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**AN EMBEDDING OF SEMIPRIME P.I.-RINGS**

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Let us say an extension  $R'$  of a ring  $R$  is a *quotient ring* of  $R$  if every regular element of  $R$  is invertible in  $R'$ . In this note we construct a class of quotient rings of semiprime *P.I.*-rings and use this construction to find rapid proofs of several facts about semiprime *P.I.*-rings.

1. Preliminaries. Throughout this paper  $R$  will denote a *semiprime P.I.-ring* with unity and center  $C$ , i.e.,  $R$  has no nonzero nilpotent ideals and the standard polynomial

$$S_{2n}(X_1, \dots, X_{2n}) = \sum_{\pi} (\text{sgn } \pi) X_{\pi(1)} \cdots X_{\pi(2n)},$$

the sum taken over all permutations  $\pi$  of  $(1, \dots, 2n)$ , is an identity of  $R$  for suitable  $n$  (the minimal such  $n$  is the *degree* of  $R$ ). Formanek [5] has constructed a polynomial  $g_n(X_1, \dots, X_{n+1})$  which is central for all semiprime *P.I.*-rings of degree  $n$ , and Rowen [11] has used these central polynomials to prove

**THEOREM A.** *Any nonzero ideal of  $R$  intersects  $C$  nontrivially.*

Let  $S = \{c \in C: cr \neq 0 \text{ for all nonzero } r \text{ in } R\}$ . Define an equivalence relation on  $R \times S$  by saying  $(r_1, s_1) \sim (r_2, s_2)$  if  $r_1 s_2 = r_2 s_1$ , and let  $rs^{-1}$  denote the equivalence class of  $(r, s)$ . Then  $R_S = \{rs^{-1}: (r, s) \in R \times S\}$  is a ring when endowed with the (well-defined) operations  $r_1 s_1^{-1} + r_2 s_2^{-1} = (r_1 s_2 + r_2 s_1)(s_1 s_2)^{-1}$ , called the *ring of central quotients* of  $R$ . The following theorem is a direct consequence of Theorem A (cf., Rowen [11, §2]):

**THEOREM B.** *If  $R$  is a prime *P.I.*-ring of degree  $n$ , then  $R_S$  is simple Artinian of dimension  $n^2$  over its center  $C_S$ ,  $C_S$  is the quotient field of  $C$ , and  $R_S$  satisfies the identities of  $R$ .*

Theorem B often enables us to study  $R$  by examining  $R_S$ . If  $R$  is a semiprime *P.I.*-ring of degree  $n$  and satisfies the ascending chain condition on annihilators of two-sided ideals, then  $R_S$  is the classical semisimple Artinian ring of left and right quotients of  $R$  (cf., [12]). Unfortunately, this situation fails for semiprime *P.I.*-rings in general, so one is led to study other extensions of  $R$ . The purpose of this paper is to introduce a straightforward type of extension of  $R$  and to deduce from it properties of semiprime *P.I.*-rings and their classical quotient rings (if these exist). This paper

subsumes Fisher [4]. First we shall derive some easy known properties of  $R$ .

For a subset  $A$  of  $R$ , let  $\text{Ann}_R(A)$  denote  $\{r \in R \mid Ar = 0\}$ . Also we say an ideal  $A$  of  $R$  is *essential* if for every nonzero ideal  $B$  of  $R$ ,  $A \cap B \neq 0$ . Since  $R$  is semiprime,  $A \cap B = 0$  if and only if  $AB = 0$ . The following lemma is known by Martindale [9].

LEMMA 1. (i) *If  $E$  is an essential ideal of  $C$ , then  $ER$  is an essential ideal of  $R$ .*

(ii) *If  $J$  is a left ideal of  $R$  with  $\text{Ann}_R(J) = 0$ , then  $J \cap C$  is essential in  $C$ , so  $J$  contains an essential ideal of  $R$ .*

*Proof.* (i) Suppose that  $A \cap E = 0$  for some ideal  $A$  of  $R$ . Then  $(A \cap C) \cap E = A \cap (C \cap E) = A \cap E = 0$ , implying  $A \cap C = 0$ . Hence  $A = 0$  by Theorem A and thus  $ER$  is essential.

