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TTF CLASSES AND QUASI-GENERATORS

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Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for ${}_A\mathcal{M}$, the category of left A -modules. In this paper the property that the torsionfree class \mathcal{F} be closed under homomorphic images is investigated. Particular attention is given to the case where the torsion class \mathcal{T} is torsion-torsionfree (TTF). Applications to projective quasi-generators are given.

When \mathcal{F} is a TTF class the question naturally arises as to when A_t , the \mathcal{T} -torsion submodule of A , is contained in a certain idempotent topologizing filter of right ideals of A . This condition is shown to be equivalent to the property that the torsionfree class \mathcal{F} be closed under homomorphic images. Our results generalize results of Jans [6] and Bernhardt [2] characterizing the property that the torsion theory $(\mathcal{T}, \mathcal{F})$ is centrally splitting. Dropping the assumption that \mathcal{T} is TTF, further investigation of the property that \mathcal{F} is closed under homomorphic images yields information as to when \mathcal{F} is TTF, generalizing a result due to Rutter [10]. Finally, our methods are applied to the TTF class $\mathcal{F} = \{{}_A X \mid P \otimes {}_A X = 0\}$ where P_A is a projective right A -module. The definition of P_A being a quasi-generator is given and characterizations are obtained.

Section 2 of this paper was taken from the author's doctoral dissertation, under the direction of Professor F. L. Sandomierski, at the University of Wisconsin. Section 1 provides a generalization of the material in §2 to arbitrary TTF classes. The author expresses his gratitude to Professor Sandomierski for his guidance and encouragement.

In this paper A will be an associative ring with unit and all modules will be unitary. The left (right) annihilator of I in X will be denoted by $l_X(I)$ ($r_X(I)$). It is easy to see that for a left A -module X and a two-sided ideal I of A , $r_X(I) \cong \text{Hom}_A(A/I, X)$.

Dickson [4] has defined a torsion theory for ${}_A\mathcal{M}$ to be a pair $(\mathcal{T}, \mathcal{F})$ of classes of left A -modules satisfying

- (1) $\mathcal{T} \cap \mathcal{F} = \{0\}$.
- (2) \mathcal{T} is closed under homomorphic images.
- (3) \mathcal{F} is closed under submodules.
- (4) For each $X \in {}_A\mathcal{M}$ there exists a (unique) submodule $X_t \in \mathcal{T}$ such that $X/X_t \in \mathcal{F}$.

A class $\mathcal{T}(\mathcal{F})$ of left modules is called a torsion (torsionfree) class provided there is a (unique) class $\mathcal{F}(\mathcal{T})$ such that $(\mathcal{T}, \mathcal{F})$ is a torsion theory. A torsion theory $(\mathcal{T}, \mathcal{F})$ is said to be hereditary if

the torsion class \mathcal{T} is closed under submodules. For further information on torsion and torsionfree classes the reader is referred to [4].

Gabriel [5] has shown that for a ring A there is a one-to-one correspondence between the hereditary torsion classes of ${}_A\mathcal{M}$ and the idempotent topologizing filters F of left ideals of A given by

$$\mathcal{T} \longrightarrow F(\mathcal{T}) = \{ {}_A I \subseteq {}_A A \mid A/I \in \mathcal{T} \} .$$

The inverse correspondence is given by

$$F \longrightarrow \mathcal{T}(F) = \{ X \in {}_A\mathcal{M} \mid l_A(x) \in F \text{ for all } x \in X \} .$$

Jans [6] has called a torsion class \mathcal{T} which is also a torsionfree class for some torsion class \mathcal{C} , a torsion-torsionfree (TTF) class. In this case $(\mathcal{T}, \mathcal{T})$ and $(\mathcal{C}, \mathcal{T})$ are called the torsion theories associated with \mathcal{T} . In [6] it is shown that \mathcal{T} is a TTF class if and only if $F(\mathcal{T})$ contains a unique minimal left ideal T . Furthermore, T is an idempotent two-sided ideal, $T = A_e$ (the \mathcal{C} -torsion submodule of A), and there is a one-to-one correspondence between the TTF classes \mathcal{T} of ${}_A\mathcal{M}$ and the idempotent two-sided ideals of A given by

