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If the outer boundary of the compact plane set E is the union of finitely many disjoint analytic Jordan curves, the Garabedian function of E is a familiar object. J. Garnett and S. Y. Havinson have each asked whether the Garabedian functions of a decreasing sequence of such sets must converge. The present paper shows that they do converge. This fact leads to a natural definition of the Garabedian function of an arbitrary compact plane set. As an intermediate step, an approximate formula is obtained for the analytic capacity of the union of a compact set E and a small disc not intersecting E .

1. Prerequisites and notation. Good introductions to Analytic Capacity are given in [2], pp. 1-26, and [1], Ch. 8; and so we shall give only a brief outline.

C denotes the complex plane. S^2 denotes the extended complex plane with its usual topology. $D(z; r)$ denotes the closed disc with centre z and radius r .

Let E be a compact subset of C . $\Omega(E)$ denotes the component of $S^2 \setminus E$ containing ∞ . The *outer boundary* of E is the boundary $\partial\Omega(E)$ of $\Omega(E)$. The *analytic capacity* of E is:

$$\gamma(E) = \sup \{ |g'(\infty)| : g \text{ analytic on } \Omega(E), |g| < 1 \}.$$

This supremum is attained by a unique function, the *Ahlfors function* of E ([1], p. 197).

\mathcal{S} will denote the class of all compact plane sets whose outer boundary is the union of finitely many pairwise disjoint analytic Jordan curves. Let $E \in \mathcal{S}$, and write $\Omega = \Omega(E)$. The *Hardy space* $H^p(\Omega)$ ($0 < p < \infty$) is the space of all analytic functions g on Ω such that there exists a harmonic function u on Ω with $|g|^p < u$. If $g \in H^p(\Omega)$ then g has a finite nontangential limit $g(z)$ at almost every point $z \in \partial\Omega$. $H^2(\Omega)$ is a separable Hilbert space with the inner product:

$$(g, h) = \int_{\partial\Omega} g(z) \overline{h(z)}^* ds \quad (g, h \in H^2(\Omega)).$$

If $\zeta \in \Omega$ there is a unique function $K(z, \zeta)$ in $H^2(\Omega)$, the *Szegö kernel function*, such that:

$$g(\zeta) = \int_{\partial\Omega} g(z) K(z, \zeta)^* ds \quad (g \in H^2(\Omega)).$$

$K(z, \zeta)$ is the inner product between the functionals on $H^2(\Omega)$ given

by evaluation at z and ζ , so that $K(z, \zeta) = \sum u_n(z)u_n(\zeta)^*$, whenever $\{u_n\}$ is an orthonormal basis for $H^2(\Omega)$. The *Garabedian function* is most easily defined for our purpose as:

$$\psi(z) = \frac{2\pi}{i} \gamma(E)^2 K(z, \infty)^2.$$

See [2], pp. 13-23.

Throughout, E will be a compact plane set, $\Omega = \Omega(E)$, and f will be the Ahlfors function of E . If $E \in \mathcal{S}$, $K(z, \zeta)$ will denote its Szegő kernel function, and ψ its Garabedian function.

We shall use the following results.

1.1. Let $\{E_n\}$ be a decreasing sequence of compact sets with intersection E . Let f_n be the Ahlfors function of E_n . Then $f_n \rightarrow f$ uniformly on compact subsets of Ω , and $\gamma(E_n) \rightarrow \gamma(E)$ ([1], p. 198).

1.2. Let $E \in \mathcal{S}$. Then:

(1) f and ψ are analytic across $\partial\Omega$.

(2) $|f| = 1$ on $\partial\Omega$.

(3) $f(z)\psi(z)dz \geq 0$ on $\partial\Omega$.

(4) $\psi(\infty) = 1/(2\pi i)$.

(5) $K(\infty, \infty) = 1/(2\pi\gamma(E))$.

([2], pp. 18-23).

1.3. Let E, F be compact, $\gamma(E) = 0$. Then $\gamma(E \cup F) = \gamma(F)$ (an immediate consequence of [2], Theorem 1.4, pp. 10-11).

1.4. Let E be compact, $0 \notin E$, $E \subset D(0; R)$. Denote by E_* the inversion of E in the unit circle. Then:

$$\gamma(E_*) \geq \gamma(E)/8R^2$$

(proof similar to [1], Lemma 12.2, p. 229).

Finally we need the following result on Hilbert spaces.

PROPOSITION 1.5. *Let h be a separable Hilbert space, and let $\{u_n\}$ be a sequence of vectors in h whose closed linear span is h . Suppose that the infinite matrix T given by $T_{ij} = (u_j, u_i)$ is bounded and invertible (as an operator on l_2). Then for every bounded linear functional f on h the sequence $\{f(u_i)\}$ is square-summable and:*

$$\|f\|^2 = \sum_{i,j=1}^{\infty} (T^{-1})_{ij} f(u_i) f(u_j)^*.$$

Proof. T is positive, and so is the matrix of a positive operator

$P \in B(l_2)$. P has a positive square root $P^{1/2}$, which is invertible since P is invertible. For $i = 1, 2, 3, \dots$, write $w_i = P^{1/2}e_i$, where e_i is the vector with 1 in its i th place and 0 elsewhere. Since $P^{1/2}$ is invertible, l_2 is the closed linear span of the w_i . $(w_j, w_i) = (P^{1/2}e_j, P^{1/2}e_i) = (Pe_j, e_i) = T_{ij} = (u_j, u_i)$; so we can define a unitary $J: l_2 \rightarrow h$ by $J(w_i) = u_i$ for all i , extended to the whole of l_2 by linearity and continuity. The bounded linear functional J^*f on l_2 is represented by some $s \in l_2$. $(e_i, P^{1/2}s) = (P^{1/2}e_i, s) = (w_i, s) = (J^*f)(w_i) = f(Jw_i) = f(u_i)$. Hence $\{f(u_i)\}$ is square-summable. Also:

$$\begin{aligned} \|f\|^2 &= \|s\|^2 = (P^{-1}(P^{1/2}s), (P^{1/2}s)) \\ &= \sum_{i,j=1}^{\infty} (T^{-1})_{ij}(e_i, P^{1/2}s)(e_j, P^{1/2}s)^* = \sum_{i,j=1}^{\infty} (T^{-1})_{ij}f(u_i)f(u_j)^* . \end{aligned}$$

