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The general setting of this paper is that of a von Neumann algebra, M, with weight, φ, and a group, G, of automorphisms which commute with the modular automorphism group associated with this weight.

The first section is devoted to the question of when the given weight is invariant under the action of G. Should G leave the center of M elementwise fixed results have been obtained by Pedersen, Størmer, Takesaki, and the present author. If, alternatively, it is assumed that φ is invariant under a subgroup, H, of G, then by requiring an ergodic action of H on the center of M it is shown here that φ must be (semi-) invariant under the action of G. This is done with the aid of some technical assumptions H. Less demanding hypothesis are shown to lead to a G-invariant weight bicommuting with the given weight.

Section two is mostly devoted to a discussion of ergodicity in time or the requirement that the centralizer of a given state coincide with the center of the von Neumann algebra in question. In particular, we show that if the fixed point algebra of a given group of automorphisms, commuting with the modular automorphism group of the given state, is semifinite, then it is contained in the center if we have ergodicity in time. We also show, in the spirit of [6], that if the centralizer is a factor, then there is no semifinite von Neumann algebra which properly contains the centralizer and is invariant under the modular automorphism group.

I. The general setting for this section will be that of a von Neumann algebra, M, acted upon by two commuting automorphism groups, $\{\alpha_g: g \in G\}$ and $\{\sigma_t^\varphi: t \in R\}$. The latter will be the unique modular group associated with a normal, faithful semifinite, weight φ, and the former will always be assumed continuous in that $g \rightarrow \psi(\alpha_g(x))$ is continuous on G for arbitrary $\psi \in M_*$, $x \in M$.

Recall that a weight, φ, on a von Neumann algebra M, is a mapping from M^+ to $[0, \infty]$ satisfying

(i) $\varphi(\alpha x) = \alpha\varphi(x)$; $\alpha \in R^+$, $x \in M^+$

(ii) $\varphi(x + y) = \varphi(x) + \varphi(y)$; $x, y \in M^+$.

The weight is said to be normal if in addition there is a set of bounded normal positive functionals on M such that

(iii) $\varphi(x) = \sup \omega_i(x)$ for each $x \in M^+$.

Further, the weight is semifinite if the linear span $\mathfrak{m}$ of the set

$$\mathfrak{m}^+ = \{x \in M+ \mid \varphi(x) < \infty\}$$

is σ-weakly dense in M. Faithfulness requires that $\varphi(x) = 0$ implies $x = 0$ for all $x \in M^+$.

It turns out the most important additional stipulation on the group of automorphisms $\{\alpha_g\}$ is its action on the center, $\mathscr{Z}$, of M. The case where the center is pointwise fixed has been discussed in [5] and [12]. The action of $\{\alpha_g\}$ is sain to be *ergodic* on $\mathscr{Z}$ if there are no elements in $\mathscr{Z}$ fixed by all α_g, $g \in G$, save the scalars.

THEOREM 1. *Let φ, σ_t^φ and α_g be as above. Suppose that for a normal subgroup H of G we have $\varphi \circ \alpha_h = \varphi$ for $h \in H$. If $\{\alpha_h: h \in H\}$ acts ergodically on $\mathscr{Z}$ then $\varphi(\alpha_g(x)) = \lambda_g \varphi(x)$ for $x \in M$ where λ_g is a continuous homomorphism of $G \to R^+$.*

Proof. Since α_g commutes with σ_t^φ it follows, [12], that

$$\varphi(\alpha_g(x)) = \varphi(z_g x)$$

where z_g is a unique positive element affiliated with $\mathscr{Z}$.

If $h \in H$ then

$$\begin{aligned}\varphi(\alpha_g(\alpha_h(x))) &= \varphi(\alpha_{gh}(x))) \\ &= \varphi(\alpha_{h'g}(x)) \text{ for some } h' \in H \text{ by the normality of } H \\ &= \varphi(\alpha_g(x))\ .\end{aligned}$$

Thus the weight $\varphi \circ \alpha_g$ is invariant under $\{\alpha_h: h \in H\}$. Its Radon-Nikodym derivative, z_g, is then affiliated with those elements in $\mathscr{Z}$ fixed by $\{\alpha_h: h \in H\}$. This is the scalars by assumption. Hence $\varphi(\alpha_g(x)) = \lambda_g \varphi(x)$ and the homomorphic nature of $g \to \lambda_g$ follows easily. The continuity is then a consequence of the lower semicontinuity of $g \to \lambda_g$ which follows from (iii).

We would like to avoid the normality restriction on H. This can be done if we restrict our attention to state and assume H is "large" in G.