$$\mathcal{T} \longrightarrow T .$$

The inverse correspondence is given by

$$T \longrightarrow \{ X \in {}_A\mathcal{M} \mid TX = 0 \} .$$

1. TTF classes. Let $\mathcal{T} \subseteq {}_A\mathcal{M}$ be a TTF class with associated torsion theories $(\mathcal{T}, \mathcal{T})$ and $(\mathcal{C}, \mathcal{T})$. Let T be the minimal, idempotent, two-sided ideal in $F(\mathcal{T})$. One easily checks that

$$\mathcal{T} = \{ {}_A X \mid TX = 0 \} ,$$

$$\mathcal{C} = \{ {}_A X \mid \text{Hom}(A/T, X) = 0 \} ,$$

and

$$\mathcal{C} = \{ {}_A X \mid A/T \otimes {}_A X = 0 \} .$$

Note that the \mathcal{T} -torsion submodule of ${}_A X$ is $X_t = r_X(T)$ while the \mathcal{C} -torsion submodule is $X_e = TX$. Furthermore, let

$$\mathcal{H} = \{ {}_A X \mid A/T \otimes {}_A M = 0 \text{ for every submodule } M \text{ of } X \} .$$

LEMMA 1.1. \mathcal{H} is a hereditary torsion class and $F(\mathcal{H}) = \{ {}_A I \subseteq {}_A A \mid (I : a) + T = A \text{ for all } a \in A \}$ where $(I : a) = \{ x \in A \mid xa \in I \}$.

Proof. It is left to the reader to check that ${}_A X \in \mathcal{H}$ if and only

if $x \in Tx$ for all $x \in X$. If ${}_A I \in F(\mathcal{H})$, then $A/I \in \mathcal{H}$. Hence $a + I \in T(a + I)$ for all $a \in A$. Thus given $a \in A$ there exists $t \in T$ such that $ta - a = (t - 1)a \in I$. Therefore, $t - 1 \in (I : a)$, which implies that $1 \in (I : a) + T$.

Conversely, for a left ideal I suppose that $(I : a) + T = A$ for all $a \in A$. Hence given $a \in A$, we have that $1 = x + t$ where $x \in (I : a)$ and $t \in T$. Thus $a + I = x(a + I) + t(a + I)$, and so $a + I \in T(a + I)$.

Lemma 1.1 can also be deduced from [9, Lemma 1.1].

LEMMA 1.2. *For the torsion class \mathcal{C} the following statements are equivalent.*

- (1) $(A/T)_A$ is flat.
- (2) \mathcal{C} is hereditary (i.e., $\mathcal{C} = \mathcal{H}$).
- (3) $\bigcap_{i=1}^n l_A(t_i) + T = A$ for all $t_i \in T$.
- (4) $l_A(t) + T = A$ for all $t \in T$.

Furthermore, if \mathcal{C} is hereditary, then $F(\mathcal{C}) = \{{}_A I \subseteq {}_A A \mid I + T = A\}$.

Proof. (1) \rightarrow (2) Take ${}_A X \in \mathcal{C}$. Since $(A/T)_A$ is flat M/TM may be viewed as a submodule of X/TX . Thus $M/TM = 0$ as $X/TX = 0$.

(2) \rightarrow (3) Since $T \in \mathcal{C}$ we have that $l_A(t) \in F(\mathcal{C})$ for all $t \in T$. Thus $\bigcap_{i=1}^n l_A(t_i) \in F(\mathcal{C})$ for all $t_i \in T$ as $F(\mathcal{C})$ is closed under finite intersections. Hence $\bigcap_{i=1}^n l_A(t_i) + T = A$ for all $t_i \in T$ by Lemma 1.1.

(3) \rightarrow (4) Trivial.

(4) \rightarrow (1) For $t \in T$, by assumption we have that $1 = a + t'$ where $a \in l_A(t)$ and $t' \in T$. So $t = at + t't = t't \in Tt$. That $(A/T)_A$ is flat now follows by [8, Lemma 4.1].

If \mathcal{C} is hereditary then ${}_A I \in F(\mathcal{C})$ if and only if $T(A/I) = A/I$; i.e., if and only if $I + T = A$.

