \times \mathrm{SO}(n^2 - k)$. Let σ be a smooth path in H from L to the identity. Then, for $0 \leqq t \leqq 1$ and $g \in T(G)$, $\nu_j(g)_t = (d\rho_g)(\sigma(t)(\nu_j))$ defines a homotopy of one normal framing to the other through normal framings, so the two framings represent the same element of Ω_k^{fr} (see, e.g., p. 190 of [1]).

Note that it is important to maintain orientation; if the orientation is changed the sign of $[T]$ changes. It is also easy to see that if G is not connected but has, say, $q > 1$ components, and if each component G_α is identified with G_0, the component of the identity, by right multiplication by a fixed g_α^{-1}, $g_\alpha \in G_\alpha$, and if $T: G \to \mathrm{O}(n)$ is a faithful representation and $T_0 = T \mid G_0$, then $\nu(G)$ may be framed as before, and in this case $[T] = q[T_0]$. The nonconnected case will not be explicitly considered further, although much of the discussion below remains valid in that case.

DEFINITION. Let S and T be two (not necessarily faithful) orthogonal representations of the group G. S and T are centrally equivalent means:

(i) S and T have the same rank n.

(ii) There exists $A \in \mathrm{O}(n)$ such that $S = ATA^{-1}$.

(iii) There exists $B \in \mathrm{O}(n)$ such that $\det A = \det B$ and B centralizes $T(G)$, i.e., $BT(g) = T(g)B$ for all $g \in G$.

We state some obvious observations as a lemma.

LEMMA 3. (i) *Central equivalence is an equivalence relation which implies the usual equivalence relation.*

(ii) *If the rank is odd, central equivalence coincides with the usual equivalence relation.*

(iii) *If S and T are centrally equivalent, so are $S+1$ and $T+1$; if S and T are equivalent, $S+1$ and $T+1$ are centrally equivalent.*

(iv) *If S and T are two representations of G, $S+T$ and $T+S$ are centrally equivalent.*

(v) *If S and T are equivalent unitary represention of G, their "realifications" are centrally equivalent.*

PROPOSITION 3. (i) *If S and T are centrally equivalent faithful representations of the compact, oriented Lie group G, then $[S]=[T]$.*

(ii) *If S and T are two faithful orthogonal representations of G, $[S+T]=[T+S]$.*

(iii) *If S and T are equivalent faithful unitary representations of G, then $[S]=[T]$.*

Proof. Let $A, B \in O(n)$ be such that $S = ATA^{-1}$, $\det A = \det B$, and B centralizes $T(G)$. Let $\sigma: I \to O(n)$ be a smooth path from B to A. Let $T_t: G \to SO(n)$ be defined by $T_t = \sigma(t)T\sigma(t)^{-1}$, and let $\hat{\sigma}: M(n) \to M(n)$ be defined by $\hat{\sigma}_t(M) = \sigma(t)M\sigma(t)^{-1}$. Observe that for every $t \in I$, $\hat{\sigma} \in SO(n^2)$. Choose a tubular neighborhood ν of $T(G)$ and, if $x \in \nu$, set $x_t = \hat{\sigma}_t B^{-1} x B$. Note that $x_0 = x$, and $T_0(g) = T(g)$ for all $g \in G$. Choose a basis for the normal bundle of $T(G)$ at the identity following the orientation convention described above. Then, for $t \in I$ and $g \in G$, framing $T_t(G)$ by applying $dT_t(g)$ to the basis chosen above defines an isotopy of framed manifolds. Since $T_1 = S$ and Lemma 2 shows that $[S]$ can be defined using the framing defined here, it follows that $[S]=[T]$. This proves (i).

The other two parts follow from (i) and Lemma 3.

A slight extension of these ideas may be useful in considering tensor products. If $T: G \to SO(n)$ is a representation with finite kernel of a compact, connected, oriented Lie group, then T induces a faithful representation T' of $G' = G/\ker T$, where G' is again a compact connected Lie group which inherits an orientation for G. We may define $[T]=[T']$. If G is semi-simple in addition, and S and T are two representations of G which have finite kernel, then so does $S \otimes T$, although $S \otimes T$ need not be faithful even if S and T are. Thus, $[S \otimes T]$ can be defined, and it can be shown as in the proof of Lemma 4 that $[S \otimes T] = [T \otimes S]$. If S and T are faithful representations

of G (or representations with finite kernel) it would be of interest to obtain general formulas for $[S + T]$ and, if G is semi-simple, $[S \otimes T]$, in terms of $[S]$ and $[T]$.