(ii) Viewed as a ring (without 1),  $J$  is clearly a *P.I.*-ring and can easily be shown to be semiprime. We claim that  $J \cap C = \text{cent } J$ . Indeed  $J \cap C \subseteq \text{cent } J$  and if  $a \in \text{cent } J$ , then for all  $r$  in  $R$  and for all  $x$  in  $J$ ,  $(ra - ar)x = rax - a(rx) = rax - r(xa) = rax - rax = 0$ . Hence  $(ra - ar) \in \text{Ann}_R(J) = 0$  and so  $a \in C$ .

Now let  $B$  be an ideal of  $C$  such that  $(J \cap C) \cap B = 0$ . Then  $(J \cap C \cap BR)^2 \subseteq (J \cap C)BR = B(J \cap C)R \subseteq (B \cap (J \cap C))R = 0$  and so  $(J \cap C \cap BR)^2 = 0$ . Since  $J \cap C$  has no nonzero nilpotent elements, we have  $J \cap C \cap BR = 0$ , i.e.,  $(J \cap C) \cap (J \cap BR) = 0$ . But by Theorem A applied to the semiprime ring  $J$  (with center  $J \cap C$ ),  $J \cap BR = 0$ . This implies  $RJB = BRJ \subseteq J \cap BR = 0$ , so  $B \subseteq \text{Ann}_R(RJ) = \text{Ann } J = 0$ . Hence  $J \cap C$  is essential in  $C$ . The rest of the lemma follows from (i).

2. Definition and elementary properties of  $T(R)$ . For the remainder of this paper, we assume that the semiprime *P.I.*-ring  $R$  has degree  $n$ . This implies that every prime factor ring of  $R$  has degree equal to or less than  $n$ . The *degree* of a prime ideal  $P$  of  $R$  is defined as the degree of  $R/P$ .

Let  $\mathcal{S}$  be a collection (indexed by  $\Lambda$ ) of prime ideals  $P_\lambda$  of  $R$  such that  $\bigcap \{P_\lambda : \lambda \in \Lambda\} = 0$ . For each  $\lambda$  in  $\Lambda$ , set  $R_\lambda = R/P_\lambda$ , let  $Q_\lambda$  equal the simple Artinian ring of central quotients of  $R_\lambda$ , and let  $Q$  be the complete direct product  $\prod \{Q_\lambda : \lambda \in \Lambda\}$ . There is a natural embedding  $R \rightarrow \prod R_\lambda \rightarrow Q$  and we shall often view  $R$  as a subring of  $Q$  under this embedding. Hence  $R$  satisfies the identities of  $Q$ . On the other hand, any identity  $f$  of  $R$  is an identity of each  $R_\lambda$ , and is an identity of each  $Q_\lambda$  by Theorem B; hence  $f$  is an identity of  $Q = \prod Q_\lambda$ . Consequently,  $R$  and  $Q$  satisfy the same identities.

Clearly  $Q$  is *von Neumann regular*, i.e., for any  $x \in Q$ , there is some  $y$  in  $Q$  such that  $xyx = x$ .

As remarked above, each  $Q_\lambda$  has degree  $\leq n$ . Let  $A_j = \{\lambda \in A: Q_\lambda \text{ has degree } j\}$  and let  $\bar{Q}_j = \prod\{Q_\lambda: \lambda \in A_j\}$ . Then  $\bar{Q}_j$  is a semiprimitive ring of degree  $j$  with the property that every nonzero homomorphic image of  $\bar{Q}_j$  has degree  $j$ . This is equivalent to saying, by the Artin [2]-Procesi [10] theorem, that  $\bar{Q}_j$  is an Azumaya algebra of rank  $j$ . Hence  $Q$  is a finite direct sum of the Azumaya algebras  $\bar{Q}_j$  of finite rank  $j$ .

