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**THEOREMS OF KOROVKIN TYPE FOR  $L_p$ -SPACES**

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**Suppose  $(X, \Sigma, \mu)$  is a measure space,  $1 < p < \infty$ ,  $p \neq 2$ , and that  $(T_n)$  is a net of linear contractions on (real or complex)  $L_p(X, \Sigma, \mu)$ . Let  $M = \{x \in L_p: T_n x \rightarrow x\}$  ( $M$  is the convergence set for  $(T_n)$ ). It is obvious that  $M$  is a closed subspace of  $L_p$ ; indeed this would be true for an arbitrary normed space. In this paper we shall show that  $M$  is the range of a contractive projection on  $L_p$  and hence is itself isometrically isomorphic to an  $L_p$ -space. If  $S \subset L_p(X, \Sigma, \mu)$  we can define the shadow,  $\mathcal{S}(S)$  of  $S$  to be the set of all  $x$  in  $L_p$  such that  $T_n x \rightarrow x$  for every net of linear contractions  $(T_n)$  such that  $T_n y \rightarrow y$  for all  $y \in S$ . We shall also give a complete description of  $\mathcal{S}(S)$  (for  $p \neq 1, 2, \infty$ ).**

Our results are new for finite  $p$  not equal to 1 or 2. In the case  $p = 2$  the assertions about  $M$  are trivial and  $\mathcal{S}(S)$  is the closed subspace spanned by  $S$ . The case  $p = 1$  was first considered by Wulbert [9] for Lebesgue measure on  $[0, 1]$ . He showed that if  $S = \{1, x, x^2\}$  then  $\mathcal{S}(S) = L_1[0, 1]$ . (Actually he considered sequences of contractions and required only  $T_n 1 \rightarrow 1$  and  $T_n f$  weakly convergent to  $f$  for  $f = x$  and  $f = x^2$ .) Wulbert's results were inspired by and generalized the classical theorem of Korovkin [7] which contains the result that if  $S = \{1, x, x^2\}$  then the shadow of  $S$  in  $C[0, 1]$  is  $C[0, 1]$ . In [8] Lorentz considered separable  $L_1$  spaces on finite measure spaces. He showed that for sequences of contractions such that  $T_n 1 \rightarrow 1$  the convergence set is a closed sublattice of  $L_1$ . A corollary of this, which he noted, is that for  $L_1[0, 1]$ ,  $\mathcal{S}(S) = L_1$  if  $S = \{1, x\}$ . This last result and some further discussion of  $L_1(X, \Sigma, \mu)$ , with  $\mu(X) = 1$  is also contained in [1].

The methods we use are suggested by the methods used in [3] in discussing contractive projections. I am very grateful to Professors Lorentz and Berens for discussions of this material and for supplying me with preprints of [1], [2], [8]. My first introduction to this circle of ideas was a colloquium lecture by Professor Lorentz in which some of the results from [2] and [8] were presented.

2. The convergence set. We shall fix notation as in the first paragraph of the introduction. It does not seem to matter whether our measure space is taken over a  $\sigma$ -ring,  $\sigma$ -algebra or  $\delta$ -ring. For definiteness we shall assume that  $\Sigma$  is a  $\sigma$ -ring and measurability is as defined by Halmos [5]. We shall let  $q$  be the conjugate index to  $p$ , defined by  $1/p + 1/q = 1$ . Since  $p \neq 1, 2, \infty$ , the same is true

for  $q$  and  $L_q(X, \Sigma, \mu)$  is the topological dual of  $L_p(X, \Sigma, \mu)$  with the usual identifications. We shall consider the complex case; i.e.,  $L_p$  (and  $L_q$ ) are (equivalence classes of) complex valued functions. The real case is a little easier, but the methods are the same. If  $T$  is a bounded linear operator on  $L_p$ , the conjugate operator  $T^*$  is defined on  $L_q$  by the identity

$$\int x \cdot (T^*y) d\mu = \int (Tx) \cdot y d\mu \quad (x \in L_p, y \in L_q).$$

DEFINITION. The *conjugate convergence set*  $M^*$  for the net of contractions  $(T_n)$  is defined by  $M^* = \{y \in L_q : T_n^*y \rightarrow y\}$ .