THEOREM 2. *Let G be a topological group with closed subgroup H such that the homogeneous space of right cosets, $H|G$, is locally compact and supports a finite G-invariant, regular, Borel measure μ. Let $\{\alpha_g\}$ and $\{\sigma_t^\varphi\}$ be commuting automorphisms of M. If φ is a normal, faithful, state on M and $\{\alpha_h: h \in H\}$ acts ergodically on $\mathscr{Z}$ and leaves φ invariant, then φ is invariant under the action of all of $\{\alpha_g: g \in G\}$.*

Proof. Let $\dot{g}$ denote the coset H_g. Then we have

$$\varphi(\alpha_{hg}(x)) = \varphi(\alpha_g(x)) \quad \text{and so}$$

$\dot{g} \to \varphi(\alpha_{\dot{g}}(x))$ $(= \varphi(\alpha_g(x)))$ defines a continuous function in $H|G$.

Let

$$\psi(x) = \int_{H|G} \varphi(\alpha_g(x)) d\mu(g) .$$

Then ψ is a normal [7], G-invariant, faithful state on M, which is easily seen to satisfy the KMS boundary condition with respect to $t \to \sigma_t^\varphi$ (ψ is in the norm closed convex hull of $\{\varphi \circ \alpha_g\}$).

Thus $\psi(x) = \varphi(zx)$ where $z \geqq 0$ and $z \in \mathscr{Z}$.

As in Theorem 1 we conclude that z is a scalar $(= 1)$ and since ψ is G-invariant, so is φ.

DEFINITION. A von Neumann algebra acted upon by a group of automorphism $\{\alpha_g : g \in G\}$ is said to be *G-finite* if for $x \in M^+$, $x \neq 0$, there exists a G-invariant state ψ such that $\psi(x) > 0$. In this case there exists a normal faithful projection, ε_G, of M onto $M_G = \{x : \alpha_g(x) = x, g \in G\}$, [8].

In the above we have sought to obtain the G-invariance of a given weight, φ. Suppose instead that we are willing to settle for obtaining a G-invariant weight "closely related" to φ. Suppose then that φ is a normal, faithful *state* on M and M is G-finite. If α_g commutes with σ_t^φ then we can easily see that $\varphi \circ \varepsilon_G$ is a faithful, normal state satisfying the KMS boundary condition with respect to $t \to \sigma_t^\varphi$. However, if φ is only assumed to be a semifinite weight then one cannot necessarily conclude the semifiniteness of $\varphi \circ \varepsilon_G$.

With some additional restrictions and work of Størmer we can obtain a result in this direction.

First some definitions are needed.

DEFINITION. A faithful normal semifinite weight φ is said to be strictly semifinite if

(a) M is $\{\sigma_t^\varphi : t \in R\}$-finite in the sense of the above definition equivalently (b). The restriction of φ to $M_\varphi = \{x : \sigma_t^\varphi(x) = x, t \in R\}$ is a faithful, normal, semifinite trace. (See [1] for other equivalent statements.)

DEFINITION. Two faithful, normal, semifinite weights, φ and ψ are said to bicommute if $\varphi(x) = \psi(zx)$ for all $x \in M$, where z is positive, nonsingular operator affiliated with $\mathscr{Z}_\psi = M_\psi \cap M'_\psi$. (It is frequently the case that $\mathscr{Z}_\psi$ coincides with the relative commutant of M_ψ, [2].)

THEOREM 3. *Let φ be a strictly semifinite, normal, faithful weight on M with modular automorphism group $t \to \sigma_t^\varphi$. Suppose* *that, $\{\alpha_g\colon g\in G\}$ is a group of automorphism of M commuting with the modular automorphism group of φ and that M is a G-finite von Neumann algebra. If G acts ergodically on the center of M_φ $(=\mathscr{Z}_\varphi)$ or leaves $\mathscr{Z}_\varphi$ elementwise invariant, then there exists a strictly semifinite, normal, faithful weight, ψ, bicommuting with φ such that ψ is G-invariant. If the action of G is ergodic on $\mathscr{Z}_\varphi$ then the weight ψ is unique up to a positive multiple.*

Proof. Since φ is strictly semifinite, M_φ is a semifinite von Neumann algebra. Further, the fact that α_g commutes with σ_t^φ means that α_g is an automorphism of M_φ, and thus that M_φ is G-finite. The action of $\{\alpha_g\colon g\in G\}$ on M_φ allows us to apply results of Størmer [13, 14, 15] to obtain a G-invariant, faithful, normal, semifinite trace, τ, on M_φ. But φ is such a trace and so

$$\tau(x)=\varphi(zx)\quad\text{for all}\quad x\in M_\varphi\quad\text{where}$$

$z\geqq 0$ and $z\eta\mathscr{Z}_\varphi$.