**LEMMA 2.** *Any nonzero homomorphic image  $\psi(Q)$  of  $Q$  is von Neumann regular. Moreover,  $\psi(Q)$  is the finite direct sum of the Azumaya algebras  $\psi(\bar{Q}_j)$  of finite rank  $j$ , and each identity of  $R$  is an identity of  $\psi(Q)$ .*

*Proof.* Every homomorphic image of a von Neumann ring is von Neumann regular. Also, every homomorphic image of  $\psi(\bar{Q}_j)$  is a homomorphic image of  $\bar{Q}_j$ , thereby having rank  $j$ ; hence  $\psi(\bar{Q}_j)$  is Azumaya of rank  $j$ , and clearly  $\psi(Q)$  is the direct sum of  $\psi(\bar{Q}_j)$  for  $j = 1, \dots, n$ . The last assertion is immediate.

For any  $x$  in  $Q$ , let  $x_\lambda$  denote the component of  $x$  in  $Q_\lambda$  and let  $W_x = \{\lambda \in A: x_\lambda \neq 0\}$ . Set  $V = \{x \in Q: \bigcap \{P_\lambda: \lambda \in W_x\} \text{ is an essential ideal of } R\}$ . Now  $V$  is an ideal of  $Q$  because, taking  $x, y$  in  $V$  and  $q$  in  $Q$ ,  $W_{x+y} \subseteq W_x \cup W_y$ ;  $W_{qx} \subseteq W_x$ ;  $W_{xq} \subseteq W_x$ . Let us define  $T(R, \mathcal{P}) = Q/V$ . From Lemma 2 we have that  $T(R, \mathcal{P})$  is a finite direct sum of Azumaya algebras of finite rank and is von Neumann regular.

**THEOREM 1.** (i) *There is a canonical imbedding  $R \rightarrow T(R, \mathcal{P})$  given by  $R \rightarrow Q \rightarrow Q/V$ .*

(ii) *Half regular elements of  $R$  are both left and right invertible in  $T(R, \mathcal{P})$ .*

(iii)  *$T(R, \mathcal{P})$  satisfies precisely the same identities as  $R$ .*

*Proof.* (i) We need show only that  $R \cap V = 0$ . If  $r \in R \cap V$ , then  $\bigcap \{P_\lambda: \lambda \in W_r\}$  is essential in  $R$  and so  $\bigcap \{P_\lambda: r \in P_\lambda\} = 0$ . Hence  $r = 0$ .

(ii) Let  $r$  in  $R$  have right annihilator zero. Then  $\text{Ann}_R(Rr) = 0$  and  $Rr$  contains an essential ideal  $E$  of  $C$  by Lemma 1(ii). Let  $W'_r = \{\lambda: P_\lambda \not\subseteq E\}$ . Clearly  $W'_r \subseteq W_r$ . Moreover, for any  $\lambda$  in  $W'_r$  there is an  $x_\lambda$  in  $Q_\lambda$  such that  $0 \neq x_\lambda r_\lambda \in \text{cent } Q_\lambda$ . Since  $\text{cent } Q_\lambda$  is a field, there is  $d_\lambda$  in  $\text{cent } Q_\lambda$  such that  $d_\lambda x_\lambda r_\lambda = 1_\lambda$ . Furthermore,  $r_\lambda d_\lambda x_\lambda = 1_\lambda$  because  $Q_\lambda$  is simple Artinian. Define  $y$  in  $Q$  as follows:  $y_\lambda = 0$  for  $\lambda \notin W'_r$  and  $y_\lambda = d_\lambda x_\lambda$  for  $\lambda \in W'_r$ . Then  $(yr - 1)_\lambda = 0$  and  $(ry - 1)_\lambda = 0$  for all  $\lambda$  in  $W'_r$ . Thus  $\bigcap \{P_\lambda: \lambda \in W_{yr-1}\} \supseteq \bigcap \{P_\lambda: \lambda \in W'_r\} \supseteq$

*E.* It follows from Lemma 1(i) that  $yr - 1 \in V$ ; likewise  $ry - 1 \in V$ . Hence, for  $\bar{y}$  the image of  $y$  in  $T(R, \mathcal{P})$ , we have  $\bar{y}r = 1$  and  $r\bar{y} = 1$  in  $T(R, \mathcal{P})$ .

