Pacific Journal of Mathematics

LOCAL IDEALS IN A TOPOLOGICAL ALGEBRA OF ENTIRE FUNCTIONS CHARACTERIZED BY A NON-RADIAL RATE OF GROWTH

James Jerome Metzger

Vol. 51, No. 1 November 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 1, 1974

LOCAL IDEALS IN A TOPOLOGICAL ALGEBRA OF ENTIRE FUNCTIONS CHARACTERIZED BY A NON-RADIAL RATE OF GROWTH

JAMES J. METZGER

In this paper a class of locally convex algebras of entire functions is considered: For fixed $\rho > 0$, $\sigma > 0$, and n a positive integer, let $E[\rho, \sigma; n]$ denote the space of all entire functions f in n variables which satisfy $|f(x+iy)| = O\{\exp[A(\|x\|^\rho + \|y\|^\sigma)]\}$ for some $A > 0$. Sufficient conditions are given in order that the local ideal generated by a family in $E[\rho, \sigma; n]$ coincides with the closed ideal generated by the family.

For $z = x + iy = (x_1 + iy_1, x_2 + iy_2, \cdots, x_n + iy_n) \in \boldsymbol{C}^n$, write $\|z\|^2 = \|x\|^2 + \|y\|^2 = \sum_{k=1}^n (x_k^2 + y_k^2)$. For $f: \boldsymbol{C}^n \to \boldsymbol{C}$ and $A > 0$, let $\|f\|_A = \sup\{|f(z)| \exp[-A(\|x\|^\rho + \|y\|^\sigma)]: z \in \boldsymbol{C}^n\}$. The space $E = E[\rho, \sigma; n]$ is a locally convex algebra over $\boldsymbol{C}$, with the natural inductive limit topology from the Banach spaces $\{f \text{ entire}: \|f\|_A < \infty\}$, $A > 0$.

For $\mathscr{F}$ a family of functions in E, write $\mathscr{I}(\mathscr{F})$, $\mathscr{I}^-(\mathscr{F})$, and $\mathscr{I}_{\text{loc}}(\mathscr{F})$, respectively, for the ideal, closed ideal, and local ideal in E generated by $\mathscr{F}$. The local ideal $\mathscr{I}_{\text{loc}}(\mathscr{F})$ consists of all $H \in E$ such that in a neighborhood of each $z_0 \in \boldsymbol{C}^n$, H has the form $H = \sum_{j=1}^r h_j F_j$ for some $F_1, F_2, \cdots, F_r \in \mathscr{F}$ and $h_1, h_2, \cdots, h_r$ analytic in a neighborhood of z_0. The ideal $\mathscr{I}_{\text{loc}}(\mathscr{F})$ is closed in E and contains $\mathscr{F}$; hence $\mathscr{I}(\mathscr{F}) \subseteqq \mathscr{I}^-(\mathscr{F}) \subseteqq \mathscr{I}_{\text{loc}}(\mathscr{F})$. The problem to be considered is: Under what conditions is $\mathscr{I}^-(\mathscr{F}) = \mathscr{I}_{\text{loc}}(\mathscr{F})$ in the space $E = E[\rho, \sigma; n]$?

Problems of this type have been studied in various algebras E by many authors, among them: L. Ehrenpreis [2, 3], L. Schwartz [14], H. Cartan [1], L. Hörmander [5, 6], B. A. Taylor [15], J. J. Kelleher and B. A. Taylor [7, 8, 9], J. Metzger [11], I. F. Krasičkov [10], P. K. Raševskiĭ [13], and K. V. Rajeswara Rao [12].