LEMMA 2.1. (Compare [3, Lemma 2.2].) *Let  $x \in L_p$ , then  $x \in M$  if and only if  $|x|^{p-1} \operatorname{sgn} \bar{x} \in M^*$ .*

*Proof.* Suppose  $x \in M$  and write  $u = |x|^{p-1} \operatorname{sgn} \bar{x}$ . Then  $\|u\|_q = \|x\|_p^{p/q}$  and  $(T_n^*u)$  is a bounded net in  $L_q$ . Let  $w$  be a weak-\* limit point of this net. We have

$$\begin{aligned} \int x \cdot (w - T_n^*u) d\mu &= \int x \cdot w d\mu - \int (T_n x) \cdot u d\mu \longrightarrow \int x \cdot w d\mu - \int x \cdot u d\mu \\ &= \int x \cdot (w - u) d\mu. \end{aligned}$$

Taking a subnet such that  $T_n^*u \rightarrow w$  (weak-\*), we conclude that

$$\int x \cdot w d\mu = \int x \cdot u d\mu = \int |x|^p = \|x\|_p^p.$$

Since the  $T_n^*$  are contractions,  $\|T_n^*u\|_q \leq \|u\|_q = \|x\|_p^{p/q}$  and hence  $\|w\|_q \leq \|x\|_p^{p/q}$ . Hölders' inequality now gives

$$\|x\|_p^p = \int x \cdot w d\mu \leq \|x\|_p \|w\|_q \leq \|x\|_p \|x\|_p^{p/q} = \|x\|_p^p.$$

This gives equality throughout so [6, § 13.5] we have

$$w = |x|^{p-1} \operatorname{sgn} \bar{x} = u.$$

Thus  $u$  is the unique weak-\* limit point of the net  $(T_n^*u)$ .

Since every subnet of  $(T_n^*u)$  has a convergent subnet (by weak-\* compactness of the unit ball in  $L_q$ ), we see that  $T_n^*u$  is weak-\* convergent to  $u$ . Hence

$$\|u\|_q \leq \liminf \|T_n^*u\|_q \leq \limsup \|T_n^*u\|_q \leq \|u\|_q,$$

because the  $T_n^*$  are contractions; and we also have  $\|u\| = \lim \|T_n^*u\|$ . Because  $L_q$  is uniformly convex [4; 6, § 15.17] it follows that  $T_n^*u \rightarrow u$

in the norm of  $L_q$ , which gives  $|x|^{p-1} \operatorname{sgn} \bar{x} \in M^*$  as required.

The same argument applied to  $L_q$  shows that if

$$u = |x|^{p-1} \operatorname{sgn} \bar{x} \in M^*, \quad x = |u|^{q-1} \operatorname{sgn} \bar{u} \in M^{**} = M,$$

so we are done.

We now apply differentiation arguments like those in [3, Lemma 2.3]. Recall that if  $z, w$  are complex,  $\lambda$  is real and  $h(\lambda) = |z + \lambda w|$  then, if  $z + \lambda w \neq 0$ ,  $h$  is differentiable at  $\lambda$  with

$$h'(\lambda) = \operatorname{Re} [w \operatorname{sgn} (\overline{z + \lambda w})].$$

**LEMMA 2.2.** (Compare [3, Lemma 2.3(i)].) *If  $x, y \in M$ , then  $|x| \operatorname{sgn} y \in M$ .*

*Proof.* Assume first that  $p > 2$  and define, for  $\lambda \in R$ , and  $0 < \lambda < 1$ ,

$$\begin{aligned} z_\lambda &= \lambda^{-1} [|x + \lambda y|^{p-1} \operatorname{sgn} (\overline{x + \lambda y}) - |x|^{p-1} \operatorname{sgn} \bar{x}] \\ &= \lambda^{-1} [(|x + \lambda y|^{p-2} - |x|^{p-2})(\overline{x + \lambda y})] + |x|^{p-2} \bar{y}. \end{aligned}$$

Now, make a fixed choice of functions from the equivalence classes determined by  $x, y$  and observe that, except for the null-set where  $x$  or  $y$  is infinite, our differentiation result quoted above shows that as  $\lambda \rightarrow 0$ ,

$$z_\lambda \rightarrow (p-2) |x|^{p-3} \cdot \operatorname{Re} [y \operatorname{sgn} \bar{x}] \cdot \bar{x} + |x|^{p-2} \bar{y}$$

at all points where  $x \neq 0$ . Also, since  $p > 2$ ,  $z_\lambda \rightarrow 0$  ( $\lambda \rightarrow 0$ ) at points where  $x = 0$ . Let  $z_1$  denote this (almost everywhere) pointwise limit of  $z_\lambda$  as  $\lambda \rightarrow 0$ .