(iii)  $T(R, \mathcal{P})$  satisfies each identity of  $R$  by Lemma 2; conversely, by (i), each identity of  $T(R, \mathcal{P})$  is an identity of  $R$ .

The following theorem of Herstein-Small [8] is a consequence of Theorem 1.

**COROLLARY 1.** *Half regular elements of  $R$  are regular.*

*Proof.* If  $r$  in  $R$  is, say, right regular, then for some  $y \in T(R, \mathcal{P})$  we have  $ry = 1$ . Hence  $r$  is left regular.

**COROLLARY 2.** *If  $R$  has a classical left ring of quotients  $R'$ , then  $R'$  satisfies the same polynomial identities as  $R$ .*

*Proof.* In view of Theorem 1(ii) the canonical embedding of  $R$  into  $T(R, \mathcal{P})$  extends to an embedding of  $R'$  into  $T(R, \mathcal{P})$ . Hence  $R'$  satisfies the identities of  $T(R, \mathcal{P})$  which are precisely the identities of  $R$ .

Note that this construction of  $T(R, \mathcal{P})$  is related to constructions of Amitsur [1] and Goldie [7]. Also, those versed in logic may wish to regard  $T(R, \mathcal{P})$  as the "reduced product" (cf., [6]) of the simple Artinian rings  $\{Q_\lambda: \lambda \in \Lambda\}$  by the filter  $\{A - W_x: x \in V\}$ .

**3. Definition and structure of  $T(R)$ .** Now we consider an interesting special case of  $T(R, \mathcal{P})$ . Index the set of all the prime ideals of  $R$  by a set  $\bar{\Lambda}$  with  $\bar{\Lambda}_i = \{\lambda \in \bar{\Lambda}: P_\lambda \text{ has degree } i\}$  for  $i = 1, \dots, n$ . Set  $\bar{N}_i = \bigcap \{P_\lambda: \lambda \in \bar{\Lambda}_i\}$  (if  $\bar{\Lambda}_i = \emptyset$  then  $\bar{N}_i = R$ ),  $\Lambda_i = \{\lambda \in \bar{\Lambda}_i: P_\lambda \not\subseteq \bigcap_{j=i+1}^n \bar{N}_j\}$ ,  $\mathcal{P}_i = \{P_\lambda: \lambda \in \Lambda_i\}$ ,  $\mathcal{P} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_n$ ,  $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_n$ . Clearly  $\bigcap \{P: P \in \mathcal{P}\} = \bar{N}_1 \cap \dots \cap \bar{N}_n = 0$ . We define  $T(R)$  to be  $T(R, \mathcal{P})$ . Note that  $\Lambda_n = \bar{\Lambda}_n$  and that  $\Lambda = \Lambda_n$  if and only if  $\bar{N}_n = 0$ .

Let  $N_i = \bigcap \{P: P \in \mathcal{P}_i\}$  and let  $R_i = R/N_i$ . Note that  $N_n = \bar{N}_n$ . Clearly  $R$  is a subdirect product of the  $R_i$  and this subdirect decomposition is unique with respect to the properties that each of the nonzero subdirect factors has a degree different from each of the other subdirect factors and that for any subdirect factor of degree  $j$ , the intersection of its prime ideals of degree  $j$  is zero. Our aim is to show how the structure of  $T(R)$  is linked to this decomposition. As in Rowen [12], let a polynomial be called *regular* if it is linear in some indeterminant, and let the *central kernel* of a ring be the additive subgroup generated by the values taken (in the center) by regular central polynomials of the ring. The central kernel is an ideal of the center  $C$ . If the central kernel is essential in  $C$ , we

say that  $R$  has *essential central kernel*. Let  $I$  be the central of  $R$ , let  $B = N_1 \cap \cdots \cap N_{n-1}$ , and let  $R'_n = R/B$ . It is shown in Rowen [12] that for  $\lambda \in \bar{A}$ ,  $I \not\subseteq P_\lambda$  if and only if  $\lambda \in A_n$ .