Let $\mathscr{F} \subseteqq E = E[\rho, \sigma; n]$. It is known (see B. A. Taylor [15]) that for $n = 1$ and $\rho = \sigma \geqq 1$, $\mathscr{I}^-(\mathscr{F}) = \mathscr{I}_{\text{loc}}(\mathscr{F})$ in E for any $\mathscr{F}$. If $\rho = \sigma$ and $\mathscr{F} = \{F\}$, but n is arbitrary, then $\mathscr{I}(F) = \mathscr{I}^-(F) = \mathscr{I}_{\text{loc}}(F)$ (see L. Ehrenpreis [2]). In [11] this author proved that if $n = 1$, and $\rho \geqq 1$ or $\sigma \geqq 1$, then $\mathscr{I}^-(F) = \mathscr{I}_{\text{loc}}(F)$ for any $F \in E$; if in addition $\rho \neq \sigma$, there exists an $F \in E$ for which $\mathscr{I}(F) \neq \mathscr{I}^-(F)$. Concerning the more general case where n is arbitrary, and ρ and σ do not necessarily agree: Ehrenpreis's Quotient Structure Theorem (see [3]) implies that if $\rho > 1$ and $\sigma > 1$, and if $\mathscr{F} = \{F_1, F_2, \cdots, F_r\}$ consists of polynomials, then $\mathscr{I}(\mathscr{F}) = \mathscr{I}^-(\mathscr{F}) = \mathscr{I}_{\text{loc}}(\mathscr{F})$ in E. Also, a result of Hörmander [5] implies that when $\rho \geqq 1$ and $\sigma \geqq 1$, a family $\mathscr{F} = \{F_1, F_2, \cdots, F_r\}$ in E satisfies $\mathscr{I}(\mathscr{F}) = E$ if and only if there exist $\varepsilon > 0$ and $A > 0$ such that

$$\sum_{j=1}^{r} | F_j(z) | \geqq \varepsilon \exp [-A(\| x \|^\rho + \| y \|^\sigma)]$$

for all $z \in C^n$.

In this papar the following result is proved:

THEOREM 1. *Let n be a positive integer, $\rho > 0$, $\sigma > 0$, and $\tau = \max(\rho, \sigma) \geqq 1$; and let $\mathscr{F} \subseteqq E[\rho, \sigma; n]$. If $\mathscr{I}(\mathscr{F}) = \mathscr{I}_{\text{loc}}(\mathscr{F})$ in $E[\tau, \tau; n]$, then $\mathscr{I}^-(\mathscr{F}) = \mathscr{I}_{\text{loc}}(\mathscr{F})$ in $E[\rho, \sigma; n]$.*

Since $\mathscr{I}(F) = \mathscr{I}_{\text{loc}}(F)$ in $E[\tau, \tau; n]$, a consequence of Theorem 1 is:

COROLLARY. *Let n be a positive integer, $\rho > 0$, $\sigma > 0$, with $\max(\rho, \sigma) \geqq 1$. Then $\mathscr{I}^-(F) = \mathscr{I}_{\text{loc}}(F)$ in $E = E[\rho, \sigma; n]$ for any $F \in E$.*

This corollary generalizes to several variables the result proved by this author in [11] for the case of one variable.

Theorem 1 follows immediately from an approximation theorem which is proved in the next section. In the third section Theorem 1 is applied to several examples.

2. The main theorem. The approximation theorem stated below, Theorem 2, yields Theorem 1 as an immediate corollary. The proof of Theorem 2 is based on a technique of L. Hörmander given in [6], which in turn involves the solution of the $\bar{\partial}$ equation (see [4, Chapter IV]).

THEOREM 2. *Let $\sigma \geqq 1$, and $H, F_1, F_2, \cdots, F_r, G_1, G_2, \cdots, G_r$ be entire functions in n variables, with $H = \sum_{j=1}^{r} G_j F_j$ and*

$$| G_j(z) | \leqq C \exp(A \| z \|^\sigma) \tag{1}$$

for all $z \in C^n$, $j = 1, 2, \cdots, r$, where A, C denote positive constants. Then there exist positive constants B, K, and M, and entire functions $g_{j,t}$, $0 < t < 1$, $j = 1, 2, \cdots, r$, such that:

$$\begin{aligned} &\left| H(z) - \sum_{j=1}^{r} g_{j,t}(z) F_j(z) \right| \\ &\leqq tK(1 + \| z \|^2)^M \left\{ | H(z) | + \left[\sum_{j=1}^{r} | F_j(z) | \exp(B \| y \|^\sigma) \right] \right\} \end{aligned} \tag{2}$$

for all $z \in C^n$, $0 < t < 1$, and

$$(3) \qquad |g_{j,t}(z)| \leqq L(t)(1+\|z\|^2)^M \exp(B\|y\|^\sigma)$$

for all $z \in \boldsymbol{C}^n$, $0 < t < 1$, $j = 1, 2, \cdots, r$, *where* $L(t) > 0$ *may depend on* t *but not on* z.

The proof of Theorem 2 is facilitated by the following:

LEMMA. *Let* n *and* N *be positive integers, with* N *even. There exist* $\alpha > 0$ *and* $\varepsilon > 0$ *such that: If* $z \in \boldsymbol{C}^n$ *with* $\alpha\|x\| \geqq \|y\|$, *then* $\operatorname{Re}(z^N) \geqq \varepsilon\|z\|^N$.

Here $z^N = \sum_{k=1}^n (x_k + iy_k)^N$.

Proof. Write $q = N/2$; then

$$\operatorname{Re}[(x_k+iy_k)^N] = x_k^{2q} + \sum_{m=1}^{q} a_m x_k^{2(q-m)} y_k^{2m}$$

for all $x_k + iy_k \in \boldsymbol{C}$, where $a_1, a_2, \cdots, a_q$ are integers depending only on N. Hence for $z \in \boldsymbol{C}^n$,

$$\begin{aligned}\operatorname{Re}(z^N) &\geqq \sum_{k=1}^{n} x_k^{2q} - \sum_{m=1}^{n} |a_m| \left(\sum_{k=1}^{n} x_k^{2(q-m)} y_k^{2m}\right) \\ &\geqq 2^{-(n-1)(q-1)}\|x\|^{2q} - \sum_{m=1}^{q} |a_m|\,\|x\|^{2(q-m)}\,\|y\|^{2m}\,.\end{aligned}$$

The required condition is then satisfied with $\varepsilon = 2^{-[(n-1)(q-1)]-(q-2)}$, and $0 < \alpha < 1$ sufficiently small that $\sum_{m=1}^{q} |a_m| \alpha^{2m} < 2^{-[(n-1)(q-1)]-1}$.

Proof of Theorem 2. Let N be an even integer, $N > \sigma$. By the lemma there exist $\alpha = \alpha(n, N) > 0$ and $\varepsilon = \varepsilon(n, N) > 0$ such that $\alpha\|x\| \geqq \|y\|$ implies $\operatorname{Re}(z^N) \geqq \varepsilon\|z\|^N$. Set $S = \{z \in \boldsymbol{C}^n : \alpha\|x\| \geqq \|y\|$ and $\operatorname{Re}(z^N) \geqq 1\}$. The bounds (1) imply that for some $B > 0$ and $K_1 > 0$,

$$(4) \qquad |G_j(z)| \leqq K_1 \exp(B\|y\|^\sigma)$$

for all $z \in \boldsymbol{C}^n \backslash S$, $j = 1, 2, \cdots, r$.

Let $\varphi: \boldsymbol{R} \to \boldsymbol{R}$ be a C^∞ function such that

$$\begin{aligned}\varphi(u) &= 0 \quad \text{if} \quad u \leqq 0\,, \\ &= 1 \quad \text{if} \quad u \geqq 1\,,\end{aligned}$$

and $0 \leqq \varphi(u) \leqq 1$ if $0 \leqq u \leqq 1$. For $0 < t < 1$ and $z \in \boldsymbol{C}^n$, set

$$\omega_t(z) = [\varphi(\operatorname{Re}(z^N))]\exp[-t(z^N)] + [1 - \varphi(\operatorname{Re}(z^N))]\,.$$