Proof of Proposition 1. Suppose $T: G \to \mathrm{SO}(n)$ is a faithful representation, and consider $T + m: G \to \mathrm{SO}(n + m)$. Linearly embed $M(n)$ in $M(n + m)$ by the map $\theta: e_{ij} \to e_{ij}$. Then, if $\tau_1, \cdots, \tau_k$ and $\nu_1, \cdots, \nu_{n^2-k}$ are properly oriented frames in τ_I and ν_I respectively, $\theta(\tau_1), \cdots, \theta(\tau_k)$ is a tangential frame of $(T + m)(G)$ at the identity, and $\theta(\nu_1), \cdots, \theta(\nu_{n^2-k})$, e_{rs}, $r > n$ or $s > n$, with the e_{rs} in lexicographic order, is a normal frame at the identity of $(T + m)(G)$. These frames do not, in general, give the proper orientation to the normal frame according to our conventions; a computation shows that this orientation is $(-1)^f$ times the proper orientation, where $f = 1/2\, mn^2(n - 1)$. Thus, normally framing $(T + m)(G)$ by moving the given frame at I by $d\rho_g$ for each $g \in G$ gives a framed manifold representing $(-1)^f [T + m]$. Applying Lemma 1 to the representation $T + m$ shows that, if $g \in G$, $d\rho_g$ fixes e_{rs} whenever $s > n$. Hence, if $V \subset M(n + m)$ is the $m(n + m)$ dimensional subspace of $M(n + m)$ spanned by the e_{rs}, $s > n$, G may be viewed as embedded in $V^\perp$ by orthogonal projection of $(T + m)G$ into $V^\perp$, and the normal frame at g may be taken as $d\rho_g$ applied to $\theta(\nu_1), \cdots, \theta(\nu_{n^2-k})$, e_{rs}, $r > n$, $s < n$, in lexicographic order, all of which vectors lie in $V^\perp$. Identifying $V^\perp$ with Euclidean $n(n + m)$ space shows that G is now embedded in $R^{n(n+m)}$ as a framed manifold, and that, with this framing, G represents $(-1)^f[T + m]$.

Now consider the situation of the proposition. Let $\tau_1, \cdots, \tau_{k_1}$ and $\tau'_1, \cdots, \tau'_{k_2}$ be tangential frames at the identity for $T_1(G_1)$ and $T_2(G_2)$ respectively, and let corresponding normal frames be $\nu_1, \cdots, \nu_{n_1^2-k_1}$ and $\nu'_1, \cdots, \nu'_{n_2^2-k_2}$. Define $\theta: M(n_1) \to M(n_1 + n_2)$ by $\theta(e_{ij}) = e_{ij}$, and $\theta': M(n_2) \to M(n_1 + n_2)$ by $\theta'(e_{st}) = e_{n_1+s, n_1+t}$. A tangential frame for $(T_1 \oplus T_2)(G_1 \times G_2)$ at the identity is given by $\theta(\tau_1), \cdots, \theta(\tau_{k_1})$, $\theta'(\tau'_1), \cdots, \theta'(\tau'_{k_2})$, and a normal frame by $\theta(\nu_1), \cdots, \theta(\nu_{n_1^2-k_1})$, e_{rs}, $\theta'(\nu'_1), \cdots, \theta'(\nu'^2_{n_2-k_2})$, e_{tu}, where $n_1 + 1 \leqq r \leqq n_1 + n_2$, $1 \leqq s \leqq n_1$, $1 \leqq t \leqq n_1$, $n_1 + 1 \leqq u \leqq n_1 + n_2$ and the e_{rs} and e_{tu} are taken in lexicographic order. View $M(n_1 + n_2)$ as the direct sum of two Euclidean spaces V and W, V of dimension $n_1(n_1 + n_2)$ and spanned by the e_{ij} with $1 \leqq j \leqq n_1$, and W of dimension $n_2(n_1 + n_2)$ and spanned by the e_{ij} with $n_1 + 1 \leqq j \leqq n_1 + n_2$. Apply Lemma 1 to the case of $T_1 \oplus T_2$, and observe that the action of $d\rho(g_1, g_2)$ for any $(g_1, g_2) \in G_1 \times G_2$ gives rise to the framing discussed above in the case of $T_1 + n_2$ when applied to $\theta(\nu_1), \cdots, \theta(\nu_{n_1^2-k_1})$, e_{rs}; $n_1 + 1 \leqq r \leqq n_1 + n_2$, $1 \leqq s \leqq n_1$, where G_1 is viewed as embedded in V, and similarly gives rise to the framing discussed above in the case of $T_2 + n_1$ when G_2 is viewed as embedded in W. Therefore, the framing on $(T_1 \oplus T_2)(G_1 \times G_2)$ represents, on