LEMMA 3. (i)  $(RI + N_n)/N_n$  is an essential ideal of  $R_n$ .

(ii)  $(N_n + B)/B$  is an essential ideal of  $R'_n$ .

(iii) A semiprime ring  $R$  of degree  $j$  has essential central kernel if and only if the intersection of its prime ideals of degree  $j$  is zero.

*Proof.* (i) Suppose that  $[(A + N_n)/N_n] \cap [(RI + N_n)/N_n] = 0$  for some ideal  $A$  of  $R$ . Then  $ARI \subseteq N_n \subseteq P_\lambda$  for each  $\lambda \in A_n$ . Since  $I \not\subseteq P_\lambda$  for  $\lambda \in A_n$ , we have  $A \subseteq \bigcap \{P_\lambda : \lambda \in A_n\} = N_n$ . So

$$(A + N_n)/N_n = 0.$$

(ii) Suppose that  $[(A + B)/B] \cap [(N_n + B)/B] = 0$  for some ideal  $A$  of  $R$ . Then  $AN_n \subseteq B = N_1 \cap \cdots \cap N_{n-1} \subseteq P_\lambda$  for each  $\lambda \in A - A_n$ . By definition  $P_\lambda \not\subseteq N_n$  for  $\lambda \in A - A_n$ , so  $A \subseteq \bigcap \{P_\lambda : \lambda \in A - A_n\} = B$ . So  $(A + B)/B = 0$ .

(iii) Let  $\bar{N}_j$  be the intersection of the prime ideals of degree  $j$ . Since every prime ideal of degree  $< j$  contains  $I$ , we have  $I \cap \bar{N}_j = 0$ . Since  $I$  is essential in  $C$ , we have  $\bar{N}_j \cap C = 0$ , hence  $N_j = 0$  by Theorem A. The reverse implication is immediate from (i) and Lemma 1.

Lemma 3(iii) gives us a neater characterization of  $R_1, \dots, R_n$ . Namely, the nonzero  $R_i$  are uniquely determined if we are to express  $R$  as a subdirect product of minimal length of rings with essential central kernel.

LEMMA 4. (i) Suppose that  $J$  is an ideal of  $R$  and  $N_n \subseteq J$ . Then  $J$  is essential in  $R$  if and only if  $J/N_n$  is essential in  $R_n$ .

(ii) Suppose  $B \subseteq J$ . Then  $J$  is essential in  $R$  if and only if  $J/B$  is essential in  $R'_n$ .

*Proof.* (i)  $(\Rightarrow)$  Suppose that  $J/N_n \cap [(A + N_n)/N_n] = 0$  for some ideal  $A$  of  $R$ . Then  $JA \subseteq N_n$  and so  $B \cap JA = 0$ . Now since  $I \subseteq P_\lambda$  for each  $\lambda \in A - A_n$ , we have  $RI \subseteq \bigcap \{P_\lambda : \lambda \in A - A_n\} \subseteq B$  and  $RI \cap JA = 0$ , or  $IJA = 0$ . Hence  $(J \cap AI)^2 \subseteq (JAI)^2 = 0$  and  $J \cap AI = 0$  since  $R$  is semiprime. By hypothesis, we then see  $AI = 0$ , so  $A \subseteq N_n$  by Lemma 3(i). Consequently  $(A + N_n)/N_n = 0$ .

Conversely suppose that  $J \cap A = 0$  for some ideal  $A$  of  $R$ . Then  $JA = 0 \subseteq N_n$ , so  $A \subseteq N_n$  by hypothesis. Thus  $A^2 \subseteq N_n A \subseteq JA = 0$  and so  $A = 0$ .

(ii)  $(\Rightarrow)$  Suppose that  $J/B \cap [(A + B)/B] = 0$ . Then  $JA \subseteq B$ , or

$JAN_n \subseteq B \cap N_n = 0$  which implies  $AN_n = 0$ . Hence  $A \subseteq B$  by Lemma 3(ii) and so  $(A + B)/B = 0$ . The proof of the converse is analogous to that in (i).

