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**PROJECTIVE PSEUDO-COMPLEMENTED SEMILATTICES**

GEOFFREY THOMAS JONES

# PROJECTIVE PSEUDO COMPLEMENTED SEMILATTICES

G. T. JONES

This paper is concerned with the properties of free, and projective pseudo complemented semilattices (PCSL).

It is proved that a projective PCSL is complemented and all its chains and disjointed subsets are countable, and that a Boolean algebra is projective in the category of PCSL if and only if it is projective in the category of Boolean algebras. Further, necessary and sufficient conditions are established for a finite PCSL to be projective.

1. Preliminaries. A *semilattice*  $A$  is a partially ordered set closed under meets. If  $A$  has a least element we will denote it by  $0$ . We say that  $a^*$  is the *pseudo complement* of  $a \in A$ ,  $A$  a semilattice with  $0$ , if we have (i)  $a \cdot a^* = 0$ , (ii) If  $ab = 0$  then  $b \leq a^*$ , for  $b \in A$ . Clearly pseudo complements are unique when they exist. A semilattice with  $0$  called a *pseudocomplemented semilattice* (PCSL) if each element has a pseudo-complement. A PCSL has a greatest element,  $0^*$ , which we denote by  $1$ . A function  $f: A \rightarrow B$ ,  $A, B$  PCSL's, is called a homomorphism if  $f(ab) = f(a) \cdot f(b)$ ,  $f(a^*) = f(a)^*$  for  $a, b \in A$ . We observe that  $f(0) = 0$ , and  $f(1) = 1$ . For  $S \subseteq A$  let  $S^* = \{x^*: x \in S\}$ .

It is easily shown that the following identities are true in any PCSL.

- |  |   |
|--|---|
| (1) $xy = yx$                                  | (13) $(xy)^* = (x^{**}y^{**})^*$                  |
| (2) $x(yz) = (xy)z$                            | (14) $x^*y^{**} = 0 \leftrightarrow x^*y^* = x^*$ |
| (3) $xx = x$                                   | (15) $xy = 0 \leftrightarrow x \leq y^*$          |
| (4) $0 \cdot x = 0$                            | (16) $x(xy)^* = xy^*$                             |
| (5) $x(xy)^* = xy^*$                           | (17) $x(x^*y)^* = x$                              |
| (6) $x0^* = x$                                 | (18) $x^*(xy)^* = x^*$                            |
| (7) $0^{**} = 0$                               | (19) $x^*(x^*y)^* = x^*y^*$                       |
| (8) $x \leq x^{**}$                            | (20) $x^{**}(x^*y)^* = x^{**}$                    |
| (9) $x \leq y \rightarrow y^* \leq x^*$        | (21) $x^{**}(xy)^* = x^{**}y^*$                   |
| (10) $x \leq y \rightarrow x^{**} \leq y^{**}$ | (22) $(xy)^*(xy^*)^* = x^*$                       |
| (11) $x^{***} = x^*$                           | (23) $(xy)^{**} = x^{**}y^{**}$                   |
| (12) $x^*y^* = (x^*y^*)^{**}$                  |   |

The definitions of the concepts discussed in this paper may be found in References 3, 4, 5, and 7.

## 2. Free PCSL.

LEMMA 2.1. *Let  $X$  freely generate the PCSL  $F$ . Then*

- (1)  $0 \notin X, 1 \notin X$ .
- (2) If  $S \subseteq X$ ,  $S$  finite then  $\Pi(S) \neq 0$ .
- (3) If  $S \subseteq X^*$ ,  $S$  finite then  $\Pi(S) \neq 0$ .
- (4) If  $x \in X$  then  $x \neq x^{**}$ .
- (5) If  $x_1, x_2 \in X$ ,  $x_1 \neq x_2$ , then  $x_1^* \neq x_2^*$ ,  $x_1^{**} \neq x_2^{**}$ .
- (6)  $S \subseteq X$ , then  $|S| = |S^*| = |S^{**}|$ .
- (7)  $x_1, x_2 \in X$  and  $x_1 \leq x_2$  then  $x_1 = x_2$ .
- (8) If  $x \geq \Pi(T)$ ,  $T \subseteq X$  then  $x \in T$ , where  $x \in X$ .

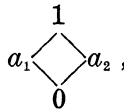
*Proof.* (1) If  $0 \in X$ , then let  $f$  be a homomorphism  $f: F \rightarrow 2$  such that  $f(0) = 1$ . But  $f(0) = f(a \cdot a^*) = f(a) \cdot f(a)^* = 0$ . Thus  $0 = 1$  in  $2$ , a contradiction. Also, suppose  $1 \in X$ . Let  $g$  be a homomorphism  $g: F \rightarrow 2$  such that  $g(1) = 0$ . But  $1 = 0^* = g(1)^* = g(0) = 0$ , again, a contradiction.

(2) Suppose  $\Pi(S) = 0$ .  $S$  finite,  $S \subseteq X$ . Then there is a homomorphism  $f: F \rightarrow 2$  so that  $f(x) = 1$  all  $x \in S$ . Thus  $1 = f(\Pi(S)) = f(0) = 0$  a contradiction.

(3) Suppose  $\Pi(S) = 0$ ,  $S$  finite  $S \subseteq X^*$ . Then there is a homomorphism  $f: F \rightarrow 2$  so that  $f(x) = 0$  for all  $x^* \in S$ . Thus  $1 = f(\Pi(S)) = f(0) = 0$  a contradiction.

(4) Suppose  $x = x^{**}$ . Then there is a homomorphism  $f: F \rightarrow 3$ , the 3 element chain  $3 = (0, a, 1)$  such that  $f(x) = a$ . But  $a^{**} = f(x^{**}) = f(x) = a$  is false since  $a^{**} = 0$  in  $3$ .

(5) Let  $x_1 \neq x_2$  and suppose  $x_1^* = x_2^*$ . Since  $F$  is free let  $f$  be the homomorphism from  $F$  onto the boolean algebra, such that  $x_1 \rightarrow a_1$ ,



and  $x_2 \rightarrow a_2$ . Since  $x_1^* = x_2^*$  then  $a_1^* = a_2^*$ . That is  $a_2 = a_1$ , a contradiction. Thus  $x_1^* \neq x_2^*$ .

If  $x_1^{**} = x_2^{**}$  then we have  $x_1^{***} = x_2^{***}$ , i.e.,  $x_1^* = x_2^*$ , a contradiction. Thus  $x_1^{**} \neq x_2^{**}$ .