the one hand, $(-1)^g[T_1 + n_2][T_2 + n_1]$, $g = 1/2\, n_1n_2(n_1^2 + n_2^2 - n_1 - n_2)$, and on the other $\varepsilon[T_1 \oplus T_2]$, where $\varepsilon = \pm 1$. This sign is positive if the orientation of the normal frame at the identity coincides with the one determined by our orientation convention, and negative otherwise. An exercise in changing bases and computing the signs of permutations then shows that $\varepsilon = (-1)^h$, where $h = n_1n_2(n_2^2 + n_1n_2) + k_2(n_1^2 - k_1) + k_2n_1n_2 + g$, and the proposition follows.

COROLLARY 1. $[T_1 \oplus T_2] = (-1)^m[T_2 \oplus T_1]$, *where* $m = k_1n_2(n_1 + 1) + k_2n_1(n_2 + 1)$.

Before proving Proposition 2 some discussion and a lemma are needed. Suppose M^k is a k-dimensional stably parallelizable compact manifold embedded in R^N with stable normal bundle ν^{N-k} and that φ and ψ are trivializations of ν^{N-k}. In [1] a difference element $d(\varphi, \psi)$ in $K^{-1}(M)$ (real K-theory) was defined. Let $T: G \to \mathrm{SO}(n)$ be a faithful representation of a compact, connected, oriented Lie group. The tangential trivialization of G discussed above gives rise [2] to a unique (in the sense of equivalence of trivializations [1] [2]) trivialization of the normal bundle of G for any embedding of G in a Euclidean space of sufficiently high dimension. In particular, n^2 and $2n^2$ are sufficiently high dimensions. Call this trivialization $\tilde{G}$. The representation T also gives rise to a trivialization of the normal bundle, as has been described. Call this trivialization $\tilde{T}$. As a map of G into $\mathrm{SO}(n) \subset O$, T represents an element of $K^{-1}(G)$, which we call $\langle T \rangle$.

LEMMA 4. *In* $K^{-1}(G)$, $d(\tilde{G}, \tilde{T}) = n\langle T \rangle$.

Proof. The tangential framing of G is independent of T, and hence $\tilde{G}$, while it may be described in terms of T, will be independent of T as trivialization of the normal bundle (see the discussion on p. 23 of [2] for the relationship between tangential and normal structures). The trivialization of the normal bundle is obtained as follows. If M is a compact, k-dimensional, stably parallelizable manifold embedded in $R^N \subset R^{N+1}$, where N is sufficiently large, let $\tau_1(x), \cdots, \tau_{k+1}(x)$ be a trivialization of $\tau(M) \oplus 1$, where at each point $x \in M$ this set of vectors is an orthonormal basis for $\tau_x \oplus R$ (a Riemannian structure on M is assumed). Then a trivialization $\nu_1(x), \cdots, \nu_{N-k}(x)$ of the normal bundle of M corresponds to the given tangential trivialization precisely when the map $x \mapsto (\tau_1(x), \cdots, \tau_{k+1}(x), \nu_1(x), \cdots, \nu_{N-k}(x))$, which is a map of M into $O(N + 1) \subset O$ represents zero in $K^{-1}(M)$, i.e., is null homotopic. If M is parallelizable, the same thing may be done in R^N with k tangent vector fields.

In our case such a framing of the normal bundle may be obtained

by viewing Euclidean $2n^2$-space as $M(n) \oplus M(n)$, and embedding $M(n)$ in it as the first summand. This embeds $T(G)$ in $M(n) \oplus M(n)$. Again, choose a tangential framing $\tau_1, \cdots, \tau_k$ and a normal framing $\nu_1, \cdots, \nu_{n^2-k}$ at the identity of $T(G)$ in $M(n)$. In $M(n) \oplus M(n)$ a tangential framing at the identity is then given by $(\tau_1, 0), \cdots, (\tau_k, 0)$ and a normal one by $(\nu_1, 0), \cdots, (\nu_{n^2-k}, 0), (0, e_{ij})$, the e_{ij} being taken, say in lexicographic order. If these framings at the identity are then translated by the action of the elements

$$A_g = \begin{pmatrix} d\rho_g & 0 \\ 0 & (d\rho_g)^t \end{pmatrix} \quad \text{in} \quad \mathrm{SO}(2n^2) ,$$

tangential and normal framings of $T(G)$ are obtained which correspond to each other as described above. Hence the framing of the normal bundle is, as framing, independent of T.