Define

$$\psi(x)=\tau\circ\varepsilon_\varphi(x)\quad\text{for all}\quad x\in M$$

where ε_φ is the normal, faithful projection of M onto M_φ the existence of which follows from the strict semifiniteness of φ, (see definitions above).

Recall, [8], that for $x\in M$, $\varepsilon_\varphi(x)$ is the unique point in $\overline{co}^{st}\{\sigma_t^\varphi(x)\}\cap M_\varphi$.

Then

$$\psi(x)=\tau(\varepsilon_\varphi(x))=\varphi(z\varepsilon_\varphi(x))\ .$$

For $y\in M$; $\varphi(zy)$ is defined to be $\lim_{\varepsilon\downarrow 0}\varphi(z_\varepsilon y)$ where $z_\varepsilon=z(1+\varepsilon z)^{-1}$. (See the remarks preceding Proposition 4.2 of [12].) With this comment we conclude that

$$\psi(x)=\varphi(zx)\ .$$

Now ψ is G-invariant, since $\tau\circ\varepsilon_\varphi$ is. From [12] we have $\sigma_t^\psi(x)=z^{it}\sigma_t^\varphi(x)z^{-it}$, so that $M_\psi\supseteqq M_\varphi$. We conclude that every σ_t^φ-invariant state is σ_t^ψ-invariant and thus that ψ is strictly semifinite.

Suppose $\tilde{\psi}$ is another such weight and G acts ergodically in $\mathscr{Z}_\varphi$. We claim $\tilde{\psi}$ restricted to M_φ is a (normal) semifinite, trace. Since $M_{\tilde{\psi}}\supseteqq M_\varphi$, $\tilde{\psi}$ is clearly a normal trace in M_φ. But $\tilde{\psi}(x)=\varphi(z_1x)$ where z_1 is a positive operator affiliated with the center of M_φ. Hence the argument of Proposition 4.2 of [12] shows that $\tilde{\psi}$ restricted to M_φ is semifinite. Considering the restrictions of $\tilde{\psi}$ and ψ to M_φ we see, by the invariance of $\tilde{\psi}$ and ψ, that the Radon-Nikodym derivative of $\tilde{\psi}$ with respect to ψ is affiliated with the fixed points in the center of M_φ and thus $\tilde{\psi}$ is a positive multiple of ψ. (This last argument is given by Størmer in [13].)

THEOREM 4. *With the setting as in Theorem 3, a G-invariant weight, Ψ, bicommuting with φ exists without assumptions concerning the action of G on M_φ, if M_* is separable.*

Proof. Nest, [10], has shown that a semifinite, G-finite von Neumann algebra with separable predual admits a faithful, normal semifinite G-invariant trace.

COROLLARY 5. *With the same hypothesis as in Theorem 3, if $\mathscr{Z}_\varphi = \{\lambda I\}$ (e.g., if φ is maximal [6]) then φ is G-invariant.*

The following is in the spirit of Theorems 1 and 2 and is immediate from Theorem 3 but we offer an alternative proof.

COROLLARY 6. *Suppose the setting to be the same as in Theorem 3 and in addtion suppose there exists a subgroup $H \subseteqq G$ such that $\varphi(\alpha_h(x)) = \varphi(x)$, $h \in H$, $x \in M$. If H acts ergodically on $\mathscr{Z}_\varphi$ then $\varphi(\alpha_g(x)) = \varphi(x)$ for all $g \in G$, $x \in M$.*

Proof. Again we restrict our attention to M_φ and obtain two semifinite, normal, faithful traces φ and τ where τ is G-invariant. Let $\psi = \varphi + \tau$. Then ψ is a semifinite, H-invariant, trace. There exists unique $z_1, z_2 \in \mathscr{Z}_\varphi$ such that

$$\varphi(\cdot) = \psi(z_1 \cdot)$$
$$\tau(\cdot) = \psi(z_2 \cdot) .$$

But, φ and τ are H-invariant. This puts $z_1, z_2 \in M_H \cap \mathscr{Z}_\varphi = \{\lambda I\}$. The weight φ is then a multiple of τ on M_φ and hence is G-invariant there. But

$$\varphi(\alpha_g(x)) = \varphi(\varepsilon_\varphi(\alpha_g(x))) = \varphi(\alpha_g(\varepsilon_\varphi(((x)))$$
$$= \varphi(\varepsilon_\varphi(x)) = \varphi(x) .$$

II. In § I we considered the affect of an automorphism group which commuted with a given modular automorphism group and in some sense acted ergodically. Of course the center of M is always elementwise fixed be any modular automorphism group and so we shall consider consequences of what is referred to as "ergodicity in time" namely the requirement that $M_\varphi = \mathscr{Z}$. There seems to be no advantage to working with weights at this point and so we stick to states.