(6) Let  $S \subseteq X$ . Then  $S^* = \{x^*: x \in S\}$ . Let  $f: S \rightarrow S^*$  be defined by  $f(x) = x^*$ . Clearly  $f$  is onto. Suppose  $f(x_1) = f(x_2)$ , i.e.,  $x_1^* = x_2^*$ .  $\therefore x_1 = x_2$ , i.e.,  $f$  is 1-1. Thus  $|S| = |S^*|$ . Also let  $g: S \rightarrow S^{**}$  be defined by  $g(x) = x^{**}$ . If  $g(x_1) = g(x_2)$  hence  $x_1^{**} = x_2^{**}$  and  $x_1 = x_2$ , i.e.,  $g$  is 1-1. Thus  $|S| = |S^{**}|$ .

(7) Suppose  $x_1 \neq x_2$ . Let  $f: F \rightarrow 2$  be a homomorphism such that  $f(x_1) = 1$  and  $f(x_2) = 0$ . But since  $x_1 < x_2$  thus  $1 \leq 0$  — a contradiction.

(8) Let  $x \geq \Pi(T)$  and suppose  $x \notin T$ . Let  $f: F \rightarrow 2$  be a homomorphism such that  $f(x) = 0$  and  $f(x_i) = 1$ ,  $x_i \in T$ . Then we have  $1 \leq 0$  — a contradiction.

LEMMA 2.2. *If  $A$  is a PCSL then  $A^*$  is a retract of  $A$ .*

*Proof.*  $\varphi: A \rightarrow A^*$  defined by  $\varphi(x) = x^{**}$  is a homomorphism onto  $A^*$ . If  $x \in A^*$  then  $\varphi(x) = x$ , since  $x^{***} = x^*$ . Hence,  $A^*$  is a retract of  $A$ .

THEOREM 2.1. *If  $F$  is a PCSL freely generated by  $X$ , then  $F^*$  is freely Boolean generated by  $X^{**}$ ; i.e.,  $F^*$  is free in the class of boolean algebras.*

*Proof.* Let  $\varphi: F \rightarrow F^*$  the homomorphism  $\varphi(x) = x^{**}$ , and let  $\psi: F^* \rightarrow F$  be the inclusion map. Then  $\varphi\psi = I_{F^*}$ . Let  $X = \{x_i: i \in I\}$  and let  $B$  be any Boolean algebra and suppose  $b_i \in B$ , for  $i \in I$  then there exists a homomorphism  $f: F \rightarrow B$  such that  $f(x_i) = b_i$ . Let  $h = f\psi: F^* \rightarrow B$ . Then  $h(x_i^{**}) = f(x_i^{**}) = b_i^{**} = b_i$ . Also we note that  $h$  is a Boolean homomorphism.

THEOREM 2.2. *Let  $A$  be any free PCSL and let  $X$  freely generate  $A$ . Then every element of  $A$  is of the form  $\Pi(T) \cdot (\Pi(P_1))^* \cdots (\Pi(P_n))^*$ , where  $T \subseteq X$ ,  $P_i = R_i \cup S_i^*$ ,  $R_i \cup S_i \subseteq X$ ,  $R_i \cap S_i = \emptyset$ ,  $P_i$  finite for  $i = 1, 2, \dots, n$ ,  $n \geq 0$ , using the convention that  $\Pi(\emptyset) = 1$ .*

*Proof.* Let  $B = \{\Pi(T) \cdot r: T \subseteq X, r \in A^*, T \text{ finite}\}$ . Then  $B$  is a subalgebra of  $A$ , since  $0 \in B$ , and  $B$  is closed under meets. Also if  $b \in B$ , then  $b^* \in A^*$ , and thus  $b^* \in B$ . Further, we note that  $X \subseteq B$ , hence  $B = A$ . Since the homomorphism  $\varphi: A \rightarrow A^*$  given by  $\varphi(x) = x^{**}$  is onto, then  $A^*$  is freely Boolean generated by  $X^{**}$ . Hence any element  $r \neq 1$  of  $A^*$  is a product of elements of the form  $\alpha = \sum_{A^*} (U \cup V^*)$  where  $U$  and  $V$  are finite disjoint subsets of  $X^{**}$ . But  $U = S^{**}$  and  $V = R^{**}$  for some  $R, S$  subsets of  $X$ . Clearly  $R \cap S = \emptyset$  and  $V^* = R^*$ . Hence

$$\alpha = \sum_{A^*} (S^{**} \cup R^*) = (\Pi(S^* \cup R^{**}))^* = (\Pi(R \cup S^*))^*,$$

by [2, Theorem. 2] and (13) of § 1. Since  $x(xy)^* = xy^*$ ,  $x(x^*y)^* = x$ , ((16), (17) of § 1) we may assume that  $T \cap R_i = T \cap S_i = \emptyset$  for all  $i \leq n$ .

THEOREM 2.3. *Let  $X, Y$  freely generate a PCSL  $F$ . Then  $X = Y$ .*

*Proof.* Let  $x \in X$ . Then  $x = \Pi(T) \cdot r$  where  $\emptyset \neq T \subseteq Y$  and  $r \in F^*$ . Then  $x \leq y_i$  for all  $y_i \in T = \{y_1, y_2, \dots, y_n\}$ . Also,  $y_i = \Pi(T_i) \cdot r_i$  for  $\emptyset \neq T_i \subseteq X$  and  $r_i \in F^*$ . Hence  $x = \Pi(T) \cdot r = \Pi(\bigcup T_i)(\Pi r_i) \cdot r$  from which we see that  $x \leq \Pi(\bigcup T_i)$ , and conclude that  $\bigcup T_i = \{x\}$ ,

using Lemma 2.1(7). Hence  $y_i = x \cdot r_i$  and thus  $y_i \leq x$  and hence  $x = y_i$ , i.e.,  $x \in Y$ . Thus  $X \subseteq Y$ , and by a similar argument  $Y \subseteq X$ .