2. The slope function. The purpose of this section is to establish Theorem 2.2, which gives an expression, up to first order in ε , for the analytic capacity of a set of the form $E \cup D(z; \varepsilon)$, where $E \in \mathcal{S}$ and $z \in \Omega(E)$. This will be extended to arbitrary compact sets E in §3. First we need a lemma which gives bounds on the Szegő kernel function.

LEMMA 2.1. *Let $E \in \mathcal{S}$, $\zeta \in \Omega(E)$, $\zeta \neq \infty$. Let r, R be the least and greatest distances of points of E from ζ . Then:*

$$\frac{r^2}{16\pi R^2\gamma(E)} \leq K(\zeta, \zeta) \leq \frac{8R^2}{2\pi r^2\gamma(E)} .$$

Proof. We prove the upper bound: the lower one is similar. We may assume that $\zeta = 0$. Let $g \in H^2(\Omega)$, $\|g\| \leq 1$. Denote inversion in the unit circle by $*$. Define g_* on Ω_* by $g_*(z) = g(z_*)^*$. Clearly $g_* \in H^2(\Omega_*)$ and $\|g_*\| \leq 1/r$. Hence:

$$|g(0)|^2 = |g_*(\infty)|^2 \leq \frac{\|g_*\|^2}{2\pi\gamma(E_*)} \leq \frac{1}{2\pi r^2\gamma(E_*)} \leq \frac{8R^2}{2\pi r^2\gamma(E)}$$

by 1.4. So:

$$K(0, 0) = \sup \{|g(0)|^2: g \in H^2(\Omega), \|g\| \leq 1\} \leq \frac{8R^2}{2\pi r^2\gamma(E)} .$$

There is a simpler bound for the Garabedian function: for, in the above notation:

$$\left| \psi(\zeta) - \frac{1}{2\pi i} \right| = \left| \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\psi(z)dz}{z - \zeta} \right| \leq \frac{1}{2\pi r} \int_{\partial\Omega} |\psi| ds = \frac{\gamma(E)}{2\pi r} .$$

THEOREM 2.2. *Let $E \in \mathcal{S}$. There is a positive real-valued func-*

tion $a_E(\zeta)$, the slope function of E , defined on Ω , with the property that for all $\zeta \in \Omega$:

$$\gamma(E \cup D(\zeta; \varepsilon)) = \gamma(E) + \varepsilon a_E(\zeta) + O(\varepsilon^2).$$

$a_E(\zeta)$ is given explicitly by:

$$a_E(\zeta) = 2\pi |\psi(\zeta)| \{1 - |f(\zeta)|^2\}.$$

The bound in the error term depends only on $\gamma(E)$ and on the ratio of the greatest and least distances of points of E from ζ .

Proof. We may suppose that $\zeta = 0$. Let r, R be the least and greatest distances of points of E from 0. Note that $E \subset D(0; R)$, so that $\gamma(E) \leq R$. We shall prove the theorem by showing that:

$$\begin{aligned} \varepsilon < 10^{-5}(r/R)^5 \gamma(E) &\implies |\gamma(E \cup D(0; \varepsilon)) - \gamma(E) - \varepsilon a_E(0)| \\ &\leq 10^9(R/r)^{10} \gamma(E)^{-1} \varepsilon^2. \end{aligned}$$

Fix $\varepsilon < 10^{-5}(r/R)^5 \gamma(E)$. Since $r < R$ and $\gamma(E) \leq R$, we have $\varepsilon < 10^{-5}r$; so $D(0; \varepsilon)$ does not meet E . Write $E_1 = E \cup D(0; \varepsilon)$, $\Omega_1 = \Omega(E_1)$, $H^2 = H^2(\Omega)$, $H_1^2 = H^2(\Omega_1)$, $\gamma = \gamma(E)$, $\gamma_1 = \gamma(E_1)$. Choose an orthonormal basis $\{u_n\}$ for H^2 . Now we can use the Cauchy integral to express any element of H_1^2 as the sum of an element of H^2 and an element of $H^2(S^1 \setminus D(0; \varepsilon))$. The latter space is the closed linear span of $\{z^{-n}; n \geq 0\}$. It follows that if, for $n \geq 1$, v_n is any function analytic on $\bar{\Omega}$ except for a pole of order n at 0, then H_1^2 is the closed linear span of $\{u_n\} \cup \{v_n\}$. To be specific, we shall put:

$$v_n(z) = \frac{\varepsilon^{n-1/2}}{\sqrt{(2\pi)} K(0, 0)} \frac{K(z, 0)}{z^n}.$$

1.2 (5) says that $1/(2\pi\gamma_1)$ is the square of the norm of evaluation at ∞ in H_1^2 . Our proof consists of calculating this by applying Proposition 1.5 to $\{u_n\} \cup \{v_n\}$.