We first prove a related result regarding the centralizer, M_φ, the proof of some being in the spirit of what follows (cf. [6]).

THEOREM 7. *Suppose φ is a normal, faithful state and $M_\varphi \subseteqq N \subseteqq M$ where N is a von Neumann algebra invariant under $t \to \sigma_t^\varphi$. If M_φ is a factor then*

(i) *N is a factor.*

(ii) *If N is semifinite then $N = M_\varphi$*, i.e., *there are no invariant semifinite von Neumann algebras properly containing M_φ.*

Proof. Since N is invariant, the center of N is elementwise fixed by $t \to \sigma_t^\varphi$. Thus $N \cap N' \subseteqq M_\varphi \cap M_\varphi' = \{\lambda I\}$.

Suppose now that N is semifinite. By the invariance of N, $\sigma_t^\varphi(x) = u_t x u_{-t}$ for $x \in N$, where the u_t form a one parameter unitary group in N, [16]. Since u_s, $s \in R$ is fixed by σ_t^φ, $u_s \in M_\varphi$. But $N \supseteqq M_\varphi$, so $u_s \in \mathscr{Z}_\varphi = \{\lambda I\}$ and thus $\sigma_t^\varphi(x) = x$ for all $x \in N$.

We remark here that if M is in case III_λ, $0 \leqq \lambda < 1$ this result is known [2], [6]. For under these circumstances, M_φ is a factor if and only if $M_\varphi' \cap M = \{\lambda I\}$, [2].

Now let $\{\alpha_g: g \in G\}$ be an automorphism group of M, commuting with the modular automorphism group $t \to \sigma_t^\varphi$, of a given faithful, normal state φ.

THEOREM 8. *If in the setting just described $M_\varphi = \mathscr{Z}$ and M_G is semifinite then $M_G \subseteqq \mathscr{Z}$.*

Proof. Again we have σ_t^φ is inner on M_G viz; there exist $t \to v_t \in M_G$ such that $\sigma_t^\varphi(x) = v_t x v_{-t}$ for $x \in M_G$ as in Theorem 7 the $v_s \in M_\varphi$. But $M_\varphi = \mathscr{Z}$ so that all of M_G is fixed.

COROLLARY 9. *With the same setting, if $M_\varphi = \{\lambda I\}$ then M_G is a factor so that either M_G is type III or $M_G = \{\lambda I\}$.*

Proof. Any central projection in M_G is fixed by $t \to \sigma_t^\varphi$.

Let us now assume that $\{\alpha_g: g \in G\}$ fixes the states φ. Then, it follows that α_g commutes with σ_t^φ [5]. Moreover suppose, without any loss of generality, that M acts on a Hilbert space $\mathfrak{H}$ and $\varphi(x) = (x\xi_\varphi \mid \xi_\varphi)$ where ξ_φ is a cyclic and separating vector for M. Then there exists a unitary group $g \to u_g$ such that $\alpha_g(x) = u_g x u_{-g}$ and $u_g \xi_\varphi = \xi_\varphi$ for all $g \in G$. Let E_0 denote the projection onto the subspace of $\mathfrak{H}$ given by $\{\xi: u_g \xi = \xi$ for all $g \in G\}$.

DEFINITION. If E_0ME_0 is abelian then the system $\{M, \varphi, \alpha_g\}$ is said to be G-abelian [4] and [9].

COROLLARY 10. *Let* $\{M, \varphi, \sigma_t^\varphi, \alpha_g\}$ *be as just described then if* M *is G-abelian,* $M_G \subseteqq \mathscr{Z}$.

Proof. E_0ME_0 equals, [4], M_GE_0 which is isomorphic to M_G. The latter being abelian is a fortiori semifinite, hence Theorem 8 applies.

1. The semifiniteness of M_G is equivalent to that of $\mathscr{R}(M, U_G)$. For the latter one should see a recent paper of Pedersen and Størmer [11].