Since the minimal idempotent ideal T in $F(\mathcal{T})$ is two-sided let

$$\mathcal{H}' = \{X_A \mid M \otimes {}_A A/T = 0 \text{ for every submodule } M \text{ of } X\},$$

a hereditary torsion class in \mathcal{M}_A . Let I be a right ideal of A contained in $F(\mathcal{H}')$. Then $I + T = A$. If $x \in A_t = r_A(T)$, then $Ax = Ix + Tx = Ix \subseteq I$. That is, A_t is contained in every right ideal in the filter $F(\mathcal{H}')$. Hence the question arises as to when A_t is itself in $F(\mathcal{H}')$?

THEOREM 1.3. *Let $\mathcal{T} \subseteq {}_A \mathcal{M}$ be a TTF class with associated torsion theories $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{T})$. Using the above notation the following statements are equivalent.*

- (1) $A_t \in F(\mathcal{H}')$.
- (2) $A = A_t + A_c$.
- (3) $X = X_t + X_c$ for all $X \in {}_A \mathcal{M}$.
- (4) ${}_A(A/A_c)$ is projective.

- (5) \mathcal{F} is closed under homomorphic images.
- (6) $\mathcal{F} = \mathcal{H}$.
- (7) $\mathcal{F} \subseteq \mathcal{C}$.

Proof. (1) \rightarrow (2) This is immediate from Lemma 1.1.

(2) \leftrightarrow (3) If $A = A_t + A_c$, then for every ${}_A X$ we have ${}_A X = A_t X + A_c X \subseteq X_t + X_c$. The converse is trivial.

(2) \rightarrow (4) This is Lemma 4.10 of [8].

(4) \rightarrow (5) Let ${}_A X \in \mathcal{F}$ and let $\alpha: {}_A X \rightarrow {}_A Y$ be an epimorphism. Since ${}_A(A/T)$ is projective we have an epimorphism $\alpha^*: \text{Hom}_A(A/T, X) \rightarrow \text{Hom}_A(A/T, Y)$. Thus $\text{Hom}_A(A/T, Y) = 0$; hence $Y \in \mathcal{F}$.

(5) \rightarrow (6) For ${}_A X \in \mathcal{H}$ consider $X_t \subseteq X$. Now $TX_t = 0$; but also $TX_t = X_t$ since $X \in \mathcal{H}$. Thus $X \in \mathcal{F}$ as $X_t = 0$.

Conversely, let ${}_A X \in \mathcal{F}$. For $x \in X$, $Ax \in \mathcal{F}$ since \mathcal{F} is closed under submodules. Thus $Ax/Tx \in \mathcal{F}$ by assumption. But $Ax/Tx \in \mathcal{F}$ since $T(Ax/Tx) = 0$. Hence $Tx = Ax$ for all $x \in X$, which implies that $X \in \mathcal{H}$.

(6) \rightarrow (7) Trivial.

(7) \rightarrow (2) By assumption $A/A_t \in \mathcal{C}$. Thus $A/A_t = T(A/A_t) = T + A_t/A_t$; i.e., $A_t + A_c = A$.

(2) \rightarrow (1) For all $t \in T$, $r_A(t) + T = A$ since $A_t \subseteq r_A(t)$. Thus by Lemma 1.2 $F(\mathcal{H}') = \{I_A \subseteq A_A \mid I + T = A\}$. Hence $A_t \in F(\mathcal{H}')$.

REMARK. The equivalence of (2), (3), and (7) was recently shown by Kurata in [7]. He also shows that (5) \rightarrow (7).

A torsion theory $(\mathcal{T}, \mathcal{F})$ is centrally splitting if \mathcal{T} is a TTF class (with associated torsion theories $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{T})$) and $X = X_t \oplus X_c$ for all $X \in {}_A \mathcal{M}$. The following corollary characterizing centrally splitting torsion theories is due to Jans [6] and Bernhardt [2].

COROLLARY 1.4. Let $\mathcal{T} \subseteq {}_A \mathcal{M}$ be a TTF class with associated torsion theories $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{T})$. The following statements are equivalent.

- (1) $A = A_t \oplus A_c$ (ring direct sum).
- (2) $X = X_t \oplus X_c$ for all $X \in {}_A \mathcal{M}$.
- (3) $\mathcal{F} = \mathcal{C}$.
- (4) \mathcal{T} is stable and A_c is a direct summand of A .
- (5) \mathcal{F} is closed under homomorphic images and A_t is a direct summand of A .
- (6) A_c is a ring direct summand of A .