**THEOREM 2.**  $T(R) \cong T(R_1) \oplus \cdots \oplus T(R_n)$ .

*Proof.* We use induction on  $n = \text{degree of } R$ . The assertion is true for  $n = 2$ . Since  $R'_n$  has degree  $\leq n - 1$ , we have by our induction hypothesis that  $T(R'_n) \cong T(R_1) \oplus \cdots \oplus T(R_{n-1})$ . Let  $\bar{Q}_n = \Pi\{Q_i: \lambda \in A_n\}$ ,  $\bar{Q}'_n = \Pi\{Q_i: \lambda \in A - A_n\}$ ,  $V_n = V \cap \bar{Q}_n$ , and  $V'_n = V_n \cap \bar{Q}'_n$ . Clearly  $V = V_n \oplus V'_n$  and  $T(R) = Q/V \cong \bar{Q}_n \oplus \bar{Q}'_n/V \cong \bar{Q}'_n/V_n \oplus \bar{Q}'_n/V'_n$ . But Lemma 4(i) shows  $\bar{Q}_n/V_n \cong T(R_n)$  and Lemma 4(ii) shows  $\bar{Q}'_n/V'_n \cong T(R'_n)$ . Thus  $T(R) \cong T(R_n) \oplus T(R'_n) \cong T(R_1) \oplus \cdots \oplus T(R_{n-1}) \oplus T(R_n)$ .

Theorem 2 enables us to reduce the study of  $T(R)$  to rings with essential central kernel.

**THEOREM 3.** *Let  $R$  be a semiprime P.I.-ring of degree  $n$  with essential central kernel. Then  $T(R)$  is an Azumaya algebra of rank  $n^2$  and  $T(C) \cong \text{center}(T(R))$ .*

*Proof.* By Lemma 3(iii),  $N_n = 0$ . Hence  $T(R)$  is a homomorphic image of  $\Pi\{Q_i: \lambda \in A_n\}$ . Therefore,  $T(R)$  is Azumaya of rank  $n^2$ . Write  $C_\lambda = \text{center } Q_\lambda$  for  $\lambda \in A$ . Since  $\Pi\{Q_i: \lambda \in A_n\}$  is an Azumaya algebra of rank  $n^2$ , we have the following fact which we will need later,  $\text{cent}[(\Pi_{\lambda \in A_n} Q_\lambda)/(V \cap \Pi_{\lambda \in A_n} Q_\lambda)] = (\Pi_{\lambda \in A_n} C_\lambda + V \cap \Pi_{\lambda \in A_n} Q_\lambda)/(V \cap \Pi_{\lambda \in A_n} Q_\lambda)$ .

We claim that the homomorphism  $\varphi: (\Pi_{\lambda \in A} Q_\lambda)/V \rightarrow (\Pi_{\lambda \in A_n} Q_\lambda)/(V \cap \Pi_{\lambda \in A_n} Q_\lambda)$ , induced by the projection,  $\Pi_{\lambda \in A} Q_\lambda \rightarrow \Pi_{\lambda \in A_n} Q_\lambda$ , is an isomorphism. Indeed, suppose that  $0 \neq x + V$  for  $x$  in  $\Pi_{\lambda \in A} Q_\lambda$ . Then  $\bigcap \{P_i: \lambda \in W_x\}$  is not essential. Since each prime of degree  $< n$  contains  $I$  and  $I \subseteq \bigcap \{P_i: \lambda \in W_x \cap (A - A_n)\}$  is essential, we conclude that  $\bigcap \{P_i: \lambda \in W_x \cap A_n\}$  is not essential and  $0 \neq x + (V \cap \Pi_{\lambda \in A_n} Q_\lambda)$ . Consequently  $\varphi$  is an isomorphism.