We shall calculate various bounds now, so as not to break continuity later. Throughout, “ $\| \cdot \|$ ” and “norm” will refer to the norm of an element of a Hilbert space, or the norm of an infinite matrix considered as a bounded operator on l_2 ; and “ $\| \cdot \|_\infty$ ” will denote the supremum of the absolute value of a function on the set $D(0; \varepsilon)$.

Let $z_0 \in C$, $|z_0| \leq \varepsilon$. For $n \geq 1$:

$$u_n(z_0) = \frac{1}{2\pi i} \int_{|z|=r/2} \frac{u_n(z) dz}{z - z_0}.$$

Hence:

$$\|u_n\|_\infty \leq \frac{1}{2\pi(r/2 - \varepsilon)} \int_{|z|=r/2} |u_n| ds,$$

so that by Schwarz's inequality:

$$\|u_n\|_\infty^2 \leq \frac{\pi r}{4\pi^2(r/2 - \varepsilon)^2} \int_{|z|=r/2} |u_n|^2 ds.$$

Summing over n and using Lemma 2.1 gives:

$$\begin{aligned} \sum \|u_n\|_\infty^2 &\leq \frac{r}{4\pi(r/2 - \varepsilon)^2} \int_{|z|=r/2} K(z, z) ds \\ (1) \quad &\leq \frac{r \cdot \pi r}{4\pi(r/2 - \varepsilon)^2} \frac{8(R + r/2)^2}{2\pi(r/2)^2\gamma} \leq 3R^2 r^{-2}\gamma^{-1} \end{aligned}$$

since $\varepsilon < 10^{-5}r$ and $r < R$. Analogous computation gives:

$$(2) \quad \sum \|u'_n\|_\infty^2 \leq 50R^2 r^{-4}\gamma^{-1}.$$

In particular:

$$(3) \quad \sum |u'_n(0)|^2 \leq 50R^2 r^{-4}\gamma^{-1}.$$

Next we want a bound for $\|d^k/dz^k K(z, 0)\|_\infty$. Let $z_0 \in C$, $|z_0| \leq \varepsilon$. Then for $k \geq 1$ and for all $s < r$:

$$\begin{aligned} \left| \frac{d^k}{dz^k} K(z, 0) \right|_{z=z_0} &= \left| \frac{k!}{2\pi i} \int_{|z|=s} \frac{K(z, 0) dz}{(z - z_0)^{k+1}} \right| \\ &\leq \frac{k!}{2\pi} \frac{2\pi s}{(s - \varepsilon)^{k+1}} \frac{8R(R + s)}{2\pi r(r - s)\gamma}. \end{aligned}$$

(Here we have estimated $|K(z, 0)|$ by $K(z, z)^{1/2} K(0, 0)^{1/2}$ and then used Lemma 2.1.) In particular, putting $s = kr/(k + 1)$:

$$\begin{aligned} \left| \frac{d^k}{dz^k} K(z, 0) \right|_{z=z_0} &\leq \frac{k!}{2\pi} \frac{8R \cdot 2R}{r(r - (k + 1)\varepsilon/k)^{k+1}\gamma} \frac{(k + 1)^{k+1}}{k^k} \\ &\leq \frac{(k + 1)!}{2\pi} \frac{16R^2}{r(r - 2\varepsilon)^{k+1}\gamma} e \leq \frac{7R^2(k + 1)!}{r(r - 2\varepsilon)^{k+1}\gamma}. \end{aligned}$$

This holds also for $k = 0$ by Lemma 2.1. Hence for $k = 0, 1, 2, \dots$:

$$(4) \quad \left\| \frac{d^k}{dz^k} K(z, 0) \right\|_\infty \leq \frac{7R^2(k + 1)!}{r(r - 2\varepsilon)^{k+1}\gamma}.$$

We need one more estimate. Since $\varepsilon < 10^{-5}r$, (4) gives:

$$\left\| \frac{dK(z, 0)}{dz} \right\|_\infty \leq \frac{15R^2}{r^3\gamma}.$$

Hence, using Lemma 2.1 and the fact that $\varepsilon < 10^{-5}R^{-4}r^5$, we have:

$$(5) \quad \|K(z, 0)\|_{\infty} \leq K(0, 0) + \varepsilon \left\| \frac{dK(z, 0)}{dz} \right\|_{\infty} \leq 1.01K(0, 0).$$

We shall imagine the basis $\{u_n\} \cup \{v_n\}$ to be partitioned into three sections. The first section consists of all the u_n , the second section consists of v_1 alone, and the third section consists of v_2, v_3, v_4, \dots . The corresponding matrix T of inner products will be in block form:

$$(6) \quad T = I + M, \quad M = \begin{bmatrix} A & B^H & C^H \\ B & D & E^H \\ C & E & F \end{bmatrix}.$$

Next we calculate the inner products. Denote inner products in H_1 by (\cdot, \cdot) . By a statement of the form “ $X = Y$ with error Z ” we shall mean $|X - Y| \leq Z$, or $\|X - Y\| \leq Z$, according to context.

$$\begin{aligned} (u_n, u_m) &= \int_{\partial\Omega_1} u_n u_m^* ds = \int_{\partial\Omega} u_n u_m^* ds + \int_{|z|=\varepsilon} u_n u_m^* ds = \delta_{mn} + 2\pi\varepsilon u_m(0)^* u_n(0) \\ &+ \varepsilon \int_0^{2\pi} [u_m(0)^* (u_n(\varepsilon e^{i\theta}) - u_n(0)) + (u_m(\varepsilon e^{i\theta})^* - u_m(0)^*) u_n(\varepsilon e^{i\theta})] d\theta. \end{aligned}$$

$$|(u_n, u_m) - \delta_{mn} - 2\pi\varepsilon u_m(0)^* u_n(0)| \leq 2\pi\varepsilon^2 (\|u_m\|_{\infty} \|u'_n\|_{\infty} + \|u'_m\|_{\infty} \|u_n\|_{\infty}).$$