**LEMMA 2.3.** *Suppose  $X$  freely generates a PCSL  $F$ , and let  $x \in X$ ,  $R \cup S \cup T \subseteq X$ ,  $R \cup S \cup T$  finite. If  $0 \neq \Pi(T \cup R^{**} \cup S^*) \leq x$ , then  $x \in T$ .*

*Proof.* Since  $0 \neq \Pi(T \cup R^{**} \cup S^*)$ , then  $T \cap S = R \cap S = \emptyset$ . Clearly  $x \notin S$ . Suppose  $x \notin T$ . Let  $f: F \rightarrow F$  be a homomorphism such that

$$f(y) = \begin{cases} 1 & \text{if } y \in R \cup T - \{x\} \\ x & \text{if } y = x \\ 0 & \text{if } y \in S. \end{cases}$$

This is possible since  $X$  freely generates  $F$ . Then

$$f(\Pi(T \cup R^{**} \cup S^*)) = \begin{cases} 1 & \text{if } x \notin R \\ x^{**} & \text{if } x \in R. \end{cases}$$

Hence  $1 \leq x$  or  $x^{**} \leq x$ , so  $x = 1$ , or  $x = x^{**}$ . But this is impossible by Lemma 2.1, and the result follows.

**LEMMA 2.4.** *Let  $X$  freely generate  $F$ , and  $T \subseteq X$ , and  $r \in F^*$ ,  $x \in X$ . If  $0 < \Pi(T) \cdot r \leq x$ . Then  $x \in T$ .*

*Proof.*  $r$  is a sum in  $F^*$  of elements of the form  $\Pi(R^{**} \cup S^*)$ , where  $R \cup S \subseteq X$ ,  $R \cap S = \emptyset$ . Hence for some  $R$  and  $S$  we have  $0 < \Pi(T \cup R^{**} \cup S^*) \leq x$ . Then by Lemma 2,  $x \in T$ .

**THEOREM 2.4.** *Let  $X$  freely generate a PCSL  $F$ . Then the elements of  $X$  are super-meet irreducible. That is, let  $a_1, a_2, \dots, a_n \in F$ ,  $x \in X$ , and  $0 < a_1 a_2 \dots a_n \leq x$ , then  $a_i \leq x$  for some  $i$ .*

*Proof.* For each  $i$ ,  $a_i = \Pi(P_i) \cdot r_i$ ,  $P_i \subseteq X$ ,  $r_i \in F^*$ . Hence  $0 < \Pi(P_1 \cup \dots \cup P_n) \cdot r_1 \dots r_n \leq x$ , then by Lemma 2.5  $x \in P_1 \cup \dots \cup P_n$  and thus  $x \in P_i$  for some  $i$ . Therefore  $a_i \leq x$ .

**LEMMA 2.5.** *Let  $X$  freely generate  $F$ , and  $a \in F$ ,  $r \in F^*$ . If  $0 < r < a$ , then  $a \in F^*$ .*

*Proof.* Suppose  $a \notin F^*$  then  $a \leq x$ , for some  $x \in X$ . Hence  $0 < r \leq x$ . But  $r$  is a sum (in  $F^*$ ) of elements of the form  $\Pi(R^{**} \cup S^*)$ , where  $R \cup S \subseteq X$ ,  $R \cap S = \emptyset$ . Hence for some such  $R, S$ , we have,  $0 < \Pi(R^{**} \cup S^*) = \Pi(\emptyset \cup R^{**} \cup S^*) \leq x$ , and then by Lemma 2.4 we

have  $x \in \emptyset$ , a contradiction.

LEMMA 2.6. *Let  $X$  freely generate  $F$ , and  $a \in F$ . If  $a^* = 0$ , then  $a = 1$ .*

*Proof.* We have  $a = \Pi(T) \cdot r$  where  $T \subseteq X$ ,  $r \in F^*$ . Since  $a^* = 0$ , then  $1 = a^{**} = \Pi(T)^{**} \cdot r \leq r$ , thus  $r = 1$ . Hence  $a = \Pi(T)$ . If  $T \neq \emptyset$  then  $a \leq x$  for some  $x \in X$ . Thus  $1 \leq x^{**}$ . But this is impossible by Lemma 2.1(3).

THEOREM 2.5. *If  $F$  is a free PCSL, then  $F$  is complemented, i.e., if  $a \in F$ , the  $a + a^*$  exists and equals 1.*

*Proof.* Suppose  $a \leq b$ ,  $a^* \leq b$ , then  $b^* \leq a^*a^{**} = 0$ . Hence  $b = 1$  by Lemma 2.6.

THEOREM 2.6. *Let  $F$  be a free PCSL.*

(1) *Let  $S \subseteq F^*$ ,  $S$  finite, and  $a = \sum_{F^*}(S)$ . Then  $\sum_F(S)$  exists and equals  $a$ .*

(2)  *$a^* + b^* = (ab)^*$  for  $a, b \in F$ .*

*Proof.* (1) Clearly true if  $S = \{0\}$ .

We may assume  $S \neq \{0\}$ . Now  $a \geq s$  for all  $s \in S$ . If  $b \in F$  and  $b \geq s$  all  $s \in S$ , then  $b \in F^*$  by Lemma 2.6 and thus  $b \geq a$ . Thus  $\sum_F(S)$  exists and equals  $a$ .

$$\begin{aligned} (2) \quad a^* + {}_F b^* &= a^* + {}_{F^*} b^* \\ &= (a^{**}b^{**})^* \text{ since } F^* \text{ is a Boolean algebra} \\ &= (ab)^* \quad \text{by (13) of § 1.} \end{aligned}$$

LEMMA 2.7. *Let  $F$  be a free PCSL and  $r \in F^*$ . Then  $\{a \in F: a^{**} = r\}$  is finite.*

*Proof.* By Lemma 2.6,  $a^* = 0$  iff  $a = 1$ , and in any PCSL  $a^* = 1$  iff  $a = 0$ . Hence we may assume  $0 < r < 1$ . Let  $X$  freely generate  $F$ . By Theorem 2.2 there exists a finite subset  $X_1$  of  $X$  such that  $r \in F_1$ , the algebra generated by  $X_1$ . Now  $F_1$  is finite. We need only show that if  $a^{**} = r$ , then  $a \in F_1$ . If  $a \in F^*$ , then  $a = a^{**} = r \in F_1$ . Now suppose  $a \notin F^*$ . Then  $a = \Pi(T) \cdot s$ , where  $\emptyset \neq T \subseteq X$ , and  $s \in F^*$ . Further, from Theorem 2.2 we may assume that  $s$  is in the subalgebra generated by a subset of  $X$  which is disjoint from  $T$ . If  $T \not\subseteq X_1$ , then there exists an element  $x \in T - X_1$ . Let  $f: F \rightarrow F$  be a homomorphism such that  $f(x) = 0$ , and  $f(y) = y$ , for all  $y \in X - \{x\}$ . Then  $f(a) = 0$  and hence,  $0 = f(a^{**}) = f(r)$ . But  $f(r) = r$  since

$x \notin X_1$ . Then  $r = 0$ , a contradiction. This proves that  $T \subseteq X_1$ , and so  $\Pi(T) \in F_1$ . Let  $g: F \rightarrow F$  be a homomorphism such that  $g(y) = 1$  for all  $y \in T$ , and  $g(y) = y$  for  $y \in X - T$ . Then  $g(s) = s$ . Hence  $s = g(s) = g(s \cdot \Pi(T)^{**}) = g(a^{**}) = g(r)$ . But by definition of  $F_1$ , and  $g, g(r) \in F_1$ . Thus  $s \in F_1$  and hence  $a = \Pi(T) \cdot s \in F_1$ .