Each ω_t is a C^∞ function on $\boldsymbol{C}^n$; and $|\omega_t(z)| \leqq 1$ for all $z \in \boldsymbol{C}^n$, while

$|\omega_t(z)| \leqq \exp(-\varepsilon \|z\|^N)$ for all $z \in S$. Together with (1) and (4), this implies that for some $K_2 > 0$,

$$(5) \qquad |\omega_t(z)G_j(z)| \leqq K_2 \exp(B\|y\|^\sigma)$$

for all $z \in \boldsymbol{C}^n$, $0 < t < 1$, $j = 1, 2, \cdots, r$. Since $|1 - e^\zeta| \leqq |\zeta|$ if $\operatorname{Re}\zeta \leqq 0$, and since $|z^N| \leqq n\|z\|^N$, it follows that $|1 - \omega_t(z)| \leqq tn\|z\|^N$ for all $z \in \boldsymbol{C}^n$, $0 < t < 1$. Consequently,

$$(6) \qquad \left|H(z) - \sum_{j=1}^{r} (\omega_t(z)G_j(z))F_j(z)\right| \leqq tn\|z\|^N |H(z)|$$

for all $z \in \boldsymbol{C}^n$, $0 < t < 1$. Thus the functions $\omega_t G_j$ satisfy conditions of the form (2) and (3).

As is done by Hörmander, the functions $\omega_t G_j$ will now be altered to obtain the desired analytic functions $g_{j,t}$. First of all, $\bar\partial\omega_t = 0$ if $\operatorname{Re}(z^N) \leqq 0$, and $\|\bar\partial[\varphi(\operatorname{Re}(z^N))]\| \leqq K_3\|z\|^{N-1}$ everywhere on $\boldsymbol{C}^n$; therefore, $\|\bar\partial\omega_t(z)\| \leqq tnK_3\|z\|^{2N-1}$ for all $z \in \boldsymbol{C}^n$, $0 < t < 1$. Also, $\bar\partial(\omega_t G_j) = (\bar\partial\omega_t)G_j$; and $\bar\partial\omega_t = 0$ on S. By (4) then, for $0 < t < 1$ and $j = 1, 2, \cdots, r$,

$$(7) \qquad \|\bar\partial(\omega_t(z)G_j(z))\| \leqq tK_4\|z\|^{2N-1}\exp(B\|y\|^\sigma)$$

for all $z \in \boldsymbol{C}^n$, and thus

$$\int_{C^n} \|\bar\partial(\omega_t(z)G_j(z))\|^2 e^{-\psi(z)} d\lambda(z) \leqq t^2 K_5$$

where $\psi(z) \equiv 2B\|y\|^\sigma + (2N+n)\log(1 + \|z\|^2)$, and λ denotes Lebesgue measure.

By applying Theorem 4.4.2 of Hörmander [4], functions $\nu_{j,t}$ of class C^∞ on $\boldsymbol{C}^n$ may be chosen such that $\bar\partial\nu_{j,t} = \bar\partial(\omega_t G_j)$ and

$$\int_{C^n} |\nu_{j,t}(z)|^2 \exp[-\psi(z) - 2\log(1 + \|z\|^2)]d\lambda(z) \leqq t^2 K_5$$

for $0 < t < 1$, $j = 1, 2, \cdots, r$. Together with (7), this implies (see Hörmander [6, p. 314]) that

$$(8) \qquad |\nu_{j,t}(z)| \leqq tK_6(1 + \|z\|^2)^M \exp(B\|y\|^\sigma)$$

for all $z \in \boldsymbol{C}^n$, $0 < t < 1$, $j = 1, 2, \cdots, r$, where $M = N + 1 + (1/2)n$.

Each of the functions $g_{j,t} = \omega_t G_j - \nu_{j,t}$ is then entire. Further, (3) is satisfied because of (5) and (8). Lastly $H - \sum_{j=1}^r g_{j,t}F_j = [H - \sum_{j=1}^r (\omega_t G_j)F_j] + \sum_{j=1}^r \nu_{j,t}F_j$, and thus (2) follows from (6) and (8).