Proof. \mathcal{T} being stable (closed under injective hulls) is equivalent to \mathcal{C} being hereditary (see e.g. [2]). Using Lemma 1.2 and Theorem 1.3 conditions (1), (2), (3), (4), and (6) are easily seen to be equivalent

to the condition that ${}_A(A/A_e)$ is projective and $(A/A_e)_A$ is flat. That (1) is equivalent to (5) is easy using Theorem 1.3.

An example of a TTF class \mathcal{T} for which \mathcal{T} is closed under homomorphic images, but $(\mathcal{T}, \mathcal{F})$ is not centrally splitting is given in §2. See Example 2.5 (i).

In the remainder of this section we investigate the condition that the torsionfree class of a hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ be closed under homomorphic images, dropping the condition that \mathcal{T} be a TTF class. Assuming \mathcal{F} is closed under homomorphic images, \mathcal{F} is now a torsion class; i.e., a TTF class. $(\mathcal{T}, \mathcal{F})$ is one torsion theory associated with \mathcal{F} . Let $(\mathcal{F}, \mathcal{L})$ be the other. The \mathcal{T} -torsion submodule of A , A_t , is in $F(\mathcal{F})$ since $A/A_t \in \mathcal{F}$. Furthermore, if ${}_A I \in F(\mathcal{F})$, then $A_t \cap I \in F(\mathcal{F})$, which means that $A/A_t \cap I \in \mathcal{F}$. Thus $A_t/A_t \cap I \in \mathcal{F}$. Since $A_t/A_t \cap I$ is also in \mathcal{T} we have that $A_t/A_t \cap I = 0$; i.e., $A_t \subseteq I$. Hence A_t is the minimal element of the filter $F(\mathcal{F})$. As before we have that

$$\mathcal{F} = \{{}_AX \mid A_t X = 0\},$$

$$\mathcal{L} = \{{}_AX \mid \text{Hom}_A(A/A_t, X) = 0\},$$

and

$$\mathcal{T} = \{{}_AX \mid A/A_t \otimes {}_AX = 0\}.$$

(By Theorem 1.3 if \mathcal{T} is a TTF class then \mathcal{L} is the torsionfree class associated with \mathcal{H} .) Since \mathcal{T} is hereditary, $(A/A_t)_A$ is flat by Lemma 1.2. Furthermore, $F(\mathcal{T}) = \{{}_AI \subseteq {}_AA \mid I + A_t = A\}$.

If in fact $(A/A_t)_A$ is projective, then $A = l_A(A_t) + A_t$ by [8, Lemma 4.10]. Thus $l_A(A_t) \in F(\mathcal{T})$. Furthermore, $l_A(A_t)$ is the minimal element of $F(\mathcal{T})$ by an argument similar to the one used prior to Theorem 1.3. Hence \mathcal{T} is a TTF class.

THEOREM 1.5. *For a hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ the following statements are equivalent.*

- (1) \mathcal{F} is closed under homomorphic images and $(A/A_t)_A$ has a projective cover.
- (2) \mathcal{F} is closed under homomorphic images and A_t is finitely generated as a right A -module.
- (3) \mathcal{T} is a TTF class where $A_e = l_A(A_t)$ and $(A/A_t)_A$ is projective.

Proof. By our above argument (3) will follow from (1) or (2) if we can show that the flat module $(A/A_t)_A$ is projective. In (1) this follows since $(A/A_t)_A$ has a projective cover [8, Lemma 1.2]. In (2) this follows by [3, corollary to Proposition 2.2].

By Theorem 1.3 (1) and (2) will follow from (3) if we show that $A = A_t + A_e$. Since $(A/A_t)_A$ is projective, $A = A_t + l_A(A_t)$ by [8, Lemma 4.10]. Thus $A = A_t + A_e$ by assumption.

