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## **LOCAL COMPACTNESS OF FAMILIES OF CONTINUOUS POINT-COMPACT RELATIONS**

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# LOCAL COMPACTNESS OF FAMILIES OF CONTINUOUS POINT-COMPACT RELATIONS

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**The purpose of this paper is to prove that the pointwise closure of an equicontinuous family of point-compact relations from a compact Hausdorff space to a locally compact Hausdorff uniform space is locally compact in the topology of uniform convergence. This is a generalization of a recent result of R. V. Fuller.**

1. Introduction. The purpose of this paper is to study conditions under which a family of continuous point-compact relations is locally compact. Our theorem generalizes a theorem of Fuller [2]. For the most part we use the concepts and results of Smithson ([5], [6], [7]), and Michael [4].

We use the term relation where other authors use multivalued function or multifunction. If  $F$  is a relation from  $X$  to  $Y$  and  $B \subset Y$ , we write

$$F^{-1}(B) = \{x \in X: F(x) \cap B \neq \emptyset\}.$$

A relation  $F$  from a topological space  $X$  to a topological space  $Y$  is called *continuous* iff

- (a)  $F^{-1}(A)$  is closed in  $X$  whenever  $A$  is closed in  $Y$ , and
- (b)  $F^{-1}(B)$  is open in  $X$  whenever  $B$  is open in  $Y$ .

$F$  is *point-closed* (respectively *point-compact*) iff  $F(x)$  is closed (respectively compact) for each  $x \in X$ .

We recall three topologies defined on the collection of all non-empty subsets of a topological space  $X$  (see Michael [4]). The collection of all sets of the form  $\{B \subset X: B \subset U\}$  where  $U$  is open in  $X$ , is a base for the *upper semi-finite* (u.s.f.) topology. The collection of all sets of the form  $\{B \subset X: B \cap U \neq \emptyset\}$  where  $U$  is open in  $X$ , is a subbase for the *lower semi-finite* (l.s.f.) topology. The *finite* topology is the supremum of the l.s.f. and u.s.f. topologies. Equivalently, the finite topology has as a basis all sets of the form  $\langle U_1, \dots, U_n \rangle = \{A \subset X: A \cap U_i \neq \emptyset, 1 \leq i \leq n \text{ and } A \subset \bigcup_{i=1}^n U_i\}$ , where  $U_1, \dots, U_n$  are open in  $X$ . A relation  $F: X \rightarrow Y$  is continuous if and only if the function  $F: X \rightarrow P(Y)$  (the power set of  $Y$  with the finite topology) is continuous, c.f. remark following Theorem 2.7 of [7].

Let  $X$  and  $Y$  be topological spaces and let  $\mathcal{F}$  be a set of relations from  $X$  to  $Y$ . The pointwise topology  $\mathcal{P}$  [7] on  $\mathcal{F}$  has a subbase consisting of the sets of the form  $\{F \in \mathcal{F}: F(x) \cap U \neq \emptyset\}$  or  $\{F \in \mathcal{F}: F(x) \subset V\}$  where  $x \in X$ , and  $U, V$  are open in  $Y$ . We

note that the projections  $\{\Pi_x: x \in X\}$  defined by  $\Pi_x(F) = F(x)$  are continuous functions into the collection of all nonempty subsets of  $Y$  with the finite topology.

Let  $X$  be a topological space,  $(Y, \mathcal{U})$  a uniform space and let  $\mathcal{F}$  be a set of relations from  $X$  to  $Y$ . For  $V \in \mathcal{U}$ , let  $W(V) = \{(F, G) \in \mathcal{F} \times \mathcal{F}: \text{for all } x \in X, (y, G(x)) \cap V \neq \emptyset \text{ for all } y \in F(x), \text{ and } (F(x), y') \cap V \neq \emptyset \text{ for all } y' \in G(x)\}$ . Let  $\mathcal{W}$  be the uniformity on  $\mathcal{F}$  generated by the collection of all such encourages as  $W(V)$ . The topology generated by  $\mathcal{W}$  is the topology of *uniform convergence* [5] and is denoted by  $\mathcal{UC}$ . If  $(Y, \mathcal{U})$  is a uniform space,  $\mathcal{F}$  is called equicontinuous at  $x \in X$  [5] iff for every  $V \in \mathcal{U}$  there is a nbhd.  $U$  of  $x$  such that for all  $F \in \mathcal{F}$ ,

- (a)  $F(U) \subset V(F(x))$ , and
- (b)  $F(z) \cap V(y) \neq \emptyset$  for all  $z \in U$  and for all  $y \in F(x)$ .

We now state a theorem of Smithson which we use in the final section.