At points where  $2|\lambda y| < |x|$ , the mean value theorem gives a  $\theta$ ,  $0 < \theta < 1$  such that

$$z_\lambda = (p-2) |x + \theta \lambda y|^{p-3} \operatorname{Re} [y \operatorname{sgn} (\overline{x + \theta \lambda y})] (\overline{x + \lambda y}) + |x|^{p-2} \bar{y}.$$

Since  $|x|/2 < |x + \theta \lambda y| < 2|x|$ , we have

$$\begin{aligned} |z_\lambda| &\leq (p-2) 2^{p-3} |x|^{p-3} |y| \cdot 2|x| + |x|^{p-2} |y| \\ &\leq (2^p(p-2) + 1) |x|^{p-2} |y| \in L_q. \end{aligned}$$

At points where  $|x| \leq 2|\lambda y|$ , we have

$$\begin{aligned} |z_\lambda| &\leq \lambda^{-1} (|3\lambda y|^{p-1} + |2\lambda y|^{p-1}) \\ &= (3^{p-1} + 2^{p-1}) |y|^{p-1} |\lambda|^{p-2} \\ &\leq (3^{p-1} + 2^{p-1}) |y|^{p-1} \in L_q. \end{aligned}$$

Thus the pointwise convergence of  $z_\lambda$  to

$$z_1 = (p - 2) |x|^{p-3} \bar{x} \operatorname{Re} (y \operatorname{sgn} \bar{x}) + |x|^{p-2} \bar{y}$$

is dominated by an element of  $L_q$ . The dominated convergence theorem then shows that  $\|z_\lambda - z_1\| \rightarrow 0$ . Now, by Lemma 2.1,  $z_\lambda \in M^*$  and  $M^*$  is closed. Hence  $z_1 \in M^*$ .

Apply this result to  $x$  and  $-iy$  to see that

$$z_2 = (p - 2) |x|^{p-3} \bar{x} \operatorname{Re} (-iy \operatorname{sgn} \bar{x}) + i |x|^{p-2} \bar{y} \in M^* .$$

Thus  $z_1 - iz_2 \in M^*$  and

$$\begin{aligned} z_1 - iz_2 &= (p - 2) |x|^{p-3} \bar{x} [\operatorname{Re} y \operatorname{sgn} \bar{x} - i \operatorname{Im} y \operatorname{sgn} \bar{x}] \\ &\quad + 2 |x|^{p-2} \bar{y} = p |x|^{p-2} \bar{y} . \end{aligned}$$

Use Lemma 2.1 again to conclude that

$$||x|^{p-2} \bar{y}|^{q-1} \operatorname{sgn} (|x|^{p-2} \bar{y}) = |x|^{1-(q-1)} |y|^{q-1} \operatorname{sgn} y \in M .$$

Let  $k_n = |x|^{1-(q-1)^n} |y|^{(q-1)^n} \operatorname{sgn} y$ . Observe that for each  $n$ ,  $k_{n+1} = |x|^{1-(q-1)} |k_n|^{q-1} \operatorname{sgn} k_n$ . The argument we have just provided shows inductively that  $k_n \in M$  for all  $n$ . Since  $0 < q - 1 < 1$ ,  $k_n \rightarrow |x| \operatorname{sgn} y$ ,  $\mu$ -almost everywhere, and clearly

$$|k_n| \leq \max(|x|, |y|) \in L_p .$$

By dominated convergence again,  $\|k_n - |x| \operatorname{sgn} y\|_p \rightarrow 0$  and hence,  $|x| \operatorname{sgn} y \in M$  as required.

