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LAPLACE TRANSFORM METHODS IN MULTIVARIATE SPECTRAL THEORY

ROBERT F. V. ANDERSON

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The Laplace transform of the semigroup $\exp(tA)$ generated by an operator A gives the resolvent of A . An integral formula is obtained for the Laplace transform of $\exp(tA + B)$, where B is another operator which does not commute with A . The new transform has analytic continuation to the same domain as the resolvent, but the analytic continuation is not single-valued. The integral formula is then applied to the joint spectral theory of noncommutative operators. Explicit computations with matrices of degree two illustrate the results.

1. Introduction. Any bounded linear operator A on a Banach space generates a semigroup $\exp(tA)$, $0 \leq t < \infty$, and the Laplace transform $\mathcal{L}(s, A)$ of this semigroup converges for $\operatorname{Re} s$ sufficiently large and equals the resolvent $(s - A)^{-1}$. $\mathcal{L}(s, A)$ therefore has unique analytic continuation to the component containing ∞ of the resolvent set A .

Multivariate problems requiring integration of $\exp(\sum t_i A_i)$ one variable at a time, lead us to consider the Laplace transform $\mathcal{L}(s, A, B)$ of $\exp(tA + B)$, $0 \leq t < \infty$, where B is a fixed bounded operator.

The main result is:

THEOREM 1. $\mathcal{L}(s, A, B)$ has the contour integral representation (1.1)

$$(s - A)^{-1} + \int_s^\infty (u - A)^{-1} B (u - A)^{-1} \exp[B(u - A)^{-1}(u - s)] du$$

valid for $\operatorname{Re} s$ sufficiently large. Therefore, $\mathcal{L}(s, A, B)$ can be analytically continued along any arc not intersecting $\sigma(A)$.

Examples are given in §5 which show that the analytic continuation is not always unique.

In §§3 and 4 our result is applied to problems in spectral theory. According to the Weyl functional calculus [1], [2], [3], two self-adjoint operators have a joint spectral distribution in the plane, which if A and B commute is simply the tensor product of their spectral measures. Two operators A and B which are merely bounded have instead a two-dimensional Laplace transform $\mathcal{L}(s, \sigma, A, B)$ which if A and B commute is simply the product of their resolvents. $\mathcal{L}(s, \sigma, A, B)$ may be regarded as a functional on the space of entire

functions on C^2 . Its carrier may be regarded as a joint spectrum of A, B . Although there is no unique minimal carrier in general for functionals of this type, Theorem 1 can be exploited to obtain information about the carriers in terms of the spectral properties of A and B . It turns out that the actual spectrum of A can be used to construct a carrier, if accuracy with respect to B is sacrificed.

Suppose now that A is bounded and B is self-adjoint. The Weyl calculus for the three self-adjoint operators $\operatorname{Re} A$, $\operatorname{Im} A$, and B gives a spectrum projecting onto the whole numerical range of A . However, in §4 we construct a hybrid functional for A and B which is an analytic functional with respect to A . The motivating question is whether the carrier of this functional will still be the whole numerical range of A or whether the actual spectrum of A will reappear. Theorem 6 gives the transition between the two competing theories and offers no help in shrinking the carrier. But the examples of §5 show that in some cases the actual spectrum of A does suffice as the carrier.

2. Proof of Theorem 1.

Proof of Theorem 1. The Laplace transform of $\exp(tA + B)$ cannot be computed directly unless B commutes with A , in which case the trivial result is:

$$\mathcal{L}(s, A, B) = \exp(B)\mathcal{L}(s, A).$$

We therefore resort to the following contour integral method.

Let C be a simple closed curve containing $\sigma(A)$ (spectrum of A) in its interior. Then C also encloses $\sigma(A + t^{-1}B)$ for $|t|$ greater than some constant k , and by the Riesz functional calculus [6] applied to the operator $A + t^{-1}B$, for $|t| > k$,

$$\exp(tA + B) = \exp(t(A + t^{-1}B)) = \frac{1}{2\pi i} \oint_C e^{tz} [z - (A + t^{-1}B)]^{-1} dz.$$