What remains is some commentary on the condition $M_\varphi = \mathscr{Z}$. Suppose then that $M = \pi_\varphi(\mathfrak{A})''$ arises via the GNS procedure [3], from a C^*-algebra $\mathfrak{A}$ with state φ and KMS automorphism group $t \longrightarrow \sigma_t$.

In [5] a condition that $M_\varphi = \mathscr{Z}$ was given and a class of representation of the Clifford algebra were shown to satisfy ergodicity in time. We offer

THEOREM 11. *Suppose* $\mathfrak{A}$ *is a* C^**-algebra acted upon by a one-parameter automorphism group,* $t \longrightarrow \sigma_t$ *which, together with a state* φ*, satisfies the* KMS *boundary condition. If* $M = \pi_\varphi(\mathfrak{A})''$*, then* $M_\varphi = \mathscr{Z}$ *if and only if whenever a state* ψ *on* $\mathfrak{A}$ *is invariant under the group* $\{\sigma_t\}_{t\in R}$ *and* π_ψ *is quasi-equivalent to* π_φ*, we have that* ψ *is a* KMS *state for* $\{\sigma_t\}$.

Proof. Suppose π_φ and π_ψ are quasi-equivalent and that ψ is invariant. There then exists an isomorphism Φ taking $\pi_\psi(\mathfrak{A})''$ onto $\pi_\varphi(\mathfrak{A})''$ such that $\Phi(\pi_\psi(x)) = \pi_\varphi(x)$, $x \in \mathfrak{A}$. Let ξ_ψ, ξ_φ be vectors in $\mathfrak{H}_\psi$, $\mathfrak{H}_\varphi$ respectively, such that

$$\psi(x) = (\pi_\psi(x)\xi_\psi \mid \xi_\psi);\ \varphi(x) = (\pi_\varphi(x)\xi_\varphi \mid \xi_\varphi) \quad \text{for} \quad x \in \mathfrak{A}\ .$$

Define a normal state ω on $\pi_\varphi(\mathfrak{A})''$ by

$$y \longrightarrow (\Phi^{-1}(y)\xi_\psi \mid \xi_\psi)\ .$$

Then, if $x \in \mathfrak{A}$

$$\omega(\pi_\varphi(x)) = (\Phi^{-1}(\pi_\varphi(x))\xi_\psi \mid \xi_\psi) = (\pi_\psi(x)\xi_\psi \mid \xi_\psi) = \psi(x)\ .$$

Furthermore,

$$\begin{aligned}\omega(\sigma_t(\pi_\varphi(x))) &= \omega(\pi_\varphi(\sigma_t(x))) = (\pi_\psi(\sigma_t(x))\xi_\psi \mid \xi_\psi) \\ &= \psi(\sigma_t(x)) = \psi(x) = \omega(\pi_\varphi(x))\ .\end{aligned}$$

Thus ω is an invariant state on M. Since $M_\varphi = \mathscr{Z}$,

$$\omega(y) = (yh\xi_\varphi | h\xi_\omega),\ y \in M \quad \text{and} \quad h \in \mathscr{Z}\ [16] .$$

It is clear then that ω satisfies the KMS condition on $\pi_\varphi(\mathfrak{A})''$ and thus ψ does on $\mathfrak{A}$

For the converse let $M = \pi_\varphi(\mathfrak{A})''$ and consider h, nonsingular belonging to M_φ with $h \geqq 0$. Suppose $\| h\xi_\varphi \| = 1$ and let $\xi_\psi = h\xi_\varphi$. Since ξ_φ is cyclic and separating M so is ξ_ψ. Define a state ψ on $\mathfrak{A}$ by

$$\psi(x) = (\pi_\varphi(x)\xi_\psi \mid \xi_\psi) .$$

Clearly the canonical representation π_ψ of $\mathfrak{A}$, due to ψ is unitarily equivalent to the triple $\{\pi_\varphi, \xi_\psi, \mathfrak{H}_\varphi\}$ and thus π_φ and π_ψ are quasi-equivalent [3]. Now by assumption ψ is a KMS state on $\mathfrak{A}$ and hence the vector state ω_{ξ_ψ} satisfies the KMS condition with respect to σ_t^φ on M. It follows, [16], that for $y \in M$

$$(yh\xi_\varphi \mid h\xi_\varphi) = \omega_{\xi_\psi}(y) = (yk\xi_\varphi \mid k\xi_\varphi) \quad \text{with} \quad k \geqq 0$$

bounded (since $k \in M_\varphi$, $\psi \leqq \varphi$) and belonging to $\mathscr{Z}$. Thus $h = k$ and M_φ is contained in, and hence equal to $\mathscr{Z}$.

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