If  $1 < p < 2$ , then by Lemma 2.1,  $x_1 = |x|^{p-1} \operatorname{sgn} \bar{x}$  and  $y_1 = |y|^{p-1} \operatorname{sgn} \bar{y}$  are in  $M^*$ . Since  $q > 2$ , our proof above shows that  $|x_1| \operatorname{sgn} y_1 = |x|^{p-1} \operatorname{sgn} \bar{y} \in M^*$ . Apply Lemma 2.1 again to get  $|x| \operatorname{sgn} y = ||x|^{p-1} \operatorname{sgn} \bar{y}|^{q-1} \operatorname{sgn} (|x|^{p-1} \operatorname{sgn} \bar{y}) \in M$ .

For our next result we need some terminology from [3]. A subspace  $N$  of  $L_p(X, \Sigma, \mu)$  is a *vector sublattice* if for each  $x \in N$ ,  $(\operatorname{Re} x)^+ \in N$ ; this means that  $N$  is closed under taking real (or imaginary) parts and that the set of real functions in  $N$  is a real subspace and a sublattice of  $L_p$ . For a nonempty subset  $K$  of  $L_p$ , the *polar*  $K^\perp = \{x \in L_p : |x| \wedge |y| = 0 (y \in K)\}$ . A *band* in  $L_p$  is a subset  $K$  such that  $K = K^{\perp\perp}$  (a band is necessarily a solid vector sublattice). If  $K$  is a band in  $L_p$  there is a natural direct sum decomposition  $L_p = K \oplus K^\perp$  and the associated projections are positive and contractive. We write  $J_K$  for the band projection on  $K$ . If  $K = y^{\perp\perp}$  (the only case we need) we write  $J_y$  for the band projection and note that  $J_y$  is multiplication by the characteristic function of the set on which  $y$  is nonzero.

LEMMA 2.3. *If  $y \in M$  and  $J_y$  is the associated band projection, then  $J_y M \subset M$ .*

*Proof.* Let  $x \in M$ , then  $J_y x = |x| \operatorname{sgn} y | \operatorname{sgn} x$  and this element is in  $M$  by two applications of Lemma 2.2.

LEMMA 2.4. *If  $x, y \in M$ , then  $(\operatorname{Re}(x \operatorname{sgn} \bar{y}))^+ \operatorname{sgn} y \in M$ .*

*Proof.* Suppose  $\lambda \in \mathbb{R}, \lambda \neq 0$ , then by Lemma 2.2,  $v_\lambda = \lambda^{-1}(|y + \lambda x| - |y|) \operatorname{sgn} y \in M$ . Since  $v_\lambda = 0$  at points where  $y = 0$  we see that as  $\lambda \rightarrow 0$ ,  $v_\lambda$  converges pointwise ( $\mu$ -almost everywhere) to  $(\operatorname{Re}(x \operatorname{sgn} \bar{y})) \operatorname{sgn} y$ . Since

$$|v_\lambda| \leq |\lambda|^{-1} ||y + \lambda x| - |y|| \leq |\lambda|^{-1} |y + \lambda x - y| = |x|,$$

dominated convergence shows that  $\|v_\lambda - (\operatorname{Re}(x \operatorname{sgn} \bar{y})) \operatorname{sgn} y\|_p \rightarrow 0$ ; so that  $(\operatorname{Re}(x \operatorname{sgn} \bar{y})) \operatorname{sgn} y \in M$ . Another application of Lemma 2.2 gives  $|\operatorname{Re}(x \operatorname{sgn} \bar{y})| \operatorname{sgn} y \in M$  and our results follows.

THEOREM 2.5. *The convergence set  $M$  is the range of a contractive projection on  $L_p$ .*

*Proof.* For  $y \in M$ , Lemma 2.4 shows that the map  $U_y: L_p \rightarrow L_p$ , defined by  $U_y x = x \operatorname{sgn} \bar{y}$ , is norm decreasing, linear, and maps  $M$  onto a closed vector sublattice of  $L_p$ . Such a map was called a *unitary multiplication operator* in [3]. Choose, by Zorn's lemma, a maximal subset  $Y$  of  $M$  such that  $|y_1| \wedge |y_2| = 0$  if  $y_1, y_2 \in Y$  and  $y_1 \neq y_2$  (a maximal pairwise disjoint subset of  $M$ ). If  $f \in L_p$  the set  $\{t \in X: f(t) \neq 0\}$  is  $\sigma$ -finite so the set  $\{y \in Y: U_y f \neq 0\}$  is countable. Thus we can define the direct sum  $U$  of the unitary multiplications  $U_y (y \in Y)$  by  $Uf = \sum_{y \in Y} U_y f$ , and the defining sum has at most countably many nonzero terms and is convergent in  $L_p$  norm. Clearly  $UM$  is a closed vector sublattice of  $L_p$ . We show that  $U$  is isometric on  $M$ .