If in addition

$$|t| > k_1 = \sup_{z \notin \operatorname{Int} C} \|B(z - A)^{-1}\|$$

then

$$\begin{aligned} [z - A - t^{-1}B]^{-1} &= [(I - t^{-1}B(z - A)^{-1})(z - A)]^{-1} \\ &= (z - A)^{-1} \sum_{n=0}^{\infty} [B(z - A)^{-1}]^n t^{-n} \end{aligned}$$

and

$$\begin{aligned}\exp(tA + B) &= \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} e^{zt} t^{-n} (z - A)^{-1} [B(z - A)^{-1}]^n dz \\ &= \frac{1}{2\pi i} \oint_C \sum_{n, j=0}^{\infty} \frac{z^j t^{j-n}}{j!} (z - A)^{-1} [B(z - A)^{-1}]^n dz.\end{aligned}$$

The double series in the integrand is absolutely and uniformly convergent on the domain $z \in C$, $\max(k, k_1) < |t| < k_2$ where k_2 is any constant. The contour integral can therefore be evaluated term-by-term. All terms having $j < n$ are $O(|z|^{-2})$ for large z , and so by enlargement of the contour, they vanish. The remaining terms can therefore be rewritten as the sum

$$\sum_{q=0}^{\infty} t^q \frac{1}{2\pi i} \int_C (z - A)^{-1} \sum_{p=0}^{\infty} \frac{z^{p+q}}{(p+q)!} [B(z - A)^{-1}]^p dz.$$

This power series expansion of the entire function $\exp(tA + B)$ is valid in the annular region given above, and consequently holds for all t .

Since $\exp(tA + B)$ has exponential growth rate, the Laplace transform of its power series expansion may be taken term-by-term. This fact is discussed fully in Widder's book on the Laplace transform [8]. Therefore

$$\mathcal{L}(s, A, B) = \sum_{q=0}^{\infty} \frac{q!}{s^{q+1}} \frac{1}{2\pi i} \oint_C (z - A)^{-1} \sum_{p=0}^{\infty} \frac{z^{p+q}}{(p+q)!} [B(z - A)^{-1}]^p dz.$$

Next we note that

$$\sum_{q, p=0}^{\infty} \frac{q!}{s^{q+1}} \frac{z^{p+q}}{(p+q)!} [B(z - A)^{-1}]^p$$

is absolutely and uniformly convergent when $z \in C$, $|z/s| < l < 1$, l being any constant less than one.

To reduce the double series to closed form, consider

$$F(a, b) = \sum_{q, p=0}^{\infty} a^q b^p \frac{q!}{(p+q)!}.$$

If the series $\sum_{q=0}^{\infty} a^{q+p} q!/(p+q)!$ is differentiated p times, we obtain a geometric series which converges to $(1 - a)^{-1}$. By elementary means, therefore,

$$\sum_{q=0}^{\infty} a^q \frac{q!}{(p+q)!} = \frac{1}{a^p} \int_0^a \frac{(a-t)^{p-1}}{(p-1)!} \frac{dt}{1-t}, \quad p \geq 1$$

and

$$\begin{aligned}
F(a, b) &= \frac{1}{1-a} + \sum_{p=1}^{\infty} \left(\frac{b}{a}\right)^p \int_0^a \frac{(a-t)^{p-1}}{(p-1)!} \frac{dt}{1-t} \\
&= \frac{1}{1-a} + b \sum_{p=1}^{\infty} b^{p-1} \int_0^1 \frac{(1-u)^{p-1}}{(p-1)!} \frac{du}{1-au} \\
&= \frac{1}{1-a} + b \int_0^1 e^{b(1-u)} \frac{du}{1-au}.
\end{aligned}$$

We now substitute $a = z/s$ and $b = zB(z - A)^{-1}$.

$$\begin{aligned}
\mathcal{L}(s, A, B) &= \frac{1}{2\pi is} \oint_c (z - A)^{-1} \left\{ \left(1 - \frac{z}{s}\right)^{-1} \right. \\
&\quad \left. + zB(z - A)^{-1} \int_0^1 \exp [zB(z - A)^{-1}(1 - u)] \left(1 - \frac{z}{s}u\right)^{-1} du \right\} dz \\
&= (s - A)^{-1} + \int_0^1 \frac{1}{2\pi i} \oint_c z(z - A)^{-1} B(z - A)^{-1} \\
&\quad \cdot \exp [zB(z - A)^{-1}(1 - u)] \left(\frac{s}{u} - z\right)^{-1} dz \frac{du}{u}.
\end{aligned}$$