Now the matrix $[2\pi\varepsilon u_m(0)^* u_n(0)]$ has norm $2\pi\varepsilon(\sum |u_m(0)|^2 |u_n(0)|^2)^{1/2} = 2\pi\varepsilon K(0, 0) \leq 8R^2 r^{-2} \gamma^{-1} \varepsilon$ by Lemma 2.1. The norm of the matrix $[2\pi\varepsilon^2 (\|u_m\|_{\infty} \|u'_n\|_{\infty} + \|u'_m\|_{\infty} \|u_n\|_{\infty})]$ is at most $4\pi\varepsilon^2 (\sum \|u_m\|_{\infty}^2 \|u'_n\|_{\infty}^2)^{1/2} \leq 200R^2 r^{-3} \gamma^{-1} \varepsilon^2$ by (1) and (2). So (see the format (6)):

$$(7) \quad A = [2\pi\varepsilon u_m(0)^* u_n(0)] \text{ with error } 200R^2 r^{-3} \gamma^{-1} \varepsilon^2.$$

Also, $\|A\| \leq 8R^2 r^{-2} \gamma^{-1} \varepsilon + 200R^2 r^{-3} \gamma^{-1} \varepsilon^2 \leq 9(R/r)^2 \gamma^{-1} \varepsilon \leq 9(R/r)^5 \gamma^{-1} \varepsilon$ since $\varepsilon < 10^{-5}r$ and $r < R$. In fact the cruder bound $\|A\| \leq 2500(R/r)^5 \gamma^{-1} \varepsilon$ will be sufficient. Observe that, since $\varepsilon < 10^{-5}(R/r)^5 \gamma$, we have also $\|A\| \leq 1/40$.

The m th element of B is:

$$\begin{aligned} (u_m, v_1) &= \frac{\varepsilon^{1/2}}{\sqrt{(2\pi)} K(0, 0)} \int_{\partial\Omega} \frac{K(z, 0)^* u_m ds}{z^*} \\ &+ \frac{\varepsilon^{1/2}}{\sqrt{(2\pi)} K(0, 0) \varepsilon i} \int_{|z|=\varepsilon} K(z, 0)^* u_m(z) dz. \end{aligned}$$

Now the second term on the right-hand side is:

$$\frac{\varepsilon^{-1/2}}{\sqrt{(2\pi)} K(0, 0) i} \int_{|z|=\varepsilon} (K(z, 0)^* - K(0, 0)^*) u_m(z) dz$$

by Cauchy's theorem, and is therefore bounded in magnitude by $(2\pi e^{3/2} / (\sqrt{(2\pi)} K(0, 0))) (15R^2 / r^3 \gamma) \|u_m\|_{\infty}$ by (4). So:

$$\begin{aligned}
 (8) \quad B &= \left[\frac{\varepsilon^{1/2}}{\sqrt{(2\pi)} K(0, 0)} \int_{\partial\Omega} \frac{K(z, 0)^* u_m ds}{z^*} \right] \\
 &\text{with error } \frac{2\pi\varepsilon^{3/2}}{\sqrt{(2\pi)} K(0, 0)} \frac{15R^2}{r^{3\gamma}} (\sum \|u_m\|_\infty^2)^{1/2} \\
 &\leq 66K(0, 0)^{-1} R^3 r^{-4} \gamma^{-3/2} \varepsilon^{3/2}
 \end{aligned}$$

by (1). The norm of the matrix in the square brackets is at most:

$$\begin{aligned}
 &\frac{\varepsilon^{1/2}}{\sqrt{(2\pi)} K(0, 0)} \left(\int_{\partial\Omega} \frac{|K(z, 0)|^2 ds}{|z|^2} \right)^{1/2} \\
 &\leq \frac{\varepsilon^{1/2}}{\sqrt{(2\pi)} K(0, 0)r} \left(\int_{\partial\Omega} |K(z, 0)|^2 ds \right)^{1/2} \leq \frac{\varepsilon^{1/2}}{\sqrt{(2\pi)} K(0, 0)^{1/2} r}.
 \end{aligned}$$

Hence, using Lemma 2.1 and the fact that $\varepsilon < 10^{-5}(r/R)^4\gamma$, (8) gives $\|B\| \leq 3Rr^{-2}\gamma^{1/2}\varepsilon^{1/2}$. The cruder bounds $\|B\| \leq 1/40$ and $\|B\|^2 \leq 2500(R/r)^5\gamma^{-1}\varepsilon$ will suffice. Also, using (8) and the estimates calculated in the last few lines, we have:

$$\begin{aligned}
 (9) \quad B^H B &= \left[\frac{\varepsilon}{2\pi K(0, 0)^2} \int_{\partial\Omega} \frac{K(z, 0) u_m^* ds}{z} \int_{\partial\Omega} \frac{K(z, 0)^* u_n ds}{z^*} \right] \\
 &\text{with error } 20000R^6 r^{-8} \varepsilon^2.
 \end{aligned}$$