Suppose  $x \in M, x \neq 0$ . Let  $y_1, \dots, y_n, \dots$  be an enumeration of the countable set of  $y \in Y$  such that  $|y| \wedge |x| \neq 0$ . Let  $y_0 = \sum 2^{-n} \|y_n\|^{-1} y_n$ . Then  $y_0 \in M$  and, by Lemma 2.3,  $J_{y_0} x \in M$ . Hence  $x - J_{y_0} x \in M$  and  $|x - J_{y_0} x| \wedge |y| = 0 (y \in Y)$ . By maximality of  $Y$   $x = J_{y_0} x$ . Hence

$$x = \sum_{y \in Y} J_y x \text{ and } \|x\|^p = \sum_{y \in Y} \|J_y x\|^p = \sum_{y \in Y} \|U_y x\|^p = \|Ux\|^p.$$

It now follows by Theorem 4.1 of [3] that  $M$  is the range of a contractive projection on  $L_p$  and, which is an equivalent condition, that  $M$  is isometrically isomorphic to some  $L_p(X_0, \Sigma_0, \mu_0)$ .

3. The shadow of subset  $S$ . In a certain sense characterization of shadows is trivial. Call a subspace  $M$  of  $L_p$  an *exchange subspace* if  $|x| \operatorname{sgn} y \in M$  for any  $x, y \in M$ . Clearly an intersection of (closed) exchange subspaces is again a (closed) exchange subspace. Hence for a subset  $S$  of  $L_p$  we can determine the closed exchange subspace of  $L_p$  generated by  $S$  as the intersection of all closed exchange subspaces of  $L_p$  which contains  $S$ .

**THEOREM 3.1.** *If  $S \subset L_p$  then the shadow,  $\mathcal{S}(S)$ , of  $S$  is the closed exchange subspace of  $L_p$  generated by  $S$ .*

*Proof.* Lemma 2.2 shows that  $\mathcal{S}(S)$  is a closed exchange subspace of  $L_p$  which contains  $S$ . If  $M$  denotes the closed exchange subspace of  $L_p$  generated by  $S$  a careful check of the proofs of Lemmas 2.3 and 2.4 and Theorem 2.5 show that these are valid for any closed exchange subspace of  $L_p$ . Hence  $M$  is the range of a contractive projection, say  $P$ , on  $L_p$ . Define a sequence  $(T_n)$  of linear contractions of  $L_p$  by  $T_n = P(n = 1, 2, \dots)$ . Then  $M$  is the convergence set for  $(T_n)$  and hence,  $\mathcal{S}(S) \subset M$ . This proves our theorem.

As a corollary of the proof of Theorem 3.1 we digress to state the following result.

**THEOREM 3.2.** *Suppose  $1 \leq p < \infty$ ,  $p \neq 2$ , a subspace  $M$  of  $L_p(X, \Sigma, \mu)$  is the range of a contractive projection if and only if  $M$  is a closed exchange subspace of  $L_p(X, \Sigma, \mu)$ .*

*Proof.* If  $M$  is a closed exchange subspace of  $L_p$  then just as in Theorem 3.1, Theorem 2.5 is valid for  $M$ . (This is equally true for  $p = 1$  and  $p = 2$  as can easily be checked.) By [3, Theorem 4.1] it follows that  $M$  is the range of a contractive projection on  $L_p$ . The converse result is the statement of [3, Lemma 2.3(i)] if  $p \neq 1$  and an easy consequence of [3, Lemma 3.3] if  $p = 1$ .

Returning to shadows we note that Theorem 3.1 is difficult to apply in practice. The following alternative seems a little more useful.