Since the integrand of the contour integral is holomorphic in the neighborhood of $z = \infty$ outside C , the Cauchy integral formula yields

$$\begin{aligned}
\mathcal{L}(s, A, B) &= (s - A)^{-1} \\
&\quad + \int_0^1 \frac{s}{u} \left(\frac{s}{u} - A\right)^{-1} B \left(\frac{s}{u} - A\right)^{-1} \exp \left[\frac{s}{u} B \left(\frac{s}{u} - A\right)^{-1} (1 - u)\right] \frac{du}{u}.
\end{aligned}$$

Replacing u by s/u , we get Theorem 1:

$$\begin{aligned}
\mathcal{L}(s, A, B) &= (s - A)^{-1} \\
&\quad + \int_s^\infty (u - A)^{-1} B(u - A)^{-1} \exp [B(u - A)^{-1}(u - s)] du.
\end{aligned}$$

COROLLARY 2. $\mathcal{L}(s, A, B)$ has unique analytic continuation to R_∞ , the component of the resolvent set of A containing ∞ , iff for every component σ_i of $\sigma(A)$ meeting \bar{R}_∞ , any contour C_i enclosing component σ_i only, and for all $j \geq 1$,

$$(2.1) \quad \oint_{C_i} (u - A)^{-1} [B(u - A)^{-1}]^j \exp [B(u - A)^{-1}u] du = 0.$$

Proof. Suppose the contour integral of Theorem 1 is continued along two different arcs terminating at s . The difference between the values of s so obtained is, by homotopy arguments, an integral combination of the closed contour integrals

$$\oint_{C_i} (u - A)^{-1} B(u - A)^{-1} \exp [B(u - A)^{-1}(u - s)] du$$

or

$$\frac{d}{ds} \oint_{c_i} (u - A)^{-1} \exp [B(u - A)^{-1}(u - s)] du.$$

The result follows by power series expansion in s .

COROLLARY 3. *The Laplace transform $\mathcal{L}(s, \sigma, A, B)$ of $\exp(tA + \xi B)$ is given by*

$$(2.2) \quad \sigma^{-1}(s - A)^{-1} + \int_s^\infty (u - A)^{-1} B(u - A)^{-1} [\sigma - B(u - A)^{-1}(u - s)]^{-2} du$$

for $\sigma > 0$, $s > 0$ sufficiently large.

Proof. Replace B by ξB in the formula for $\mathcal{L}(s, A, B)$. The integration with respect to ξ is elementary.

3. Analytic functionals in spectral theory. Suppose the bounded operators A_1, \dots, A_n are all self-adjoint, so that for $\xi \in R^n$, $\exp(i\xi \cdot A)$ is a unitary operator. Then by Fourier inversion a tempered distribution $\mathcal{F}^{-1} \exp(i\xi \cdot A)$ is determined. In previous papers by the present author, [1], [2], [3], this distribution was called the "joint spectral distribution" of A_1, \dots, A_n and denoted $T(A)$.

In order to gain further insight into this type of spectral distribution, we consider the slightly different case when iA_1, \dots, iA_n are assumed only to be the generators of contraction semigroups. This is equivalent to the condition that A_1, \dots, A_n have numerical range in the upper half plane. In this case, so does $\xi \cdot A$ if $\xi_1, \dots, \xi_n \geq 0$ (abbreviation $\xi \geq 0$), so $\|\exp(i\xi \cdot A)\| \leq 1$ when $\xi \geq 0$.

DEFINITION. When iA_1, \dots, iA_n generate contraction semigroups, $S(A)$ denotes the tempered distribution defined for $f \in \mathcal{S}(R^n)$ by

$$(3.1) \quad S(A)f = (2\pi)^{-n/2} \int_{\xi \geq 0} (\mathcal{F}f)(\xi) \exp(i\xi \cdot A) d\xi.$$

In one dimension, simple computation shows that

$$(3.2) \quad S(A)f = \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(x) \mathcal{L}(x, A) dx$$

where the Laplace transform $\mathcal{L}(x, A) = (x - A)^{-1}$, provided the spectrum of A does not intersect the real line.