The elements of C are, for $m \geq 1$, $n \geq 2$:

$$\begin{aligned}
 (u_m, v_n) &= \frac{\varepsilon^{n-1/2}}{\sqrt{(2\pi)} K(0, 0)} \int_{\partial\Omega} \frac{K(z, 0)^* u_m ds}{(z^*)^n} \\
 &\quad + \frac{\varepsilon^{-n+1/2}}{\sqrt{(2\pi)} K(0, 0)i} \int_{|z|=\varepsilon} K(z, 0)^* u_m(z) z^{n-1} dz.
 \end{aligned}$$

Call the first and second terms of the above expression P_{mn} and Q_{mn} respectively. Then:

$$\begin{aligned}
 \|P\| &\leq \frac{1}{\sqrt{(2\pi)} K(0, 0)} \left(\sum_{n=2}^{\infty} \int_{\partial\Omega} \frac{|K(z, 0)|^2 ds}{|z|^{2n}} \varepsilon^{2n-1} \right)^{1/2} \\
 &\leq \frac{1}{\sqrt{(2\pi)} K(0, 0)} \left(\sum_{n=2}^{\infty} \frac{\varepsilon^{2n-1}}{r^{2n}} K(0, 0) \right)^{1/2} \\
 &\leq 3Rr^{-3}\gamma^{1/2}\varepsilon^{3/2} \leq (R/r)^5\gamma^{-1}\varepsilon.
 \end{aligned}$$

We estimate the integral in the expression for Q_{mn} as follows. Replace $K(z, 0)$ by $K(z, 0)$ minus its Taylor expansion about 0 as far as the term in z^{n-1} . By Cauchy's theorem, these added terms do not affect the integral. By Taylor's theorem, $K(z, 0)$ minus its Taylor expansion is bounded on $|z| = \varepsilon$ by $(\varepsilon^n/n!) \|d^n/dz^n K(z, 0)\|_\infty$, and (4) gives an estimate for that. This procedure gives:

$$\begin{aligned}
\|Q\| &\leq \frac{14\pi R^2}{r\sqrt{(2\pi)} K(0, 0)\gamma} \left(\sum_{n=2}^{\infty} \frac{\varepsilon^{2n+1}(n+1)^2}{(r-2\varepsilon)^{2n+2}} \right)^{1/2} (\sum \|u_m\|_{\infty}^2)^{1/2} \\
&\leq \frac{14\pi R^2}{r\sqrt{(2\pi)} \gamma} \frac{16\pi R^2\gamma}{r^2} \frac{4\varepsilon^{5/2}}{r^3} \frac{\sqrt{3} R}{\sqrt{\gamma} r} \\
&\leq 6000 R^5 r^{-7} \gamma^{-1/2} \varepsilon^{5/2} \leq (R/r)^5 \gamma^{-1} \varepsilon.
\end{aligned}$$

Hence $\|C\| \leq \|P\| + \|Q\| \leq 2(R/r)^5 \gamma^{-1} \varepsilon$. Once again we shall need only $\|C\| \leq 2500(R/r)^5 \gamma^{-1} \varepsilon \leq 1/40$.

It is convenient to deal with $\begin{bmatrix} D & E^H \\ E & F \end{bmatrix}$ as a single matrix. Its (m, n) th element (see (6)) is, for $m \geq 1, n \geq 1$:

$$\frac{\varepsilon^{m+n-1}}{2\pi K(0, 0)^2} \int_{\partial\Omega} \frac{|K(z, 0)|^2 ds}{(z^*)^m z^n} + \left(\frac{\varepsilon^{m+n-1}}{2\pi K(0, 0)^2} \int_{|z|=\varepsilon} \frac{|K(z, 0)|^2 ds}{(z^*)^m z^n} - \delta_{mn} \right).$$

Denote by G_{mn} , H_{mn} respectively the first term and the bracketed term of the above expression. We have:

$$|G_{mn}| \leq \frac{\varepsilon^{m+n-1}}{2\pi K(0, 0)^2} \frac{1}{r^{m+n}} \int_{\partial\Omega} |K(z, 0)|^2 ds = \frac{\varepsilon^{m+n-1}}{2\pi K(0, 0) r^{m+n}}.$$

Hence $\|G\| \leq (1/(2\pi\varepsilon K(0, 0))) \sum_{n=1}^{\infty} \varepsilon^{2n}/r^{2n} \leq 9R^2 r^{-4} \gamma \varepsilon \leq 9(R/r)^5 \gamma^{-1} \varepsilon$. H is trickier to deal with. We have:

$$\begin{aligned}
H_{nn} &= \frac{1}{2\pi K(0, 0)^2 \varepsilon} \int_{|z|=\varepsilon} |K(z, 0)|^2 ds - 1 \\
&= \frac{1}{2\pi i K(0, 0)^2} \int_{|z|=\varepsilon} K(z, 0)(K(z, 0)^* - K(0, 0)^*) z^{-1} dz.
\end{aligned}$$

Lemma 2.1, (4) with $k=1$, and (5) now give $|H_{nn}| \leq 800R^4 r^{-5} \varepsilon$. If $m > n$, then:

$$H_{mn} = \frac{\varepsilon^{n-m}}{2\pi i K(0, 0)^2} \int_{|z|=\varepsilon} K(z, 0) K(z, 0)^* z^{m-n-1} dz.$$