Let  $\mathcal{B}$  be the smallest sub  $\sigma$ -ring of  $\Sigma$  such that the functions  $J_y x/y$  are  $\mathcal{B}$ -measurable for all  $x, y \in S$ . (To be precise here, we consider all choices of functions  $x, y$  in equivalence classes in  $S$ . The ratios are zero, by definition, wherever the denominators are zero.) Choose, by Zorn's lemma, a subset  $\mathcal{N}$  of  $\mathcal{B} \times S$  which is maximal with respect to the properties: (i) if  $(A, y) \in \mathcal{N}$ , then  $A \subset \{t \in X: y(t) \neq 0\}$ ; (ii) if  $(A, y) \in \mathcal{N}$ ,  $\mu A > 0$ ; (iii) if  $(A_1, y_1), (A_2, y_2)$  are distinct elements of  $\mathcal{N}$ , then  $\mu(A_1 \cap A_2) = 0$ . Define a measure  $\lambda$  on  $\mathcal{B}$  by

$$\lambda(B) = \sum_{(A,y) \in \mathcal{K}} \int_{A \cap B} |y|^p d\mu. \quad (B \in \mathcal{B}).$$

Also define a map  $V: L_p(X, \mathcal{B}, \lambda) \rightarrow L_p(X, \Sigma, \mu)$  by  $Vf = \sum_{(A,y) \in \mathcal{K}} f \cdot y \chi_A$  ( $f \in L_p(X, \mathcal{B}, \lambda)$ ). We note that the sets in  $\mathcal{B}$  all have  $\sigma$ -finite  $\mu$ -measure; that the sum defining  $Vf$  has, therefore, only countably many nonzero terms and is convergent in the norm of  $L_p(X, \Sigma, \mu)$ . Furthermore,  $V$  is an isometry of  $L_p(X, \mathcal{B}, \lambda)$ .

**THEOREM 3.3.** *The shadow  $\mathcal{S}(S) = VL_p(X, \mathcal{B}, \lambda)$ .*

*Proof.* Let  $U$  be a direct sum of unitary multiplications such that  $U$  is an isometry of  $\mathcal{S}(S)$  and  $U\mathcal{S}(S)$  is a closed vector sublattice of  $L_p(X, \Sigma, \mu)$ . For  $f \in L_p$  write  $T(f) = \{t \in X: f(t) \neq 0\}$ , (we allow the ambiguity of sets of measure zero here); and let  $\mathcal{B}_S = \{T(f): f \in \mathcal{S}(S)\}$ . I claim that  $\mathcal{B}_S$  is a sub  $\sigma$ -ring of  $\Sigma$ .

For this, observe that  $T(f) = T(Uf)$  so we may assume that  $\mathcal{S}(S)$  is a closed vector sublattice of  $L_p$ . Now  $T(f) = T(|f|) = T(|\operatorname{Re} f|) \cup T(|\operatorname{Im} f|)$  and  $|f| \in \mathcal{S}(S)$ . Hence

$$\bigcup_{n=1}^{\infty} T(f_n) = T\left(\sum_{n=1}^{\infty} 2^{-n} \|f_n\|^{-1} |f_n|\right) \in \mathcal{B}_S$$

so  $\mathcal{B}_S$  is closed under countable union. If  $f, g \in \mathcal{S}(S)$ ,  $J_f |g| = \lim_{n \rightarrow \infty} |g| \wedge n |f| \in \mathcal{S}(S)$  so  $T(g) \sim T(f) = T(|g| - J_f |g|) \in \mathcal{B}_S$ . This proves our claim about  $\mathcal{B}_S$ .

If  $x, y \in S$ ,  $\{t \in X: \operatorname{Re} J_y x / y > \alpha > 0\} = T((\operatorname{Re} (J_y x - \alpha y))^+) \in \mathcal{B}_S$ . Hence, every  $J_y x / y$  ( $x, y \in S$ ) is  $\mathcal{B}_S$ -measurable and  $\mathcal{B}_S \supset \mathcal{B}$ .