**COROLLARY 2.1.** *Let  $F$  be a free PCSL and let  $r \in F^*$ , then  $\{a \in F: a^* = r\}$  is finite.*

*Proof.*  $\{a \in F: a^* = r\} = \{a \in F: a^{**} = r^*\}$  which is finite.

**COROLLARY 2.2.** *Let  $F$  be an infinite free PCSL and let  $S \subseteq F$ ,  $S$  infinite. Then,  $|S^*| = |S|$ . Proof is clear.*

**THEOREM 2.7.** *If  $B$  is a free Boolean algebra, then there exists a free PCSL  $F$  such that  $F^* = B$ .*

*Proof.* Let  $X \subseteq B$ , freely Boolean generate  $B$ . Let  $F$  be the free PCSL on a set  $S$  of  $|X|$  free generators. Then  $F^*$  is a free Boolean algebra freely generated by  $S^{**}$ . Since  $|X| = |S| = |S^{**}|$ , by Lemma 2.1(6), then  $F^* = B$ .

**LEMMA 2.8.** *Every free Boolean algebra is a retract (in the category of PCSL) of a free PCSL.*

*Proof.* Let  $B$  be a free Boolean algebra. By Theorem 2.7, there exists a free PCSL  $F$  such that  $F^* = B$ . But  $F^*$  is a retract of  $F$ , hence  $B$  is a retract of  $F$ .

**THEOREM 2.8.** *In a free PCSL, all chains are countable.*

*Proof.* Let  $F$  be a free PCSL, and let  $C = \{a_i \in F: i \in I\}$  be an infinite chain. Then  $C^*$  is an infinite chain in  $F^*$  a free Boolean algebra. But chains in  $F^*$  are countable [6], and since  $|C| = |C^*|$ , hence  $C$  is a countable chain.

**THEOREM 2.9.** *All disjointed subsets of a free PCSL are countable.*

*Proof.* Let  $S$  be an infinite disjointed subset of  $F$ , a free PCSL. Now  $|S| = |S^{**}|$ . Also  $a^{**}b^{**} = (ab)^{**} = 0^{**} = 0$ , for  $a, b \in S$ . Thus  $S^{**}$  is a disjointed subset of  $F^*$ . But in a free Boolean algebra all disjointed sets are countable [7, p. 51], hence  $S$  is countable.

### 3. Projective PCSL.

**THEOREM 3.1.**  *$B$ , a Boolean algebra is projective in the category of boolean algebras, iff it is projective in the category of PCSL.*

*Proof.*  $B$  is a retract of a free Boolean algebra  $\bar{B}$ . By Theorem 2.7 there exists a free PCSL  $F$  such that  $\bar{B} = F^*$ , and thus  $\bar{B}$  is a retract of  $F$  in the category of PCSL. Hence  $B$  is a retract of  $F$  in the category of PCSL and thus  $B$  is projective. Conversely, let  $B$  be a Boolean algebra which is projective in the category of PCSL. Thus there is a free PCSL  $F$  such that  $B$  is a retract of  $F$ . Then by Lemma 2, it follows that  $B$  is a retract of  $F^*$  in the category of Boolean algebras, and the result follows.

**REMARK.** The definition of projectivity makes it clear that the results of the preceding section following Theorem 2.4, hold for projective PCSL.

### 4. Finite projective PCSL.

**DEFINITION 4.1.** If  $P$  is a partially ordered set and  $M \subseteq P$ ,  $p \in P$ , let  $M_p = \{m \in M: m \geq p\}$ .

**DEFINITION 4.2.** Let  $P$  be a finite partially ordered set and let  $M$  be the set of maximal elements of  $P$ . Then a semi-lattice  $A$  with least element, 0, is said to be *freely generated by  $P$  with the defining relation  $\Pi(M) = 0$*  if there is an order preserving function  $\theta: P \rightarrow A$  such that  $\Pi(\theta(M)) = 0$ ,  $\theta(P)$  generates  $A$ , and such that if  $B$  is any semi-lattice with 0, and  $h: P \rightarrow B$  is any order preserving function such that  $\Pi(h(M)) = 0$ , then there exists a semi-lattice homomorphism  $g: A \rightarrow B$  such that  $g(0) = 0$ , and  $g\theta = h$ . The existence of  $A$  is guaranteed by a known theorem of universal algebra.  $A$  is unique up to isomorphism. See [4, p. 182, 183].

**LEMMA 4.1.** *Let  $P$  be a finite partially ordered set and  $M$  be the set of maximal elements of  $P$ . Suppose for each  $p \in P - M$  we have  $M_p \neq M$ . Let  $A$  be a semi-lattice with 0 freely generated by  $P$  with defining relation  $\Pi(M) = 0$  and let  $\theta: P \rightarrow A$  be an in Definition 4.2. Then,*

(a)  *$\theta$  is an order isomorphism. (So we may consider  $P$  contained in  $A$ , and  $\theta$  as the inclusion function.)*

(b) *If  $p_1, \dots, p_n \in P$  then  $p_1 p_2 \dots p_n = 0$  iff  $\bigcup \{M_{p_i}: i \leq n\} = M$ .*

(c) *If  $p, p_1, \dots, p_n \in P$  and  $0 < p_1 p_2 \dots p_n \leq p$  then  $p_i \leq p$ , some  $i$ .*



(d)  $P$  is the set of meet irreducible elements of  $A$ .