**THEOREM 1.1.** ([5]). *If  $\mathcal{F}$  is an equicontinuous family of point-compact relations from a compact space  $X$  to a uniform space  $Y$ , then on  $\mathcal{F}$ ,  $\mathcal{P} = \mathcal{UC}$ .*

For further details and a survey the reader is referred to Smithson [7].

**2. Local compactness of a space of relations.** We begin by proving two lemmas.

**LEMMA 2.1.** *Let  $F$  be a point-compact relation from  $X$  to  $Y$ , and let  $A = \{(\{x\}, F(x)): x \in X\}$  be compact in  $P(X) \times P(Y)$ , where each of  $P(X)$ ,  $P(Y)$  has the finite topology. Then  $F$  is a compact subset of  $X \times Y$ .*

*Proof.* Let  $\mathcal{O}$  be an open cover of  $F$  in  $X \times Y$ . For each  $x \in X$ ,  $\{x\} \times F(x)$  is compact; so there is a finite subcollection  $V_i^x \times U_i^x$ ,  $1 \leq i \leq n$  of  $\mathcal{O}$  which covers the set. We can assume that  $x \in V_i^x$  for each  $i$  and that  $F(x) \cap U_i^x \neq \emptyset$ .  $\langle V_1^x, \dots, V_n^x \rangle \times \langle U_1^x, \dots, U_n^x \rangle$  is an open set in  $P(X) \times P(Y)$ . For each  $x \in X$ , we obtain such a set, and this leads to an open cover of  $A$ . Since  $A$  is compact, there is a finite subcover  $\langle V_1^{x_i}, \dots, V_{n_i}^{x_i} \rangle \times \langle U_1^{x_i}, \dots, U_{n_i}^{x_i} \rangle$ ,  $1 \leq i \leq k$ . Finally,  $\{V_j^{x_i} \times U_j^{x_i}: 1 \leq j \leq n_i, 1 \leq i \leq k\}$  is a cover of  $F$  and so  $F$  is compact.

**LEMMA 2.2.** *If  $F$  is a continuous relation from  $X$  to  $Y$ , then the function  $g: X \rightarrow P(X) \times P(Y)$  defined by  $g(x) = (\{x\}, F(x))$  is continuous. ( $P(X)$  and  $P(Y)$  both have finite topology.)*

*Proof.* Let  $\langle V \rangle \times \langle U_1, \dots, U_n \rangle$  be a basic open nbhd. of  $(\{x\}, F(x))$ . The function  $F: X \rightarrow P(Y)$  is continuous, hence there exists a nbhd.  $N \subset V$  of  $x$  such that  $F(N) \subset \langle U_1, \dots, U_n \rangle$ . Clearly,  $g(N) \subset \langle V \rangle \times \langle U_1, \dots, U_n \rangle$ .

From the above lemmas it follows that if  $X$  is compact,  $Y$  is  $T_2$  and  $F$  is a continuous point-compact relation, then  $F$  is a compact subset of  $X \times Y$ .

The proof of the following lemma is straightforward.

**LEMMA 2.3.** *Let  $\mathcal{F}$  be a family of relations from a topological space  $X$  to a topological space  $Y$ . Then the l.s.f. topology on  $\mathcal{F}$  is contained in  $\mathcal{P}$ .*

**LEMMA 2.4.** *Let  $X$  be compact Hausdorff,  $(Y, \mathcal{V})$  a uniform space, and  $\mathcal{F}$  a family of continuous point-compact relations from  $X$  to  $Y$ . Then on  $\mathcal{F}$  the u.s.f. topology is smaller than  $\mathcal{UC}$ .*

*Proof.* If  $F \in \mathcal{F}$ , then  $F$  is a compact subset of  $X \times Y$ . Suppose  $F \subset N$  is an open subset of  $X \times Y$ . Let  $\mathcal{U}$  be the (unique) uniformity on  $X$ . Then from [3] page 199, it follows that there exist  $U \in \mathcal{U}$ ,  $V \in \mathcal{V}$  such that  $F \subset \cup \{U(x) \times V(y) : x \in X, y \in F(x)\} \subset N$ . Then  $W(V)[F] \subset N$ , thus completing the proof.