If we translate the normal framing at the identity by the elements

$$B_g = \begin{pmatrix} d\rho_g & 0 \\ 0 & I \end{pmatrix} \quad \text{in} \quad \mathrm{SO}(2n^2) \text{ we}$$

obtain a framing of $T(G)$ which represents $[T]$; it is just the framing already described stabilized to co-dimension $2n^2 - k$. Take the first framing of the normal bundle to be $\tilde{G}$ and the second to be $\tilde{T}$.

It follows from the definition of the difference element in [1] that $d(\tilde{G}, \tilde{T}) \in K^{-1}(G)$ is represented by the map $\delta: G \to \mathrm{SO}(2n^2 - k)$ to be described next.

Fix an ordered orthonormal basis of Euclidean $(2n^2 - k)$-space. The fixed normal frame at the identity then defines a norm preserving monomorphism $F: R^{2n^2-k} \to M(n) \oplus M(n)$. Let $\pi: M(n) \oplus M(n) \to R^{2n^2-k}$ be orthogonal projection on the image of F followed by F^{-1}. Then $\delta(g) = \pi A_g B_g^t F$. If $C_g = A_g B_g^t \in \mathrm{SO}(2n^2)$, then

$$C_g = \begin{pmatrix} I & 0 \\ 0 & (d\rho_g)^t \end{pmatrix} .$$

It follows from the definitions of the frame at the identity and F that $\ker \pi$ is contained in the first summand of $M(n) \oplus M(n)$. Since C_g is the identity map when restricted to that summand and C_g sends the second summand to itself, we see that $d(\tilde{G}, \tilde{T})$ is also represented by the map $g \mapsto (d\rho_g)^t \in \mathrm{SO}(n^2)$. Lemma 4 then follows from Lemma 1.

Proof of Proposition 2. According to the theorem of [1], $[T] = [G] + \varphi(\tilde{G})\tilde{J}^{-1}d(\tilde{G}, \tilde{T})$, where $d(\tilde{G}, \tilde{T})$ is projected into $\tilde{K}^{-1}(G)$, $\tilde{J}^{-1}$ is the reduced J-morphism mapping $\tilde{K}^{-1}(G)$ to $\tilde{S}^0(G)$, the reduced stable cohomotopy group of G in degree zero (or the reduced cohomology group of G with coefficients in the sphere spectrum), and $\varphi(\tilde{G})$ is a map of $\tilde{S}^0(G)$ to Ω_k^{fr} which depends only on $\tilde{G}$ up to equivalence of

framings. In $K^{-1}(G)$, $\langle T + k\rangle = \langle T\rangle$. Then, by Lemma 4, $d(\tilde{G}, \widetilde{(T + k)}) = (n + k)\langle T + k\rangle = (n + k)\langle T\rangle$, so that $d(\tilde{G}, \widetilde{(mT + p)}) = (mn + p)\langle mT\rangle = m(mn + p)\langle T\rangle = (n + (m^2 - 1)n + mp)\langle T\rangle = d(\tilde{G}, \widetilde{(T + (m^2 - 1)n + p)})$. Hence, $[mT + p] = [T + (m^2 - 1)n + p]$, which proves part (c).

Since G is connected, $\tilde{S}^0(G)$ is a finite commutative ring unber the product in $S^*(G)$. (Note that this is the first time that the hypothesis of connectivity is really necessary.) In [1] it is shown that $\tilde{J}^{-1}\colon \tilde{K}^{-1}(G) \to \tilde{S}^0(G)$ is a homomorphism from $\tilde{K}^{-1}(G)$ to a modified abelian group structure on $\tilde{S}^0(G)$. Hence, $\tilde{J}^{-1}d(\tilde{G}, \widetilde{(T + k)}) = \tilde{J}^{-1}((n + k)\langle T\rangle)$ is the $(n + k)$-fold sum of an element in a finite abelian group with itself. Therefore, as k changes, $\tilde{J}^{-1}d(\tilde{G}, \widetilde{(T + k)})$ changes cyclically, and, for some k, $\tilde{J}^{-1}d(\tilde{G}, \widetilde{(T+k)}) = 0$. This proves parts (a) and (b).