*Proof.* (a) If  $S \subseteq P$  let  $M(S) = \bigcup \{M_p: p \in S\}$ ,  $m(S)$  be the set of minimal elements of  $S$ . Define

$$B = \{M\} \cup \{S: \emptyset \neq S \subseteq P, M(S) \neq M \text{ and for } x, y \in S, x \neq y \rightarrow x \parallel y\}$$

where  $a \parallel b$  means  $a \not\leq b$  and  $b \not\leq a$ . For  $S_1, S_2 \in B$ , define

$$S_1 \cdot S_2 = \begin{cases} M & \text{if } M(S_1 \cup S_2) = M \\ m(S_1 \cup S_2) & \text{if } M(S_1 \cup S_2) \neq M. \end{cases}$$

Then  $S_1 \cdot S_2 = S_2 \cdot S_1$ ,  $S_1 \cdot S_1 = S_1$ ,  $S_1 \cdot M = M$  for any  $S_1, S_2 \in B$ . It is easy to verify that

$$S_1 \cdot (S_2 \cdot S_3) = \begin{cases} M & \text{if } M(S_1 \cup S_2 \cup S_3) = M \\ m(S_1 \cup S_2 \cup S_3) & \text{if } (S_1 \cup S_2 \cup S_3) \neq M. \end{cases}$$

Therefore,  $(S_1 \cdot S_2) \cdot S_3 = S_3 \cdot (S_1 \cdot S_2) = S_1 \cdot (S_2 \cdot S_3)$ , and thus  $B$  is a semi-lattice with smallest element  $M$  if we define  $S_1 \leq S_2$  whenever  $S_1 \cdot S_2 = S_1$ . Note that  $S_1 \leq S_2$  iff either  $S_1 = M$ , or for any  $x \in S_2$  there exists  $y \in S_1$  such that  $x \geq y$ . Define  $h: P \rightarrow B$  by  $g(p) = \{p\}$  for  $p \in P$ . If  $p_1 \leq p_2$  then  $\{p_1\} \leq \{p_2\}$ . Also,  $\Pi(h(M)) = \Pi(\{m\}: m \in M) = M$ . Thus there exists a homomorphism  $g: A \rightarrow B$  such that  $g\theta = h$ . If  $\theta(p_1) \leq \theta(p_2)$ , then  $h(p_1) \leq h(p_2)$ . But  $\{p_1\} \leq \{p_2\}$  implies  $p_1 \leq p_2$  or  $M_{p_1} = M$ . If  $M_{p_1} = M$  then  $p_1 \in M$  and hence  $M = P = \{p_1\}$ , so  $p_1 = p_2$ . Therefore,  $\theta$  is an order isomorphism. Henceforth we may assume  $P \subseteq A$  and  $\theta(p) = p$  for all  $p \in P$ .

(b) If  $p_1 p_2 \cdots p_n = 0$ , then  $\{p_1\} \cdots \{p_n\} = M$ . Therefore,  $M = M(\{p_1, \cdots, p_n\}) = \bigcup \{M_{p_i}: i \leq n\}$ . If  $\bigcup \{M_{p_i}: i \leq n\} = M$  then  $p_1 p_2 \cdots p_n \leq \Pi(M) = 0$ .

(c) Suppose  $0 < p_1 \cdots p_n \leq p$ . Then  $\{p_1\} \cdots \{p_n\} \leq \{p\}$  and  $\{p_1\} \cdots \{p_n\} \neq M$ . Therefore,  $p \geq p_i$  for some  $i$ .

(d) Since  $P$  generates  $A$ , every element of  $A$  is a product of elements of  $P$ . Therefore, any meet irreducible element of  $A$  is in  $P$ . Conversely if  $p \in P$ , and  $p \neq 0$  then  $p$  is meet irreducible by (c) and the fact that  $P$  generates  $A$ . If  $0 \in P$  then  $0 \in M$  because  $M_p \neq M$  for  $p \in P - M$ , thus  $P = \{0\}$  and  $A = \{0\}$  and thus (d) is proved.

**LEMMA 4.2.** Let  $A$  be a finite semi-lattice,  $P$  be the set of meet irreducible elements of  $A$ , and  $M$  the set of maximal elements of  $P$ . If

(a) If  $p_1, \cdots, p_n \in P$  then  $p_1 \cdots p_n = 0$  iff  $\bigcup \{M_{p_i}: i \leq n\} = M$ .

(b) If  $p, p_1, \cdots, p_n \in P$  and  $0 < p_1 \cdots p_n \leq p$  then  $p_i \leq p$  some  $i$ .

Then for each  $p \in P - M$ ,  $M_p \neq M$  and  $A$  is freely generated by  $P$  with defining relation  $\Pi(M) = 0$ .

*Proof.* If  $p \in P - M$  and  $M_p = M$ , then by (a)  $p = 0$ . But  $\Pi(M) = 0$  by (a) hence  $p = \Pi(M)$  which contradicts the fact that  $P$  is meet irreducible. If  $x \in A$ ,  $x \neq 0$ , then  $x = \Pi(S)$  for some  $S \subseteq P$ , and we may assume the elements of  $S$  to be pairwise incomparable. If  $x \in \Pi(S')$ ,  $S' \subseteq P$  and the elements of  $S'$  incomparable, then by (b) every member of  $S'$  is greater than or equal to a member of  $S$ , and vice-versa. Therefore  $S = S'$ . Thus for  $x \in A$  there exists a unique set  $S_x$  of incomparable elements of  $P$  such that  $x = \Pi(S_x)$ .

Suppose  $B$  is any semi-lattice with 0, and  $h: P \rightarrow B$  is an order preserving function such that  $\Pi(h(M)) = 0$ .

Define  $g: A \rightarrow B$  by  $g(x) = \Pi(h(S_x))$  for  $x \neq 0$  and  $g(0) = 0$ . To show  $g$  is a semi-lattice homomorphism, first note  $g(xy) = g(x) \cdot g(y) = 0$  if  $x = 0$ , or  $y = 0$ . Suppose  $x \neq 0$  and  $y \neq 0$ . If  $xy \neq 0$  then  $S_{xy} = m(S_x \cup S_y)$ , the set of minimal elements of  $S_x \cup S_y$ . Since  $g$  is order preserving we have  $g(x) \cdot g(y) = \Pi(h(S_x)) \cdot \Pi(h(S_y)) = \Pi(h(S_x \cup S_y)) = \Pi h(S_{xy}) = g(xy)$ .