As before, we may replace the second occurrence of $K(z, 0)$ in the integral by $K(z, 0)$ minus its Taylor expansion, this time as far as the term in z^{m-n-1} . Then by (5), Lemma 2.1, and (4) with $k=m-n$:

$$\begin{aligned}
|H_{mn}| &\leq \varepsilon^{m-n} 1.01 \frac{16\pi R^2 \gamma}{r^2} \frac{7R^2(m-n+1)}{r(r-2\varepsilon)^{m-n+1} \gamma} \\
&\leq 400R^4 r^{-5} \varepsilon (|m-n|+1)(1/99998)^{|m-n|-1}
\end{aligned}$$

since $\varepsilon < 10^{-5}r$. This holds similarly for $m < n$. Combining the cases $m=n$, $m > n$, and $m < n$, we see that:

$$\begin{aligned}
\|H\| &\leq \frac{\varepsilon R^4}{r^5} \left(800 + 2 \times 400 \left(2 + \frac{3}{99998} + \frac{4}{(99998)^2} + \dots \right) \right) \\
&\leq 2401R^4 r^{-5} \varepsilon \leq 2401(R/r)^5 \gamma^{-1} \varepsilon.
\end{aligned}$$

So $\begin{bmatrix} D & E^H \\ E & F \end{bmatrix}$ has norm at most $\|G\| + \|H\| \leq 2500(R/r)^5 \gamma^{-1} \varepsilon$. Hence each of $\|D\|$, $\|E\|$, $\|F\| \leq 2500(R/r)^5 \gamma^{-1} \varepsilon \leq 1/40$.

To summarise: we have shown that:

$$(10) \quad \begin{aligned} &\|A\|, \|B\|^2, \|C\|, \|D\|, \|E\|, \|F\| \leq 2500(R/r)^5 \gamma^{-1} \varepsilon; \\ &\|A\|, \|B\|, \|C\|, \|D\|, \|E\|, \|F\| \leq 1/40. \end{aligned}$$

In particular we have verified that M is a bounded matrix: indeed that $\|M\| \leq 3/40 < 1$. Thus $T = I + M$ is invertible, and Proposition 1.5 applies.

Our next step is to calculate the top left-hand block of the inverse of T . Since $T^{-1} = I - M + M^2 - M^3 + \dots$, this top left-hand block is:

$$\begin{aligned} S = & I \\ & - A \\ & + A^2 + B^H B + C^H C \\ & - A^3 - AB^H B - AC^H C - B^H BA - B^H DB - B^H E^H C - C^H CA \\ & - C^H EB - C^H FC \\ & + \dots \end{aligned}$$

The row of this expression containing products of degree n ($n \geq 4$) consists of 3^{n-1} terms. Each of these terms has norm at most $(2500)^2(R/r)^{10} \gamma^{-2} \varepsilon^2 (1/40)^{n-4}$ by (10). Hence $S = I - A + B^H B$ with error:

$$\begin{aligned} &\frac{(2500)^2 \varepsilon^2 R^{10}}{\gamma^2 r^{10}} \left(1 + 1 + \frac{1}{40} + 1 + \frac{1}{40} + 1 + 1 + \frac{1}{40} + \frac{1}{40} + \frac{1}{40} + \frac{1}{40} \right. \\ &\quad \left. + 27 \left(1 + \frac{3}{40} + \left(\frac{3}{40} \right)^2 + \dots \right) \right) \leq 3 \cdot 10^8 (R/r)^{10} \gamma^{-2} \varepsilon^2. \end{aligned}$$

Using (7) and (9), we have:

$$S = \left[\delta_{mn} - 2\pi \varepsilon u_m(0)^* u_n(0) + \frac{\varepsilon}{2\pi K(0, 0)^2} \int \frac{K(z, 0) u_m^* ds}{z} \int \frac{K(z, 0)^* u_n ds}{z^*} \right]$$

with error $200R^2 r^{-3} \gamma^{-1} \varepsilon^2 + 20000R^6 r^{-8} \varepsilon^2 + 3 \cdot 10^8 (R/r)^{10} \gamma^{-2} \varepsilon^2 \leq 4 \cdot 10^8 (R/r)^{10} \gamma^{-2} \varepsilon^2$. Here and subsequently all integrals are taken round $\partial\Omega$.

Finally we apply Proposition 1.5, which says that $1/(2\pi\gamma_1) = \sum S_{mn} u_m(\infty) u_n(\infty)^*$ (since $v_n(\infty) = 0$ for all n). Hence:

$$\begin{aligned} \frac{1}{2\pi\gamma_1} = & \sum |u_n(\infty)|^2 - 2\pi\varepsilon \left| \sum u_n(0)^* u_n(\infty) \right|^2 \\ & + \frac{\varepsilon}{2\pi K(0, 0)^2} \left| \sum \left(u_n(\infty) \int \frac{K(z, 0) u_n^* ds}{z} \right) \right|^2 \end{aligned}$$

with error $4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2 \sum |u_m(\infty)|^2 = 4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2/(2\pi\gamma)$. Multiplying by $2\pi\gamma$ and using the fact that $\sum u_n(z)u_n(\zeta)^* = K(z, \zeta)$, we have:

$$(11) \quad \frac{\gamma}{\gamma_1} = 1 - 4\pi^2\gamma\varepsilon |K(0, \infty)|^2 + \frac{\gamma\varepsilon}{K(0, 0)^2} \left| \int \frac{K(z, 0)K(z, \infty)^* ds}{z} \right|^2$$

with error $4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2$.