However, $f \in L^2(R^1)$ may be written as $f = f_+ + f_-$, where $\mathcal{F}f_+ = \mathcal{F}f$ for $x \geq 0$, $\mathcal{F}f_- = \mathcal{F}f$ for $x \leq 0$. f_+ is the boundary value of a function $f_+(z)$ holomorphic in the upper half plane, and $|f_+(z)| =$

$0((\operatorname{Im} z)^{-1})$. If C is any contour in the upper half plane enclosing spectrum A , we obtain

$$(3.3) \quad S(A)f = \frac{1}{2\pi i} \oint_C f_+(z) \mathcal{L}(z, A) dz.$$

This is just the Riesz calculus (see Ch. XI of [6]), but in two dimensions we can similarly obtain the formula

$$(3.4) \quad S(A, B)f = \left(\frac{1}{2\pi i} \right)^2 \oint_{C_1} \oint_{C_2} f_+(s, \sigma) \mathcal{L}(s, \sigma, A, B) ds d\sigma$$

where $\mathcal{L}(s, \sigma)$ is holomorphic for s, σ outside C_1, C_2 respectively, and $\mathcal{F}f_+ = \mathcal{F}f$ for $\xi \geq 0$, $\mathcal{F}f_+ = 0$ otherwise.

Formula (3.4) defines a continuous linear functional on the space of entire functions in two complex variables, and the numerical range of A, B need not be restricted. Such functionals are discussed, for example, in Hormanders' book [5]. Such functionals are in one-to-one correspondence with entire functions of exponential growth, in our case $\exp(i\xi \cdot A)$. In one dimension, there is a canonical representation of a functional similar to (3.2), but not in higher dimensions. If K_i denotes the compact set bounded by C_i , and $K = K_1 \times K_2$, then $\|S(A, B)f\| \leq c \sup_{s, \sigma \in K} |f(s, \sigma)|$, so K is an example of a "carrier" of $S(A, B)$. In general, there is no unique minimal carrier of a functional in dimension greater than 1.

LEMMA 4. *If K_1, K_2 contain neighborhoods of the numerical ranges of A, B resp., then $K = K_1 \times K_2$ is a carrier of $S(A, B)$. If K_1 is simply connected and contains the spectrum of A in its interior, then there exists K_2 such that $K = K_1 \times K_2$ is a carrier of $S(A, B)$.*

Proof. $\mathcal{L}(s, \sigma, A, B)$ is holomorphic when $s, \sigma > 0$ if A and B have numerical range in the left half-plane. By translating and rotating A and B independently, the general result is obtained. The second result follows by inspection of formula (2.2) in Corollary 3.

4. A hybrid functional.

DEFINITION. Let iA generate a contraction semigroup and let B be self-adjoint. Then the tempered distribution $\operatorname{ST}(A, B)$ is defined for $f \in \mathcal{S}(R^2)$ by

$$(4.1) \quad \operatorname{ST}(A, B)f = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_0^{\infty} (\mathcal{F}f)(t, \xi) \exp(i(tA + \xi B)) dt d\xi.$$

NOTATION. Given $f \in \mathcal{S}(R^2)$, let $f_+(z, x)$ denote the analytic con-

tinuation to $\operatorname{Im} z \geq 0$ of the function $f_+ \in L^2(R^2)$ satisfying

$$(\mathcal{F}f_+)(t, \xi) = (\mathcal{F}f)(t, \xi), \quad t \geq 0 \\ 0 \quad t < 0.$$

Note that for each fixed $z_0, f_1(z_0, x) \in \mathcal{S}(R^1)$.

LEMMA 5. *For each z outside the closure of the numerical range of A , there is a tempered distribution $\Phi(z) \in \mathcal{S}'(R^1)$ acting on $\varphi(x) \in \mathcal{S}(R^1)$, such that $\Phi(z)$ is (weakly) holomorphic in z , and such that for a contour C enclosing the numerical range of A ,*

$$(4.2) \quad \operatorname{ST}(A, B)f = \frac{1}{2\pi i} \int_C \Phi(z) f_+ dz.$$

Proof. It is easily checked that

$$\|\exp(tA + i\xi B)\| \leq \exp(|t| \|A\|).$$

Therefore, $\exp(tA + i\xi B) = \sum_{j=0}^{\infty} t^j G_j(\xi)$ where for all j , $G_j(\xi) \in C^\infty(R^1)$ and

$$\|G_j(\xi)\| \leq \left(\frac{e\|A\|}{j}\right)^j \text{ uniformly in } \xi.$$

Therefore, for $\varphi(x) \in \mathcal{S}(R^1)$,

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} (\mathcal{F}\varphi)(\xi) \exp(tA + i\xi B) d\xi = \sum_{j=0}^{\infty} t^j \Phi_j(\varphi)$$

where $\Phi_j(\varphi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} (\mathcal{F}\varphi)(\xi) G_j(\xi) d\xi$ satisfies

$$\|\Phi_j(\varphi)\| \leq (2\pi)^{-1/2} \left(\frac{e\|A\|}{j}\right)^j \|(\mathcal{F}\varphi)\|.$$