If  $xy = 0$  then by (a)  $\bigcup \{M_p: p \in S_x\} \cup \bigcup \{M_p: p \in S_y\} = M$ . Therefore  $g(x) \cdot g(y) = \Pi(h(S_x \cup S_y)) \leq \Pi(h(M)) = 0 = g(xy)$ . Clearly  $g|P = h$ , and the proof is complete.

LEMMA 4.3. Let  $A$  be a finite semi-lattice with 1 and suppose  $A - \{1\}$  satisfies the hypothesis of Lemma 4.2. Then

(a)  $A$  is pseudo complemented and for each  $x \in A - \{1\}$ ,  $x^* = \Pi(M - M_x)$  and  $x^{**} = \Pi(M_x)$  where  $M_x, M$  as in Lemma 4.2.

(b)  $A^* - \{1\} = \{\Pi(S): S \subseteq M\}$ .

(c)  $M$  is the set of dual atoms of  $A$  which is also the set of dual atoms of  $A^*$ .

(d) If  $S \subseteq A^*$ , then  $\sum_A(S)$  exists and equals  $\sum_{A^*}(S)$ .

*Proof.* Firstly we show that if  $S \subseteq M$ ,  $m \in M$  and  $\Pi(S) \leq m$ , then  $m \in S$ . We prove this as follows: If  $\Pi(S) = 0$  then  $S = M$  by hypothesis (a), and thus  $m \in S$ . If  $\Pi(S) \neq 0$  then by hypothesis (b)  $m' \leq m$  for some  $m' \in S$ , but then  $m = m' \in S$ , so  $m \in S$ .

(a) Let  $x \in A - \{1\}$  and let  $y = \Pi(M - M_x)$ . Then

$$xy \leq \Pi(M_x) \cdot \Pi(M - M_x) = \Pi(M) = 0.$$

Now suppose  $xz = 0$  for some  $x \in A$ . Using the notation of the proof of Lemma 4.2 we have  $\Pi(S_x \cup S_z) = 0$ . Therefore by hypothesis (b),  $M = \bigcup \{M_p: p \in S_x \cup S_z\}$ . If  $m \in M - M_x$ , it follows that  $m \geq p$  for some  $p \in S_x \cup S_z$ . If  $p \in S_x$ , then  $m \geq x$  contradicting  $m \in M - M_x$ .

Therefore,  $p \in S_z$  and so  $m \geq p \geq z$ . Therefore,  $y = \Pi(M - M_x) \geq z$  and this proves that  $y = x^*$ .

Now

$$\begin{aligned} m \in M_y &\leftrightarrow y = \Pi(M - M_x) \leq m \\ &\leftrightarrow m \in M - M_x \text{ by hypothesis (a).} \end{aligned}$$

Therefore,  $M_y = M - M_x$  and  $x^{**} = y^* = \Pi(M - M_y) = \Pi(M_x)$ . This proves (a).

(b) By (a), every element of  $A^*$  is of the form  $\Pi(S)$  for some  $S \subseteq M$ . If  $m \in M$ , then  $m^{**} = \Pi(M_m) = m$ , hence  $M \subseteq A^*$ . This proves (b).

(c) If  $m \in M$  and  $m < x < 1$ ,  $x \in A$ , then  $m < p < 1$  for some  $p \in P$ . This is a contradiction and so  $m$  is a dual atom of  $A$ . If  $x$  is a dual atom of  $A$ , then  $x$  is meet irreducible and hence  $x \in M$ . By (b), the dual atoms of  $A^*$  are in  $M$ . Therefore,  $M$  is the set of dual atoms of  $A^*$ .

(d) By hypothesis (b) of Lemma 4.2, and (b) above, it is easy to see that if  $a \in A^*$ ,  $x \in A$ , and  $0 < a \leq x$  then  $x \in A^*$ . This implies (d) just as it did for a free PCSL, in the proof of Theorem 2.6.

REMARK. By Lemmas 4.2 and 4.3, the free finite PCSL  $F$  with  $n$  generators may be described as follows. Let  $P$  be the set

$$\{x_i: i \leq n\} \cup \{z_s: S \subseteq \{1, 2, \dots, n\}\},$$

and suppose  $x_i \leq z_s$  iff  $i \in S$ . Then  $F$  is the semi-lattice with 0 which is freely generated by  $P$  with defining relation  $\Pi\{z_s: S \subseteq \{1, \dots, n\}\} = 0$ .

THEOREM 4.1. Let  $A$  be a finite projective PCSL, let  $P$  be the set of meet irreducible elements of  $A - \{1\}$ , and  $M$  be the set of maximal elements of  $P$ . Then

(a) If  $S \subseteq P$ ,  $p \in P$ , and  $0 < \Pi(S) \leq p$ , then  $s \leq p$  for some  $s \in S$ .

(b) If  $S \subseteq P$ , then  $\Pi(S) = 0$  iff  $\bigcup \{M_s: s \in S\} = M$ .

(c)  $\bigcap \{M_p: p \in P - M\} \neq \emptyset$ .

*Proof.* As in the proof of Lemma 4.3, it is easy to see that  $M$  is the set of dual atoms of  $A$ .  $M$  is also the set of dual atoms of  $A^*$ . It follows that  $(P - M) \cap A^* = \emptyset$ .

(a) Let  $F$  be a PCSL freely generated by a set  $X$  such that  $|X| = |P|$ . Let  $h: X \rightarrow P$  be 1-1 and onto. Then there exists a homomorphism  $f: F \rightarrow A$  such that  $f|X = h$ . Since  $P$  generates  $A$ ,  $f$  is onto. Since  $A$  is projective, there exists a homomorphism  $g: A \rightarrow F$  such that  $fg = I_A$ . Let  $p \in P - M$  and  $x = h^{-1}(p)$ . Now  $g(p) = \Pi(T) \cdot r$

for some  $T \subseteq X$ ,  $r \in F^*$ . Hence  $p = fg(p) = \Pi(f(T)) \cdot f(r)$ . Since  $p$  is meet irreducible and  $p \notin A^*$ , it follows that  $p = f(y)$  for some  $y \in T$ . But  $p = f(x)$  and  $f|X$  is 1-1, therefore  $x = y \in T$  and so  $g(p) \leq \Pi(T) \leq x$ . We have therefore shown that for any  $p \in P - M$ ,  $g(p) \leq h^{-1}(p)$ .