Now the last term simplifies. On $\partial\Omega$, $f(z)\psi(z)dz \geq 0$, so that $ds = (\psi(z)/|\psi(z)|)f(z)dz = (K(z, \infty)/K(z, \infty)^*)(f(z)/i)dz$. Therefore:

$$\begin{aligned} \int \frac{K(z, 0)K(z, \infty)^* ds}{z} &= \frac{1}{i} \int \frac{K(z, 0)K(z, \infty)f(z)dz}{z} \\ &= -2\pi K(0, 0)K(0, \infty)f(0) \end{aligned}$$

since $K(z, 0)K(z, \infty)f(z)$ is analytic on $\bar{\Omega}$ and vanishes at ∞ . Substituting in (11), we have:

$$\frac{\gamma}{\gamma_1} = 1 - 4\pi^2\gamma\varepsilon |K(0, \infty)|^2 \{1 - |f(0)|^2\} \text{ with error } 4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2.$$

Now $4\pi^2\gamma\varepsilon |K(0, \infty)|^2 \{1 - |f(0)|^2\} \leq 4\pi^2\gamma\varepsilon K(0, 0)K(\infty, \infty) \leq 8(R/r)^2\gamma^{-1}\varepsilon < 10^{-4}$. Also $4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2 \leq 1/25$. So we can invert to obtain:

$$\begin{aligned} \frac{\gamma_1}{\gamma} &= 1 + 4\pi^2\gamma\varepsilon |K(0, \infty)|^2 \{1 - |f(0)|^2\} \text{ with error } 10^9(R/r)^{10}\gamma^{-2}\varepsilon^2; \\ \gamma_1 &= \gamma + 4\pi^2\gamma^2\varepsilon |K(0, \infty)|^2 \{1 - |f(0)|^2\} \\ &= \gamma + 2\pi\varepsilon |\psi(0)| \{1 - |f(0)|^2\} \text{ with error } 10^9(R/r)^{10}\gamma^{-1}\varepsilon^2. \end{aligned}$$

It is as well to explain the curious choice of the functions v_n in the above proof. The only essential property of v_n we used is that it vanishes at ∞ and is analytic on $\bar{\Omega}$ except for a pole at 0 near which $v_n(z) = (2\pi)^{-1/2}\varepsilon^{n-1/2}z^{-n} + \dots$. The simpler choice $v_n(z) = (2\pi)^{-1/2}\varepsilon^{n-1/2}z^{-n}$ shortens the proof but yields an error bound dependent on the length of $\partial\Omega$, which would have been unsuitable for the next section.

3. Extension to arbitrary compact sets. We shall now show how the above results extend to arbitrary compact sets E . In particular, we show how to define the Garabedian function of E , thus solving a problem considered in [2] and [3].

Let E be compact. We shall suppose meantime that $\gamma(E) > 0$. E can be expressed as the intersection of a decreasing sequence $\{E_n\}$ in \mathcal{S} . Hence ψ_{E_n} and a_{E_n} are defined. Fix $\zeta \in \Omega(E)$, and choose n_0 so that $\zeta \in \Omega(E_n)$ whenever $n > n_0$. By Theorem 2.2 there exist $\varepsilon_0 > 0$, $k > 0$, such that $\forall n > n_0$, $\forall \varepsilon < \varepsilon_0$:

$$(12) \quad |\gamma(E_n \cup D(\zeta; \varepsilon)) - \gamma(E_n) - \varepsilon a_{E_n}(\zeta)| \leq k\varepsilon^2.$$

That is, for all $\varepsilon < \varepsilon_0$, the sequence $\{\varepsilon a_{E_n}(\zeta)\}_{n > n_0}$, considered as an element of the Banach space of bounded sequences with the supremum norm, is within a distance $k\varepsilon^2$ of the sequence $\{\gamma(E_n \cup D(\zeta; \varepsilon)) - \gamma(E_n)\}_{n > n_0}$, which converges to $\gamma(E \cup D(\zeta; \varepsilon)) - \gamma(E)$ by 1.1. Thus $\{a_{E_n}(\zeta)\}$ is within a distance $k\varepsilon$ of the closed subspace c of convergent sequences, for all ε , and is therefore itself in c . Call its limit $a_E(\zeta)$. a_E is the *slope function* of E . Letting $n \rightarrow \infty$ in (12) now gives, for all $\varepsilon < \varepsilon_0$:

$$|\gamma(E \cup D(\zeta; \varepsilon)) - \gamma(E) - \varepsilon a_E(\zeta)| \leq k\varepsilon^2.$$

This shows also that the limit $a_E(\zeta)$ is independent of the choice of the sequence $\{E_n\}$.

Now, for each n , $|\psi_{E_n}(\zeta)| = a_{E_n}(\zeta)/(2\pi|1 - |f_{E_n}(\zeta)|^2|)$, and this converges pointwise in $\Omega(E)$. Moreover, $\{\psi_{E_n}\}$ is a normal sequence, since, if F is a compact subset of $\Omega(E)$, $\{\psi_{E_n}\}$ is uniformly bounded on F by the remark following Lemma 2.1. It follows that for some sequence λ_n of points on the unit circle, $\{\lambda_n \psi_{E_n}\}$ converges uniformly on compact subsets of $\Omega(E)$. In fact we may take $\lambda_n = 1$, since $\psi_{E_n}(\infty) = 1/(2\pi i)$. So $\{\psi_{E_n}\}$ converges uniformly on compact sets. Call its limit, ψ_E , the *Garabedian function* of E . Hence also $a_{E_n}(\zeta) = 2\pi|\psi_{E_n}(\zeta)|\{1 - |f_{E_n}(\zeta)|^2\}$ converges uniformly on compact sets (and not merely pointwise, as ascertained already).