In particular, Φ_j is a tempered distribution on $\mathcal{S}(R^1)$. Define

$$\Phi(z) = \mathcal{L}\left(\sum_{j=0}^{\infty} t^j \Phi_j\right) = \sum_{j=0}^{\infty} \frac{j!}{z^{j+1}} \Phi_j$$

which converges when $|z| > \|A\|$.

By trivial arguments, $\Phi(z)$ has analytic continuation to all z not in the closure of the numerical range of A . The lemma follows immediately for f of the form $\psi(z)\varphi(x)$, which suffices.

THEOREM 6. *Suppose A, B act on Hilbert space, and let $A = \operatorname{Re} A + i \operatorname{Im} A$, where $\operatorname{Re} A, \operatorname{Im} A$ are self-adjoint. Then for $\varphi \in \mathcal{S}(R^1)$ and $|z|$ large,*

$$(4.3) \quad \Phi(z)\varphi = T(\operatorname{Re} A, \operatorname{Im} A, B) \frac{\varphi(x_3)}{z - (x_i + ix_2)}$$

where T is defined for $g(x_1, x_2, x_3) \in \mathcal{S}(R^3)$ as stated at the beginning of §3 and in [1].

Note. The support of the distribution T contains only (x_1, x_2, x_3) such that $x_1 + ix_2$ is in the closure of the numerical range of A . See [1]. Therefore, (4.3) extends at least to all z outside the closed numerical range of A .

Proof. Both sides expand in Laurent series in z , with coefficient of z^{-j-1} on the left

$$= (j!)\Phi_j(\varphi) = (j!)(2\pi)^{-1/2} \int_{-\infty}^{\infty} (\mathcal{S}\varphi)(\xi) G_j(\xi) d\xi$$

and on the right

$$\begin{aligned} & T(\operatorname{Re} A, \operatorname{Im} A, B)[\varphi(x_3)(x_i + ix_2)^j] \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} (\mathcal{S}\varphi)(\xi) T(\operatorname{Re} A, \operatorname{Im} A, B)[e^{ix_3\xi}(x_i + ix_2)^j] d\xi. \end{aligned}$$

Now $G_j(\xi)$ is the coefficient of t^j in $\exp(tA + i\xi B)$ or $\sum_{n=0}^{\infty} 1/n! (tA + i\xi B)^n$ or, by the monomial substitution rule for T in [1],

$$\sum_{n=0}^{\infty} \frac{1}{n!} T(\operatorname{Re} A, \operatorname{Im} A, B)[e^{t(x_i + ix_2)} e^{i\xi x_3}].$$

That is,

$$G_j(\xi) = \frac{1}{j!} T(\operatorname{Re} A, \operatorname{Im} A, B)[e^{ix_3\xi}(x_i + ix_2)^j].$$

Therefore, the two Laurent expansions coincide.

5. Examples. We first examine the hybrid functional in the case when A and B act on the two-dimensional complex Hilbert space.

Let M_1, M_2, M_3 be the three hermitian matrices with eigenvalues ± 1 , satisfying $M_i M_j + M_j M_i = 0$, $i \neq j$. (E.g. the Pauli matrices.) Then every 2×2 complex matrix is a unique linear combination of M_1, M_2, M_3, I . Since I commutes with everything, we may as well assume that A, B are linear combinations of M_1, M_2, M_3 , and up to unitary equivalence and scale changes we can assume $B = M_1$, $A = i\omega \cdot M$ where ω is a triple of complex numbers.

By simple calculations,

$$\begin{aligned} \exp(A + i\xi B) &= I \cos \sqrt{(\xi + \omega_1)^2 + r^2} \\ &+ (\xi M_1 + \omega \cdot M) \frac{i \sin \sqrt{(\xi + \omega_1)^2 + r^2}}{\sqrt{(\xi + \omega_1)^2 + r^2}} \end{aligned}$$

where $r^2 = \omega_2^2 + \omega_3^2$. Let $\chi(x)$ denote the characteristic function of the interval $-1 \leq x \leq 1$, and let δ_1, δ_{-1} denote the unit measures concentrated at the points 1, -1 respectively.