Now suppose  $S \subseteq P$ ,  $p \in P - M$  and  $0 < \Pi(S) \leq p$ . Since  $g$  is 1-1,  $0 < \Pi(g(S)) \leq g(p) \leq h^{-1}(p)$ , so by Theorem 2.4,  $g(s) \leq h^{-1}(p)$  for some  $s \in S$ . Hence  $s = fg(s) \leq fh^{-1}(p) = p$ . This proves (a) for the case when  $p \notin M$ . If  $p \in M$  and  $\Pi(S) \leq p$  for some  $S \subseteq P$ , then  $\Pi(S^{**}) \leq p^{**} = p$ . Since  $p$  is super-meet irreducible in  $A^*$ , it follows that for some  $s \in S$ ,  $s \leq s^{**} \leq p$ , and so (a) holds.

(b) If  $S \subseteq P$  and  $\Pi(S) = 0$ , then for any  $m \in M$ ,  $\Pi(S) \leq m$  and so  $m \in M_s$  for some  $s \in S$ , by the preceding paragraph. This proves (b).

(c) We have shown that  $A$  satisfies the hypothesis of Lemmas 4.2 and 4.3. Therefore, for each  $x \in A - \{1\}$ ,  $x^* = \Pi(M - M_x)$ . Suppose  $\bigcap \{M_p: p \in P - M\} = \emptyset$ . Then

$$\begin{aligned} \Pi\{p^*: p \in P - M\} &= \Pi\{\Pi(M - M_p): p \in P - M\} \\ &= \Pi(\bigcup \{M - M_p: p \in P - M\}) \\ &= \Pi(M - \bigcap \{M_p: p \in P - M\}) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &= g(0) = g(\Pi\{p^*: p \in P - M\}) \\ &= \Pi\{g(p)^*: p \in P - M\} \\ &\geq \Pi\{h^{-1}(p)^*: p \in P - M\} \end{aligned}$$

since  $g(p) \leq h^{-1}(p)$  for all  $p \in P - M$ .

But this is impossible, because if  $T$  is any finite subset of  $X$ , then  $\Pi(T^*) \neq 0$  by Lemma 2.1.

**LEMMA 4.4.** *Suppose a PCSL  $A$  satisfied the hypotheses of Lemma 4.3. Let  $B$  be PCSL and  $g: A \rightarrow B$  is a semi-lattice homomorphism such that  $g(0) = 0$ ,  $g(p^{**}) = g(p)^{**}$  for all  $p \in P$ ,  $P$  the set of meet irreducible elements of  $A$ , and  $g(u^*) = g(u)^*$  for all  $u \in A^*$ . Then  $g$  is a PCSL homomorphism.*

*Proof.* Let  $x$  be any element of  $A$ . We first prove that  $g(x^{**}) = g(x)^{**}$ . We have  $x = \Pi\{p_i: i \leq n\}$  for some  $\{p_1, \dots, p_n\} \subseteq P$ . Hence

$$g(x^{**}) = g(\Pi\{p_i: i \leq n\}^{**}) = g(\Pi\{p_i^{**}: i \leq n\})$$

by (23) of § 1

$$\begin{aligned} &= \Pi\{g(p_i^{**}): i \leq n\} = \Pi\{g(p_i)^{**}: i \leq n\} \\ &= (\Pi\{g(p_i): i \leq n\})^{**} = g(\Pi\{p_i: i \leq n\})^{**} = g(x)^{**}. \end{aligned}$$

Let  $x \in A$  and let  $u = x^{**}$ , we have

$$g(x^*) = g(x^{***}) = g(u^*) = g(u)^* = g(x^{**})^* = g(x)^{***} = g(x)^*.$$

Hence  $g$  is a  $*$  homomorphism.

**THEOREM 4.2.** *Let  $A$  be a finite semi-lattice with 1, and let  $P$  be the meet irreducible element of  $A - \{1\}$  and  $M$  the maximal elements of  $P$ . If*

(a) *If  $p_1, \dots, p_n \in P$  then  $p_1 \cdots p_n = 0$  iff  $\bigcup \{M_{p_i} : i \leq n\} = M$ .*

(b) *If  $p, p_1, \dots, p_n \in P$  and  $0 < p_1 \cdots p_n \leq p$ , then  $p_i < p$  for some  $i$ .*

(c)  $\bigcap \{M_p : p \in P - M\} \neq \emptyset$ .

*Then  $A$  is a projective PCSL.*

*Proof.* By Lemma 4.3  $A$  is a PCSL. Let  $M = \{a_1, \dots, a_n\}$  and  $P - M = \{b_1, \dots, b_m\}$ . Let  $F$  be a PCSL freely generated by  $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_m\}$  and let  $f: F \rightarrow A$  be a homomorphism such that  $f(x_i) = a_i$ , and  $f(y_j) = b_j$  for all  $i, j$ . If  $1 \leq i \leq n$ , let

$$c_i = x_i^{**} + \sum \{x_k^* : k \neq i\} + \sum \{y_j^{**} : b_j < a_i\} \\ + \sum \{y_j^* : b_j \not< a_i\}.$$

We observe that  $c_i$  is a dual atom of  $F^*$ . Let  $D$  be the set of all dual atoms of  $F^*$  which are not in  $\{c_1, c_2, \dots, c_n\}$ . Since  $\bigcap \{M_p : p \in P - M\} \neq \emptyset$  we may assume that  $a_1 \geq b_j$  for all  $j = 1, 2, \dots, m$ . Let  $h: P \rightarrow F$  be defined by

$$\begin{aligned} h(a_i) &= c_i \Pi(D) \\ h(a_i) &= c_i & \text{for } 1 < i \leq n \\ h(b_j) &= \Pi\{y_k : b_j \leq b_k\} \cdot \Pi(D), & \text{for } 1 \leq j \leq m. \end{aligned}$$

To show  $h$  is order preserving we observe the following: If  $b_j < a_i$  then

$$\begin{aligned} h(b_j) &\leq y_j \cdot \Pi(D) \leq y_j^{**} \cdot \Pi(D) \leq \Sigma\{y_k^{**} : b_k < a_i\} \cdot \Pi(D) \\ &\leq c_i \cdot \Pi(D) \leq h(a_i). \end{aligned}$$