COROLLARY 1.6. (Rutter [10, Proposition 2].) *Let A be a semi-perfect ring and $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory in ${}_A\mathcal{M}$. If \mathcal{F} is closed under homomorphic images, then \mathcal{T} is a TTF class where $A_e = l_A(A_t)$.*

COROLLARY 1.7. *Let A be a right Noetherian ring and $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory in ${}_A\mathcal{M}$. If \mathcal{F} is closed under homomorphic images, then \mathcal{T} is a TTF class where $A_e = l_A(A_t)$.*

2. Projective quasi-generators. In this section P_A will be a projective right A -module with trace ideal $T = \sum_{f \in \text{Hom}(P, A)} \text{im } f$ and A -endomorphism ring B . For further information on projective modules and the idempotent trace ideal the reader is referred to [13]. Let $\mathcal{T} = \{{}_AX \mid P \otimes {}_AX = 0\}$, a hereditary torsion class in ${}_A\mathcal{M}$. It is easily seen that $\mathcal{T} = \{{}_AX \mid TX = 0\}$ and $F(\mathcal{T}) = \{{}_AI \subseteq {}_AA \mid T \subseteq I\}$. Hence \mathcal{T} is a TTF class, and the notation of §1 applies.

Sandomierski [12] has defined an A -module ${}_AX(X_A)$ to be T -accessible if $TX = X(XT = X)$. Note that X_A is T -accessible if and only if X_A is a homomorphic image of a direct sum of copies of P_A [12]. Define an A -module X to be strongly T -accessible if every submodule of X is T -accessible. The class of T -accessible (strongly T -accessible) left A -modules is our class $\mathcal{C}(\mathcal{H})$. From Lemma 1.1 we see that ${}_AX$ is strongly T -accessible if and only if $x \in Tx$ for all $x \in X$.

We shall call P_A a quasi-generator if every T -accessible right A -module is strongly T -accessible. That is, if every submodule of a homomorphic image of a direct sum of copies of P_A is itself a homomorphic image of a direct sum of copies of P_A . This definition is dual to the definition of self-cogenerator given in [11].

THEOREM 2.1. *For P_A projective with trace ideal T the following statements are equivalent.*

- (1) P_A is a quasi-generator.
- (2) P_A is strongly T -accessible.
- (3) T_A is strongly T -accessible.
- (4) ${}_A(A/T)$ is flat.
- (5) $r_A(p) + T = A$ for all $p \in P$.
- (6) $r_A(t) + T = A$ for all $t \in T$.

Proof. By definition P_A is a quasi-generator if and only if the torsion class $\mathcal{C}' = \{X_A \mid X \otimes {}_AA/T = 0\}$ is hereditary. Thus the equiv-

alence of (1), (4), and (6) follows by Lemma 1.2. As $\mathcal{H}' = \{X_A \mid M \otimes_A A/T = 0 \text{ for every submodule } M \text{ of } X\}$ is the class of strongly T -accessible right A -modules, the equivalence of (2) and (5), and the equivalence of (3) and (6) follows from Lemma 1.1. That (1) implies (2) is by definition.

(2) \rightarrow (1) Let X_A be T -accessible. Then X_A is a homomorphic image of P_A^I , a direct sum of copies of P_A . Now $P_A^I \in \mathcal{H}'$ as $P_A \in \mathcal{H}'$ by assumption and \mathcal{H}' is closed under direct sums. Thus $X \in \mathcal{H}'$ as \mathcal{H}' is closed under homomorphic images.

Ware [13] has called a projective module regular if every homomorphic image is flat.

COROLLARY 2.2. *Let P_A be a regular module. Then P_A is a quasi-generator. Hence over a regular ring, every projective is a quasi-generator.*

Proof. Let $x \in P_A$ and consider the exact sequence

$$0 \longrightarrow xA \longrightarrow P \longrightarrow P/xA \longrightarrow 0.$$

Since P/xA is flat there exists a map $\theta: P_A \rightarrow xA$ such that $\theta(x) = x$ [13, Lemma 2.2]. Letting T be the trace ideal of P_A , we have that $x = \sum_{i=1}^n p_i t_i$ where $p_i \in P$, $t_i \in T$, $i = 1, 2, \dots, n$. Hence $x = \theta(x) = \theta(\sum_{i=1}^n p_i t_i) = \sum_{i=1}^n \theta(p_i) t_i \in xAT = xT$. Thus P_A is strongly T -accessible, hence a quasi-generator by Theorem 2.1.

The remainder of the corollary follows from the fact that over a regular ring every projective is regular [13, Example 1, page 238].