Suppose  $B \in \mathcal{B}$  then  $B = T(f_B)$  for some  $f_B \in \mathcal{S}(S)$ . If  $\lambda(B) < \infty$ , then  $V\chi_B = \sum_{(A,y) \in \mathcal{K}} y \cdot \chi_{B \cap A}$ . Since  $B$  is  $\sigma$ -finite we can enumerate the countable set of pairs  $(A, y)$  in  $\mathcal{K}$  such that  $\mu(B \cap A) \neq 0$  as  $(A_n, y_n)$  and choose  $f_n \in \mathcal{S}(S)$  such that  $T(f_n) = A_n \cap B$ . Then  $V\chi_B = \sum_{n=1}^{\infty} J_{f_n} y_n$ . Since the  $A_n$  are disjoint and

$$\begin{aligned} \sum_{n=1}^{\infty} \|J_{f_n} y_n\|_{L_p(\mu)}^p &= \sum \int_{A_n \cap B} |y_n|^p d\mu \\ &= \sum \lambda(A_n \cap B) \\ &= \lambda B \\ &< \infty, \end{aligned}$$

the series for  $V\chi_B$  converges in  $L_p(X, \Sigma, \mu)$ . Since each  $y_n \in S$  and each  $f_n \in \mathcal{S}(S)$ , each  $J_{f_n} y_n \in \mathcal{S}(S)$  and  $V\chi_B \in \mathcal{S}(S)$ . Extending linearly to simple functions in  $L_p(X, \mathcal{B}, \mu)$  and then taking closure we conclude that  $VL_p(X, \mathcal{B}, \mu) \subset \mathcal{S}(S)$ .

If  $x \in S$  and  $(A, y) \in \mathcal{K}$  then  $\chi_A x / y$  is  $\mathcal{B}$ -measurable and



$$\int |\chi_A x/y|^p d\lambda = \int_A |x|^p d\mu < \infty.$$

Hence  $z = \sum_{(A,y) \in \mathcal{K}} \chi_A x/y \in L_p(X, \mathcal{B}, \lambda)$  and  $x = Vz \in VL_p(X, \mathcal{B}, \lambda)$ . This shows that  $VL_p(X, \mathcal{B}, \lambda) \supset S$ .

If  $f, g \in L_p(X, \mathcal{B}, \lambda)$ , then  $|f| \operatorname{sgn} g \in L_p(X, \mathcal{B}, \lambda)$  and  $|Vf| \operatorname{sgn} Vg = V(|f| \operatorname{sgn} g)$ . Thus  $VL_p(X, \mathcal{B}, \lambda)$  is a closed exchange subspace of  $L_p(X, \Sigma, \mu)$  which contains  $S$ . By Theorem 3.1 it follows that  $\mathcal{S}(S) = VL_p(X, \mathcal{B}, \lambda)$  as required.

For applications of this result we recall that a subset  $S$  of a Banach space  $E$  is a *Korovkin set* for contractions if  $\mathcal{S}(S) = E$ .

In our closing results,  $\mu$  is Lebesgue measure and  $\Sigma$  the  $\sigma$ -ring of Lebesgue measurable subsets of  $R$ , or  $R^n$  as appropriate. Also, remember that  $1 < p < \infty$ ,  $p \neq 2$ .

**THEOREM 3.4.** *If  $-1/p < \alpha < \beta$  then  $\{t^\alpha, t^\beta\}$  is a Korovkin set for  $L_p([0, 1], \Sigma, \mu)$ .*

*Proof.* In the construction preceding Theorem 3.3, we see that  $\mathcal{B}$  is the  $\sigma$ -ring of Borel subsets of  $[0, 1]$ . Take  $\mathcal{K} = \{([0, 1], t^\beta)\}$  and observe that  $V$  is an isometry of  $L_p([0, 1], \Sigma, \mu)$  and  $L_p([0, 1], \mathcal{B}, \lambda)$ .

Observe that we could also use  $\{\cos t, \sin t\}$  on  $[0, \pi/2]$  or on  $[0, \pi]$  or, of course, many other doubleton sets on many other finite intervals.

**THEOREM 3.5.** *Let  $X = [0, 1]^n \subset R^n$  and let  $S$  contain the constant function 1 and the  $n$  coordinate projections, then  $S$  is a Korovkin set for  $L_p(X, \Sigma, \mu)$ .*

One last result for a case when  $\mu$  is not finite will now suffice.

**THEOREM 3.6.** *Let  $S = \{e^{-t}, e^{-t^2}\}$ , then  $S$  is a Korovkin set for  $L_p([0, \infty], \Sigma, \mu)$ .*

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