If  $b_j \leq b_r$  then  $h(b_j) \leq h(b_r)$  since

$$\{b_k : b_j \leq b_k\} \supseteq \{b_k : b_r \leq b_k\}.$$

Also,  $h(a_1) \cdots h(a_n)$  is the product of all the dual atoms of  $F^*$ , which is 0. By Lemma 4.2, there exists a semi-lattice homomorphism  $g: A - \{1\} \rightarrow F$  such that  $g|_P = h$ . Extend  $g$  to  $A$  by defining  $g(1) = 1$ . By Lemma 4.3,  $z^* = \Pi(M - M_z)$  for all  $z \in A$ . Now  $f(c_i) = a_i^{**} + \Sigma\{a_k^* : k \neq i\} + \Sigma\{b_j^{**} : b_j < a_i\} + \Sigma\{b_j^* : b_j \not< a_i\} = a_i$  for all  $i$ , since

$a_i^{**} = a_i$ ,  $a_k^* \leq a_i$  for  $k \neq i$ ,  $b_j^{**} \leq a_i$  if  $b_j < a_i$ , and  $b_j^* \leq a_i$  if  $b_j \not< a_i$ . If  $d \in D$ , then either  $d \geq x_k^{**} + x_l^{**}$  for some  $k \neq l$ , in which case  $f(d) \geq a_k + a_l = 1$ , or  $d \geq \Sigma\{x_k^*; k \leq n\}$  in which case  $f(d) \geq \Sigma\{a_k^*; k \leq n\} = (\Pi\{a_k: k \leq n\})^* = 0^* = 1$ , or  $d \geq x_i^{**} + y_j^*$  for some  $i$  and some  $j$  such that  $b_j < a_i$ , in which case  $f(d) \geq a_i + \Pi\{a_l: b_j \not< a_l\} = \Pi\{a_i + a_l: b_j \not< a_l\} = 1$ , or  $d \geq x_i^{**} + y_j^{**}$  for some  $i$  and some  $j$  such that  $b_j \not< a_i$  in which case  $f(d) \geq a_i + \Pi\{a_l: b_j < a_l\} = \Pi\{a_i + a_l: b_j < a_l\} = 1$ . Thus  $f(d) = 1$  for all  $d \in D$ . Now

$$\begin{aligned} fg(a_i) &= f(c_i) = a_i, & \text{for } i > 1 \\ fg(a_1) &= f(c_1) \cdot \Pi(f(D)) = a_1, & \text{and} \\ fg(b_j) &= \Pi\{b_k: b_j \leq b_k\} \cdot \Pi f(D) = b_j, & \text{for all } j. \end{aligned}$$

Since  $P$  generates  $A$  we have  $fg = I_A$ . It remains to show that  $g$  is a  $*$  homomorphism.

For any  $k$ ,  $y_k^{**}$  is the product of all dual atoms of  $F^*$  which are  $\geq y_k^{**}$ . Since  $F^*$  is a free Boolean algebra, the only such dual atoms are the ones of the form  $\Sigma(S^* \cup T^{**})$  where  $y_k \in T$  and  $S \cup T = \{x_1, \dots, x_n, y_1, \dots, y_m\}$ . Thus  $y_k^{**} = \Pi\{c_i: b_k < a_i\} \cdot \Pi(D_k)$  where  $D_k \leq D$ . Therefore for any  $j$ ,

$$\begin{aligned} g(b_j)^{**} &= \Pi\{y_k^{**}: b_j \leq b_k\} \cdot \Pi(D) \\ &= \Pi\{\Pi\{c_i: b_k \leq a_i\}: b_j \leq b_k\} \cdot \Pi(D) = \Pi\{c_i: b_j < a_i\} \cdot \Pi(D) \\ &= \Pi\{g(a_i): b_j < a_i\} = g(\Pi\{a_i: b_j < a_i\}) = g(b_j^{**}). \end{aligned}$$

Also  $g(a_i)^{**} = g(a_i^{**})$  since  $a_i \in A^*$  and  $g(a_i) \in F$ . We observe that if  $R$  is the set of dual atoms of a finite Boolean algebra, then for any  $T \subseteq R$ ,  $\Pi(T)^* = \Pi(R - T)$ . Hence if  $u \in A^*$ , then  $u = \Pi(S)$ , for some  $S \subseteq M$ , and  $u^* = \Pi(M - S)$ . If  $a_i \in S$ ,

$$\begin{aligned} g(u)^* &= (\Pi\{c_i: a_i \in S\} \cdot \Pi(D))^* = \Pi\{c_i: a_i \notin S\} \\ &= g(\Pi(M - S)) = g(u^*). \end{aligned}$$

While if  $a_i \notin S$ ,  $g(u)^* = (\Pi\{c_i: a_i \in S\})^* = \Pi\{c_i: a_i \notin S\} \cdot \Pi(D) = g(\Pi(M - S)) = g(u^*)$ . We have now satisfied the hypothesis of Lemma 4.4, so  $g$  is a  $*$  homomorphism. Since  $A$  has been shown to be a retract of a free PCSL, then  $A$  is projective.

**THEOREM 4.3.** *Let  $A$  be a finite semi-lattice with 1. Let  $P$  be the set of meet irreducible elements of  $A - \{1\}$ , and  $M$  the set of maximal elements of  $P$ . Then  $A$  is projective if and only if the following hold.*

(a) *If  $Q \subseteq P$ , then  $\Pi(Q) = 0$  iff for each  $m \in M$ , there is a  $q \in Q$  such that  $m \geq q$ .*

- (b) If  $Q \subseteq P$ ,  $p \in P$  and  $0 < \Pi(Q) \leq p$ , then  $q \leq p$ , for some  $q \in Q$ .
- (c) There exists an  $m \in M$  such that  $m \geq p$  for every  $p \in P - M$ .

Proof of this follows from Theorems 4.1 and 4.2.

**THEOREM 4.4.** *If  $P$  is a finite partially ordered set and  $M$  is the set of maximal elements of  $P$ . Suppose*

- (a) *For every  $p \in P$ , there exists an  $m \in M$  such that  $p \not\leq m$ .*
- (b) *There exists a  $m \in M$  such that  $m \geq p$ , for every  $p \in P - M$ .*

*Then the semi-lattice with 0 which is freely generated by  $P$  with the defining relation  $\Pi(M) = 0$ , is a projective PCSL, and every finite projective PCSL can be so obtained. Proof of this follows from Lemma 4.1 and Theorem 4.2.*

**REMARK.** To the conditions of Theorem 4.2 and 4.3, we could add the following, though redundant condition: If  $Q \subseteq M$ , then  $\Pi(Q) = 0$  iff  $Q = M$ .

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