3. Examples and applications. In this section several examples are given where $\mathscr{I}^-(\mathscr{F}) = \mathscr{I}_{\text{loc}}(\mathscr{F})$ in spaces of the form $E[\rho, \sigma; n]$.

EXAMPLE 1. Let $E = E[\rho, \sigma; n]$, with $\tau = \max(\rho, \sigma) \geqq 1$, and let $F \in E$. The corollary to Theorem 1 implies that $\mathscr{I}^-(F) = \mathscr{I}_{\text{loc}}(F)$

in E. Also, $\mathscr{I}(F) = \mathscr{I}^{-}(F) = \mathscr{I}_{\text{loc}}(F)$ in $E[\tau, \tau; n]$. However, it need not be the case that $\mathscr{I}(F) = \mathscr{I}^{-}(F)$ in E; indeed, if $\rho \neq \sigma$ then (see [11]) there exists an $F \in E$ for which $\mathscr{I}(F) \neq \mathscr{I}^{-}(F)$.

EXAMPLE 2. Let $n = 1$, and $E = E[\rho, \sigma; 1]$, with $\tau = \max(\rho, \sigma) \geqq 1$. Let $\mathscr{T} \subseteqq E$ and suppose some $F_0 \in E$ has only finitely many zeros. Then $\mathscr{I}^{-}(\mathscr{T}) = \mathscr{I}_{\text{loc}}(\mathscr{T})$ in E. To prove this, write $F_0 = PH$ where P is a polynomial and $H \in E$ has no zeros. There exists a polynomial Q such that $\mathscr{I}_{\text{loc}}(\mathscr{T})$ in E is $\{G \in E: G/Q$ is analytic$\}$. Set $P = P_0Q$, so that $F_0 = P_0QH \in \mathscr{T} \subseteqq \mathscr{I}(\mathscr{T})$. The factors of P_0 can bo divided out (see Taylor [15]) to yield $QH \in \mathscr{I}(\mathscr{T})$ in E. Since $1/H \in E[\tau, \tau; 1]$, it follows that $Q \in \mathscr{I}(\mathscr{T})$ in $E(\tau, \tau; 1)$, which implies that $\mathscr{I}(\mathscr{T}) = \mathscr{I}_{\text{loc}}(\mathscr{T})$ in $E[\tau, \tau; 1]$, Then by Theorem 1, $\mathscr{I}^{-}(\mathscr{T}) = \mathscr{I}_{\text{loc}}(\mathscr{T})$ in $E = E[\rho, \sigma; 1]$.

Note that if $1/H \in E$—e.g., if $\rho \geqq 1$, $\sigma \geqq 1$, and F_0 is an exponential polynomial $F_0(z) \equiv P(z)e^{az}$—then $Q \in \mathscr{I}(\mathscr{T})$ in E and thus $\mathscr{I}(\mathscr{T}) = \mathscr{I}^{-}(\mathscr{T}) = \mathscr{I}_{\text{loc}}(\mathscr{T})$ trivially. On the other hand, if $1/H \notin E$ then $\mathscr{I}(\mathscr{T})$ need not coincide with $\mathscr{I}^{-}(\mathscr{T})$ in E—for instance, if $\rho = 1$, $\sigma = 2$, and $\mathscr{T} = \{e^{-(z^2)}, e^{iz} - 1\}$.

EXAMPLE 3. Let $1 \leqq \rho < \sigma$ and $E = E[\rho, \sigma; 1]$. Choose γ, $\rho < \gamma < \sigma$; let $\varepsilon_m = \exp(-(2^{m\gamma}))$, $a_m = 2^m$, $b_m = 2^m + \varepsilon_m$, $m = 1, 2, \cdots$; and let