In our present framework the question that we asked prior to Theorem 1.3 becomes the following: When is $r_A(T) = r_A(P)$ strongly T -accessible as a right A -module?

THEOREM 2.3. *Let P_A be projective with trace ideal T and $B = \text{End}(P_A)$. The following statements are equivalent.*

- (1) $r_A(P)$ is strongly T -accessible as a right A -module.
- (2) $r_A(P) + T = A$.
- (3) ${}_A(A/T)$ is projective.
- (4) $P_{\bar{A}}$ is a generator where $\bar{A} = A/r_A(P)$.
- (5) P_A is a quasi-generator and ${}_A T$ is finitely generated.
- (6) P_A is a quasi-generator and ${}_B P$ is finitely generated.

Proof. The equivalence of (1), (2), and (3) follows from Theorem 1.3. That (3) is equivalent to (5) follows since P_A is a quasi-generator if and only if ${}_A(A/T)$ is flat.

(2) \leftrightarrow (4) $P_{\bar{A}}$ is projective with trace ideal $\bar{T} = T + r_A(T)/r_A(T)$.

Thus $r_A(P) + T = A$ if and only if $\bar{T} = \bar{A}$; i.e., if and only if P_A is a generator.

(4) \rightarrow (6) Since $B = \text{End}(P_A)$, ${}_B P$ is finitely generated. Thus (6) follows by (3).

(6) \rightarrow (1) Since ${}_B P$ is finitely generated we may write $P = Bp_1 + \cdots + Bp_n$ where $p_1, \dots, p_n \in P$. Since P_A is a quasi-generator $r_A(p_i) \in F(\mathcal{H}')$ for $i = 1, \dots, n$. Thus $r_A(T) = r_A(P) = \bigcap_{i=1}^n r_A(p_i) \in F(\mathcal{H}')$ since $F(\mathcal{H}')$ is closed under finite intersections.

REMARK. While $r_A(T) \in F(\mathcal{H}')$ implies that $F(\mathcal{H}')$ has a minimal element, it is possible for $F(\mathcal{H}')$ to have a minimal element without $r_A(T)$ being contained in $F(\mathcal{H}')$. Let A be a left perfect ring and let P_A be a faithful projective that is not a generator. $F(\mathcal{H}')$ has a minimal element L [1, Corollary 1.6], but $L \neq r_A(T)$. Otherwise $L = 0$, which implies that every right A -module is torsion; i.e., P_A is a generator.

COROLLARY 2.4. Let P_A be projective with trace ideal T and $B = \text{End}(P_A)$. The following statements are equivalent.

(1) P_A is a generator (progenerator).

(2) P_A is a (finitely generated) faithful quasi-generator and ${}_B P$ is finitely generated.

(3) P_A is (finitely generated) faithful and ${}_A(A/T)$ is projective.

EXAMPLE 2.5. (i) Let A be the ring of all 2 by 2 upper triangular matrices over a field K . Let

$$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $P_A = eA$ is a projective quasi-generator. ${}_B P$ is finitely generated since ${}_B P = Be$ where $B = eAe$. However, P_A is not faithful, so P_A is not a generator.

Also, $T = AeA$ is a left direct summand; i.e., ${}_A(A/T)$ is projective. However, $(A/T)_A$ is not flat (see Corollary 1.4).

(ii) Let $A = \prod_{i \in I} K_i$ where $K_i = K$ a field and the index set I is countable infinite. Let $P_A = \bigoplus_{i \in I} K_i$. We may write $P_A = \bigoplus_{i \in I} e_i A$ where $e_i^2 = e_i \in A$ and $e_i A \cong K_i$ for $i \in I$. Then P_A is projective with trace ideal $T = \bigoplus_{i \in I} e_i A$. Since A is a regular ring ${}_A(A/T)$ is flat, and thus P_A is a quasi-generator. Furthermore, P_A is faithful, but ${}_B P$ is not finitely generated as ${}_A T$ is not finitely generated. Hence P_A is not a generator.

REMARK. By Corollary 2.4 (3) if A is a semiperfect ring, then a projective module P_A is a generator if and only if P_A is a faithful

quasi-generator. Also, by Corollary 2.4 (2) if A is a regular ring, then a projective module P_A is a generator if and only if P_A is faithful and ${}_B P$ is finitely generated.

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