Essentially, the Fourier transforms we need are

$$\mathcal{F}^{-1} \frac{\sin \sqrt{\xi^2 + r^2}}{\sqrt{\xi^2 + r^2}} = \sqrt{\frac{\pi}{2}} J_0(r\sqrt{1-x^2}) \chi(x).$$

$$\mathcal{F}^{-1}(\cos \sqrt{\xi^2 + r^2}) = \sqrt{\frac{\pi}{2}} \left[\delta_1 + \delta_{-1} + \frac{r J_1(r\sqrt{1-x^2})}{\sqrt{1-x^2}} \chi(x) \right].$$

Since these formulae are rather hard exercises in contour integration, it is worth mentioning that the first is a corollary of the standard formula (see [4]) in dimension 3:

$$\mathcal{F}^{-1} \frac{\sin |\xi|}{|\xi|} = (2\pi)^{3/2} \mu,$$

where μ is the uniformly distributed measure on the unit sphere. Our formula follows from this equation by taking the partial Fourier transform with respect to two variables.

In order to obtain $\Phi(z)$, we replace A by tA (i.e., replace ω by $t\omega$) and compute the Laplace transform. For the Laplace transforms of the Bessel functions see Watson [7], p. 386. The result is:

THEOREM 7. *If $B = M_1$, $A = i\omega \cdot M$, then*

$$\begin{aligned} \Phi(z) = & I \sqrt{\frac{\pi}{2}} \left[\frac{\delta_1}{z - i\omega_1} + \frac{\delta_{-1}}{z + i\omega_1} + \frac{\chi(x)}{((z - i\omega_1 x)^2 + r^2(1 - x^2))^{3/2}} \right] \\ & + \left(-M_1 \frac{\partial}{\partial x} + i\omega \cdot M \frac{\partial}{\partial z} \right) \sqrt{\frac{\pi}{2}} \frac{\chi(x)}{((z - i\omega_1 x)^2 + r^2(1 - x^2))^{1/2}}. \end{aligned}$$

In particular, $\Phi(z)$ can be analytically continued to the complement of the set $\{z | z = i\omega_1 x \pm ir\sqrt{1-x^2}, -1 \leq x \leq 1\}$ which is the ellipsoid parameterized by $0 \leq \theta \leq 2\pi$,

$$z = i\omega_1 \cos \theta + ir \sin \theta.$$

In the case when ω_2/ω_3 is real, this ellipsoid is the boundary of the numerical range of iA , and Lemma 5 gives the actual domain of $\Phi(z)$. The other extreme is the case when $\omega_1 = 0$ and ω_2/ω_3 is imaginary. Then $\omega_1 = r = 0$ and $\Phi(z)$ is singular only at $z = 0$, although the numerical range of iA is the nontrivial ellipsoid $\{z | z = i\omega_2 \cos \theta + i\omega_3 \sin \theta, 0 \leq \theta \leq 2\pi\}$. In the latter case, the singularities of $\Phi(z)$ coincide with the spectrum of iA (i.e., $z = 0$), but the former case shows that the analytic continuation established for $\mathcal{L}(s, A, B)$ does not carry over to the hybrid functional $\text{ST}(A, B)$.

To obtain examples of the Laplace transform, we utilize the elementary fact that when $g \in \mathcal{S}(R^1)$, the Laplace transform is obtained from the Fourier transform by the formula

$$(\mathcal{L}g)(is) = \sqrt{2\pi} \int_{-\infty}^{\infty} \frac{(\mathcal{F}^{-1}g)(x)}{s - x} dx.$$

One of the coefficients in $\mathcal{L}(s, iB, A)$, with A, B the 2×2 complex matrices described above, is therefore

$$\pi \int_{-1}^1 \frac{J_0(r\sqrt{1-x^2})}{s-x} dx.$$

This coefficient, like the others, has nonunique analytic continuation to all $s \neq \pm 1$. The difference between two values is an integer times $\pi J_0(r\sqrt{1-s^2})$.

This result is easily extended by analytic continuation to any 2×2 matrices A, B .

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