$$F_1(z) = \prod_{m=1}^{\infty}\left(1 - \frac{z}{a_m}\right)$$

$$F_2(z) = \prod_{m=1}^{\infty}\left(1 - \frac{z}{b_m}\right)$$

for all $z \in C$. Each F_j is an entire function of order 0 and thus is in E. Clearly $\mathscr{I}_{\text{loc}}(F_1, F_2) = E$. It is easily argued that for $\rho < \rho' < \gamma$, $|F_2(2^m)| = O[\exp(-(2^{m\rho'}))]$ as $m \to \infty$. Consequently $1 \notin \mathscr{I}(F_1, F_2)$ in E. On the other hand, letting $\gamma < \sigma' < \sigma$ and using standard estimates on infinite products yields:

$$|F_1(z)| \geqq \delta \exp(-|z|^{\sigma'}) \quad \text{if} \quad z \notin \bigcup_m \left\{z: |z - a_m| < \frac{1}{2}\varepsilon_m\right\},$$

$$|F_2(z)| \geqq \delta \exp(-|z|^{\sigma'}) \quad \text{if} \quad z \notin \bigcup_m \left\{z: |z - b_m| < \frac{1}{2}\varepsilon_m\right\},$$

where $\delta > 0$. Thus $|F_1(z)| + |F_2(z)| \geqq \delta \exp(-|z|^{\sigma'})$ for all $z \in C$. It then follows (Hörmander [5]) that $1 \in \mathscr{I}(F_1, F_2)$ in $E[\sigma, \sigma; 1]$. Hence $\mathscr{I}(F_1, F_2) = \mathscr{I}^{-}(F_1, F_2) = \mathscr{I}_{\text{loc}}(F_1, F_2)$ in $E[\sigma, \sigma; 1]$, while $\mathscr{I}(F_1, F_2) \subsetneqq \mathscr{I}^{-}(F_1, F_2) = \mathscr{I}_{\text{loc}}(F_1, F_2)$ in $E = F[\rho, \sigma; 1]$.

References

1. H. Cartan, *Idéaux et modules de fonctions analytiques de variables complexes*, Bull. Soc. Math. France, **78** (1950), 29-64.

2. L. Ehrenpreis, *Mean periodic functions. I*, Amer. J. Math., **77** (1955), 293-328.

3. ———, *Fourier analysis in several complex variables*, Pure and Appl. Math., **17**, Interscience, New York, 1970.

4. L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, D. Van Nostrand, Princeton, N. J., 1966.

5. ———, *Generators for some rings of analytic functions*, Bull. Amer. Math. Soc., **73** (1967), 943-949.

6. ———, *Convolution equations in convex domains*, Invent. Math., **4** (1968), 306-317.

7. J. J. Kelleher and B. A. Taylor, *An application of the Corona theorem to some rings of entire functions*, Bull. Amer. Math. Soc., **73** (1967), 246-249.

8. ———, *Finitely generated ideals in rings of analytic functions*, Math. Ann., **193** (1971), 225-237.

9. ———, *Closed ideals in locally convex algebras of analytic functions*, in preparation.

10. I. F. Krasičkov, *Closed ideals in the locally convex algebra of entire functions with an arbitrary growth majorant*, Sov. Math. Dokl., **7** (1966), 1324-1325.

11. J. Metzger, *Principal local ideals in weighted spaces of entire functions*, Trans. Amer. Math. Soc., **165** (1972), 149-158.

12. K. V. Rajeswara Rao, *On a generalized Corona problem*, J. d'Analyse Math., **18** (1967), 277-278.

13. P. K. Raševskiĭ, *Closed ideals in a countably normed algebra of analytic entire functions*, Sov. Math. Dokl., **6** (1965), 717-719.

14. L. Schwartz, *Théorie générale des fonctions moyenne-périodiques*, Ann. of Math., (2) **48** (1947), 857-929.

15. B. A. Taylor, *Some locally convex spaces of entire functions*, Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966), Amer. Math. Soc., Providence, R. I., 1968, pp. 431-467.

Received November 14, 1972 and in revised form July 20, 1973.

UNIVERSITY OF GEORGIA

Printed in Japan by Intarnational Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 51, No. 1   November, 1974