Now suppose that $\gamma(E) = 0$. We define $\psi_E(\zeta) = 1/(2\pi i)$, $a_E(\zeta) = 1$ for $\zeta \in \Omega(E)$. This is consistent with the relation $a_E(\zeta) = 2\pi|\psi_E(\zeta)|\{1 - |f_E(\zeta)|^2\}$ since $f_E(\zeta) = 0$. $\gamma(E \cup D(\zeta; \varepsilon)) = \varepsilon$ for all $\varepsilon > 0$ by 1.3, and so the relation $\gamma(E \cup D(\zeta; \varepsilon)) = \gamma(E) + \varepsilon a_E(\zeta) + O(\varepsilon^2)$ holds trivially. If $\{E_n\}$ is a sequence in \mathcal{S} decreasing to E , then $\psi_{E_n}(\zeta) \rightarrow 1/(2\pi i) = \psi_E(\zeta)$ uniformly on compact sets by the remark following Lemma 2.1.

Finally, if E is compact, and $\{E_n\}$ is *any* sequence of compact sets decreasing to E , the same working as above shows that $\psi_{E_n} \rightarrow \psi_E$ and $a_{E_n} \rightarrow a_E$ uniformly on compact sets.

We have therefore proved:

THEOREM 3.1. *The Garabedian function $\psi_E(\zeta)$ and the slope function $a_E(\zeta)$ can be defined for all compact sets E , in such a way that:*

(1) *The definitions coincide with the existing meanings if $E \in \mathcal{S}$;*

(2) *If $\{E_n\}$ is a sequence of compact sets decreasing to E , then $\psi_{E_n} \rightarrow \psi_E$ and $a_{E_n} \rightarrow a_E$ uniformly on compact subsets of $\Omega(E)$;*

(3) *$\gamma(E \cup D(\zeta; \varepsilon)) = \gamma(E) + \varepsilon a_E(\zeta) + O(\varepsilon^2)$ for all $\zeta \in \Omega(E)$, and the bound in the error term depends only on $\gamma(E)$ and on the ratio of the greatest and least distances of points of E from ζ ; and*

(4) *$a_E(\zeta) = 2\pi|\psi_E(\zeta)|\{1 - |f_E(\zeta)|^2\}$ for all $\zeta \in \Omega(E)$.*

The slope function is related to the problem of subadditivity of γ . If E is connected, then $\alpha_E(\zeta) \leq 1$: this is a re-statement of Bieberbach's distortion theorem. Subadditivity of γ would obviously imply $\alpha_E(\zeta) \leq 1$ for all compact E .

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Added in proof. N. Suita recently has independently proved the uniqueness of the Garabedian function much more simply ("On a metric induced by Analytic Capacity," Kōdai Math. Sem. Rep. 25 (1973), 215-218).

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Zvi Arad, <i>π-homogeneity and π'-closure of finite groups</i>	1
Ivan Baggs, <i>A connected Hausdorff space which is not contained in a maximal connected space</i>	11
Eric Bedford, <i>The Dirichlet problem for some overdetermined systems on the unit ball in C^n</i>	19
R. H. Bing, Woodrow Wilson Bledsoe and R. Daniel Mauldin, <i>Sets generated by rectangles</i>	27
Carlo Cecchini and Alessandro Figà-Talamanca, <i>Projections of uniqueness for $L^p(G)$</i>	37
Gokulananda Das and Ram N. Mohapatra, <i>The non absolute Nörlund summability of Fourier series</i>	49
Frank Rimi DeMeyer, <i>On separable polynomials over a commutative ring</i>	57
Richard Detmer, <i>Sets which are tame in arcs in E^3</i>	67
William Erb Dietrich, <i>Ideals in convolution algebras on Abelian groups</i>	75
Bryce L. Elkins, <i>A Galois theory for linear topological rings</i>	89
William Alan Feldman, <i>A characterization of the topology of compact convergence on $C(X)$</i>	109
Hillel Halkin Gershenson, <i>A problem in compact Lie groups and framed cobordism</i>	121
Samuel R. Gordon, <i>Associators in simple algebras</i>	131
Marvin J. Greenberg, <i>Strictly local solutions of Diophantine equations</i>	143
Jon Craig Helton, <i>Product integrals and inverses in normed rings</i>	155
Domingo Antonio Herrero, <i>Inner functions under uniform topology</i>	167
Jerry Alan Johnson, <i>Lipschitz spaces</i>	177
Marvin Stanford Keener, <i>Oscillatory solutions and multi-point boundary value functions for certain nth-order linear ordinary differential equations</i>	187
John Cronan Kieffer, <i>A simple proof of the Moy-Perez generalization of the Shannon-McMillan theorem</i>	203
Joong Ho Kim, <i>Power invariant rings</i>	207
Gangaram S. Ladde and V. Lakshmikantham, <i>On flow-invariant sets</i>	215
Roger T. Lewis, <i>Oscillation and nonoscillation criteria for some self-adjoint even order linear differential operators</i>	221
Jürg Thomas Marti, <i>On the existence of support points of solid convex sets</i>	235
John Rowlay Martin, <i>Determining knot types from diagrams of knots</i>	241
James Jerome Metzger, <i>Local ideals in a topological algebra of entire functions characterized by a non-radial rate of growth</i>	251
K. C. O'Meara, <i>Intrinsic extensions of prime rings</i>	257
Stanley Poreda, <i>A note on the continuity of best polynomial approximations</i>	271
Robert John Sacker, <i>Asymptotic approach to periodic orbits and local prolongations of maps</i>	273
Eric Peter Smith, <i>The Garabedian function of an arbitrary compact set</i>	289
Arne Stray, <i>Pointwise bounded approximation by functions satisfying a side condition</i>	301
John St. Clair Werth, Jr., <i>Maximal pure subgroups of torsion complete abelian p-groups</i>	307
Robert S. Wilson, <i>On the structure of finite rings. II</i>	317
Kari Ylinen, <i>The multiplier algebra of a convolution measure algebra</i>	327