Now by Rowen [12, Theorem 3] there exists a 1:1 correspondence of  $\{P_i: \lambda \in A_n\}$  and the set of prime ideals of  $C$ , not containing  $I$ , given by  $P_\lambda \rightarrow P_\lambda \cap C$ . We claim that  $T(C) \cong (\Pi_{\lambda \in A_n} C_\lambda)/(V \cap \Pi_{\lambda \in A_n} C_\lambda)$ . The proof of this is similar to the one in the preceding paragraph because every prime in  $C$  which is not in  $\{P_\lambda \cap C: \lambda \in A_n\}$  contains  $I$  which is essential in  $C$ .

Finally we have all the requisite pieces to obtain

$$\begin{aligned}
T(C) &\cong (\Pi_{\lambda \in A_n} C_\lambda) / (V \cap \Pi_{\lambda \in A_n} C_\lambda) \\
&\cong (\Pi_{\lambda \in A_n} C_\lambda + V \cap \Pi_{\lambda \in A_n} Q_\lambda) / (V \cap \Pi_{\lambda \in A_n} Q_\lambda) \\
&\cong \text{cent} [(\Pi_{\lambda \in A_n} Q_\lambda) / (V \cap \Pi_{\lambda \in A_n} Q_\lambda)] \\
&\cong \text{cent} ((\Pi_{\lambda \in A} Q_\lambda) / V) = \text{cent} (T(R)) .
\end{aligned}$$

REMARK 1. Given  $\mathcal{P}$  as in §2, let  $\varphi: Q \rightarrow T(R, \mathcal{P})$  be the canonical homomorphism. Then there is a partial order on {ideals  $A$  of  $Q$ :  $\text{Ker } \varphi \subseteq A$  and  $R \cap A = 0$ }. So there exists a maximal such ideal  $\bar{A}$ . Then  $Q/\bar{A} \cong T(R, \mathcal{P})/(\bar{A}/(\text{Ker } \varphi))$  is an extension of  $R$  which has all the aforementioned properties of  $T(R, \mathcal{P})$ , and, moreover, any ideal of  $Q/\bar{A}$  intersects  $R$  (viewed as a subring) nontrivially.

REMARK 2. Suppose that  $R$  has an involution  $(*)$ . Then, for any prime  $P$  of degree  $j$ , there is a prime  $P^*$  of degree  $j$  and an isomorphism  $R/P \rightarrow R/P^*$  given by  $r + P \rightarrow r^* + P^*$ . This isomorphism extends to the algebra of central quotients, and one can check that in the definition of  $T(R)$ , an involution is induced in  $Q$ . Moreover,  $V$  is stable under this involution, so  $T(R)$  inherits an involution which coincides with  $(*)$  on  $R$ . Hence the embedding  $R \rightarrow T(R)$  is actually an embedding in the category of rings with involution.

## REFERENCES

1. S. A. Amitsur, *Prime rings having polynomial identities with arbitrary coefficients*, Proc. London Math. Soc., (3), **17** (1967), 470-486.
2. M. Artin, *On Azumaya algebras and finite dimensional representations of rings*, J. Algebra, **11** (1969), 532-563.
3. J. W. Fisher, *Structure of semiprime P.I.-rings*, Proc. Amer. Math. Soc., **39** (1973), 465-467.
4. ———, *Structure of semiprime P.I.-rings, II*, 1971, (unpublished).
5. E. Formanek, *Central polynomials for matrix rings*, J. Algebra, **23** (1972), 129-132.
6. Frayne, Morel, and Scott, *Reduced direct products*, Fund. Math., **51**, 214ff.
7. A. W. Goldie, *A note on prime rings with polynomial identities*, J. London Math. Soc., (2), vol. 1 (1969), 606-608.
8. I. N. Herstein, and L. W. Small, *Regular elements in P.I.-rings*, Pacific J. Math., **36** (1971), 327-330.
9. W. S. Martindale, III, *On semiprime P.I.-rings*, Proc. Amer. Math. Soc., **40** (1973), 364-369.
10. C. Procesi, *On a theorem of M. Artin*, J. Algebra, **22** (1972), 309-316.
11. L. Rowen, *Some results on the center of a ring with polynomial identity*, Bull. Amer. Math. Soc., **79** (1973), 219-222.
12. ———, *On rings with central polynomials*, J. Algebra, (to appear).

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