Now consider statements (A) and (B), which will be proved by example. The methods used will involve Lemma 4 and the results of [1] as used in the proof of Proposition 3. In the examples, G will always be $S^3 = \mathrm{Sp}(1)$ and T_0 will be the representation of Sp(1) in SO(4) called σ on p. 116 of [3], which is defined by viewing $q \in \mathrm{Sp}(1)$ as a unit quaternion, and $q' \in R^4$ as a quaternion and setting $\sigma(q)q' = qq'$. This map defines an element called $\beta_3 \in \pi_3(\mathrm{SO}(4))$ which yields a generator of $\pi_3(\mathrm{O})$ under the inclusion $\mathrm{SO}(4) \subset \mathrm{O}$ [3]. The image of β_3 under the J-homomorphism generates Ω_3^{fr}, as is well-known, and is in fact the generator usually called ν. We shall not stop to prove this here since it is not necessary; for our purposes, we may define the generator $\nu \in \Omega_3^{fr} \approx Z_{24}$ as $\nu = J(\beta_3)$.

The proof of Proposition 1 of [1] shows that in the case where G (or the manifold, not necessarily a group, which is considered in that proposition) is a sphere, the map which we have called $\varphi(\tilde{G})$ in the case considered here is the identity (recall that $\tilde{S}^0(S^n) = \Omega_n^{fr}$) if the orientations involved are correctly chosen. Further, in the case of a sphere both group structures on $\tilde{S}^0(S^n)$ coincide, so that the formula used in the proof of Proposition 2 becomes

$$[T] = [G] + \tilde{J}^{-1}d(\tilde{G}, \tilde{T}) ,$$

where $\tilde{J}^{-1}\colon \tilde{K}^{-1}(S^3) \to \Omega_3^{fr}$ is the usual J-homomorphism.

Lemma 4 applied to this case then says that $d(\tilde{G}, \widetilde{(T_0 + k)}) = (4 + k)\langle T_0\rangle$. Since $\tilde{J}^{-1}\langle T_0\rangle = \nu$, $\tilde{J}^{-1}d(\tilde{G}, \widetilde{(T_0 + k)}) = (4 + k)\nu$, so $[T_0 + k] = [G] + (4 + k)\nu$. Therefore, $[T_0 + k] = [T_0]$ precisely when $k\nu = 0$, or $k \equiv 0 (\mathrm{mod}\ 24)$. This proves statement (B). Note that the length of the cycle of Proposition (2a) is twenty-four in this case, so that all of Ω_3^{fr} is filled up, and that $[T_0 + k] = [G]$ precisely when $k \equiv -4 (\mathrm{mod}\ 24)$.

Proposition (2c) tells us that $[mT_0] = [T_0 + 4(m^2 - 1)] = [G] + 4m^2\nu$. Then, $[pT_0] = [G] + 4p^2\nu$, and $[mT_0 + pT_0] = [(m + p)T_0] = [G] + 4(m + p)^2\nu$. Therefore, $[mT_0] + [pT_0] = [mT_0 + pT_0]$ precisely when $8mp \equiv 0 (\text{mod } 24)$ or $mp \equiv 0 (\text{mod } 3)$. In particular, $[T_0] + [T_0] \neq [T_0 + T_0]$, which proves statement (A).

In these examples it should be pointed out that in fact $[G] = \nu$ if the orientation of G is picked properly, and that this orientation can be picked so as to yield $[G] = \nu$ at the same time that $\varphi(\tilde{G})$ is the identity. The formulas of the last paragraphs then become

$$[T_0] = 5\nu$$

$$[T_0 + k] = (5 + k)\nu$$

$$[mT_0] = (4m^2 + 1)\nu \ .$$

Seeing that $[G] = \nu$ involves a very tedious matrix calculation which is not described in detail here. Briefly, by writing down the representation T_0 it is seen that S^3 is embedded in $R^{16} \subset R^{32}$ as the intersection of the 15-sphere of radius two with a certain 4-plane. Using this fact, a normal trivialization of $T_0(G)$ can be written down which represents zero in Ω_3^{fr}. The difference element between this framing and $\tilde{G}$ can then be written down and, by a suitable choice of inner automorphisms, recognized as representing the image of β_3 in $\pi_3(\text{SO}(29))$, and therefore as representing a generator of $\tilde{K}^{-1}(S^3)$. That $[G] = \nu$ then follows from an application of the theorem of [1].

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