**LEMMA 2.5.** *Let  $\mathcal{F}$  be an equicontinuous family of relations from a  $T_1$ -space  $X$  to a uniform space  $(Y, \mathcal{V})$ . Then on  $\mathcal{F}$ ,  $\mathcal{P} \subset$  the finite topology.*

*Proof.* Let  $[x, U] = \{G \in \mathcal{F} : G(x) \subset U\}$ , where  $x \in X$  and  $U$  is open in  $Y$ . If  $F \in [x, U]$ , then  $N = \langle X \times U \cup (X - \{x\}) \times Y \rangle$  is a nbhd. of  $F$  in the finite topology, and  $F \in N \subset [x, U]$ . Suppose  $F \in M = \{G \in \mathcal{F} : G(x) \cap W \neq \emptyset\}$  where  $x \in X$  and  $W$  is open in  $Y$ . If  $F(x) \subset W$ , then the above method works, and so we assume that  $F(x) \not\subset W$ . Let  $p \in F(x) \cap W$  and let  $V \in \mathcal{V}$  such that  $\overline{V^2(p)} \subset W$ . Since  $\mathcal{F}$  is equicontinuous at  $x$ , there is a nbhd.  $U$  of  $x$  such that for all  $T \in \mathcal{F}$ ,  $T(U) \subset V(T(x))$ . Now  $F \in \langle U \times [V(p)]^0, U \times (Y - \overline{V^2(p)}), U \times W, (X - \{x\}) \times Y \rangle \subset M$ , which completes the proof.

**LEMMA 2.6.** *Let  $X$  be compact Hausdorff, and let  $(Y, \mathcal{V})$  be a uniform space. Let  $\mathcal{F}$  be an equicontinuous family of point-compact relations from  $X$  to  $Y$ . If  $\hat{\mathcal{F}}$  is the  $\mathcal{P}$ -closure of  $\mathcal{F}$  in the space of all point-compact relations from  $X$  to  $Y$ , then  $\hat{\mathcal{F}}$  is closed in  $\mathcal{C}(X \times Y)$ , the space of all nonempty compact subsets of  $X \times Y$  with the finite topology.*

*Proof.* Let  $(F_\alpha)$  be a net in  $\hat{\mathcal{F}}$  converging to  $F \in \mathcal{C}(X \times Y)$ . Note that the domain of  $F$  is  $X$ ; for if  $x \in X - \text{dom } F$ , then  $\{G \in \mathcal{C}(X \times Y): G \subset X \times Y - \{x\} \times Y\}$  is an u.s.f. nbhd. of  $F$  in  $\mathcal{C}(X \times Y)$ , and  $F_\alpha$  is eventually in this nbhd., a contradiction. Clearly  $F$  is point-compact. We now show that  $(F_\alpha) \rightarrow F$  in  $\mathcal{P}$ . Let  $U$  be open in  $Y$ , and suppose  $F(x) \subset U$ . Then the set  $N = \langle X \times U \cup (X - \{x\}) \times Y \rangle$  is a nbhd. of  $F$  in the finite topology on  $\mathcal{C}(X \times Y)$ . Since  $F_\alpha$  is eventually in  $N$ , it follows that  $F_\alpha$  is eventually in  $[x, U]$ . If we are given a nbhd.  $M = \{G: G(x) \cap W = \emptyset\}$ , ( $W$  open in  $Y$ ) of  $F$  and  $F(x) \not\subset W$ , then we employ the technique used in the last part of the proof of Lemma 2.5, and use the fact that  $\hat{\mathcal{F}}$  is equicontinuous ([5], Lemma 6).

We now prove the main result.

**THEOREM 2.7.** *Let  $\mathcal{F}$  be an equicontinuous family of point-compact relations from a compact Hausdorff space  $X$  to a locally compact Hausdorff uniform space  $Y$ . Let  $\hat{\mathcal{F}}$  be the  $\mathcal{P}$ -closure of  $\mathcal{F}$  in the space of all point-compact relations from  $X$  to  $Y$ . Then  $\hat{\mathcal{F}}$  is locally compact in  $\mathcal{UC}$ .*

*Proof.* We first note that on  $\hat{\mathcal{F}}$ ,  $\mathcal{P} \subset \mathcal{UC}$  ([5], Lemma 1). From Lemmas 2.3 and 2.4 it follows that the finite topology is contained in  $\mathcal{UC}$ , and since  $\hat{\mathcal{F}}$  is equicontinuous, we have by Theorem 1.1,  $\mathcal{P} = \mathcal{UC}$  on  $\hat{\mathcal{F}}$ . From Lemma 2.5, it follows that the finite topology equals  $\mathcal{UC}$  on  $\hat{\mathcal{F}}$ . Each member of  $\hat{\mathcal{F}}$  is compact, and so  $\hat{\mathcal{F}} \subset C(X \times Y)$ . By Lemma 2.6,  $\hat{\mathcal{F}}$  is closed in the finite topology on  $C(X \times Y)$ . Since  $C(X \times Y)$  is locally compact ([4], Prop. 4.4.1),  $\hat{\mathcal{F}}$  is locally compact.

If in the above theorem, each  $F \in \mathcal{F}$  is a (single valued) function, then it is easy to verify that each member of  $\hat{\mathcal{F}}$  is also a function. Hence a recent result of R. V. Fuller [2] on the local compactness of  $\hat{\mathcal{F}}$  is a special case of the above theorem.

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