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With G a normed space, this paper provides conditions on a nonlinear function A from $R \times G$ to G in order to insure that if P is in G then there will be a (not necessarily continuous) solution Y for

$$Y(x) = P + \int_0^x d_t A(t, Y(t)) .$$

Early work in the study of the Stieltjes integral equation

$$M(x, z) = 1 + \int_x^z dF M(I, z)$$

was done by H. S. Wall [25] and T. H. Hildebrandt [8]. In Wall's paper, F is a continuous matrix valued function which is of bounded variation on each finite interval. Hildebrandt dropped the requirement of continuity and used a modified Stieltjes integral. J. S. Mac Nerney carefully analysed these ideas in a series of papers which led to the fundamental relationships found in [15], [16], and [17].

The papers [15] and [17] establish two classes OA and OM of functions and a one-to-one pairing of the classes made possible through a continuously continued sum, a continuously continued product, and a Stieltjes integral equation. In [17], if V is in OA, Mis in OM, S is a linearly ordered set, and P is contained in a complete, normed, Abelian group, then V and M are related by M(x, y)P = ${}_x \prod^{y} [1+V]P$, $V(x, y)P = {}_x \sum^{y} [M-1]P$, and $M(x, y)P = P + \int^{y} VM(I, y)P$.

The results in [15] may be identified with analogous results in ordinary differential equations associated with nonautonomous, continuous, linear systems and [17] may be identified with Lipschitz systems. An indication of the nature of the generality obtained in the Stieltjes integral equation theory is found in [16], or in David L. Lovelady's discussion of interface problems [11, p. 184], or in a recent paper by Robert H. Martin [20] which investigates a linear operator equation and which identifies the linearly ordered set as the positive integers. Additional results related to [15] were found by B. W. Helton and Davis-Chatfield (see [2] or [3]). Also, this author determines a characterization of subsets of the two classes OA and OM which give rise to invertible evolution operators M in [4], for the linear case, and in [7] for the nonlinear (but Lipschitz) case. In [9] Don Hinton and in [1] Carl Bitzer develop a theory for Stieltjes-Volterra equations. Reneke shows in [21] and [23] that much of the classical Volterra theory is contained in [15] or [17].

Questions concerning bounds for solutions of Stieltjes equations, as well as perturbations of these solutions have been investigated by Schamedeke and Sell [24], Herod [5], Martin [19], Reneke [22], and Lovelady [10], [11], and [12]. Also, Marrah and Proctor [18] have found results concerning periodic solutions.

In [6], this author extends the classes OA and OM by using some of the ideas of analytic semi-group theory. In that investigation, similar to Mac Nerney's, two classes OA and OM are paired by a continuously continued sum, a continuously continued product, and a Riemann-Stieltjes equation. (In this setting, also, Lovelady [14] has generalized earlier results of his involving perturbations of the systems.) The Lipschitz condition of [17] was dropped in [6] at the expense of requiring that $M(\cdot, y)P$, in addition to being of bounded variation on each finite interval, be continuous and that Sshould be the real line. The results which follow relax these requirements.

We suppose that S is a nondegenerate set with a linear ordering and that $\{S, \geq\}$ has the least upper bound property. Also, $\{G, +, |\cdot|\}$ denotes a complete, normed Abelian group with zero element 0. Further, suppose that D is a closed subset of G and that V is a function such that if each of x and y is in S and $x \geq y$ then V(x, y)is a function from D into G having the following properties:

(i) If $x \ge y \ge z$ and P is in D then V(x, y)P + V(y, z)P = V(x, z)P,

(ii) If a > b then there is a nondecreasing, numerical valued function β defined on S such that if $\varepsilon > 0$ and P is in D then there is a positive number δ having the property that if Q is in D such that $|Q - P| < \delta$ and $a \ge x \ge y \ge b$ then $|V(x, y)P - V(x, y)Q| \le [\beta(x) - \beta(y)]\varepsilon$,

(iii) If a > b then D is contained in the range of [1 - V(a, b)]and if P and Q are in D then $|[1 - V(a, b)]P - [1 - V(a, b)]Q| \ge |P - Q|$, and

(iv) If a > b and P is in D then there is a nondecreasing, numerical function α such that if $\{s_p\}_0^n$ is a nonincreasing sequence with values in [b, a] and $a \ge x \ge y \ge b$ then $|V(x, y) \prod_{p=1}^n [1 - V(s_{p-1}, s_p)]^{-1}P| \le \alpha(x) - \alpha(y)$.

If f is a function from S with values in G and y is in S then $f(y^-)$ is a member g of G having the property that if $\varepsilon > 0$ then there is a member x of S such that x < y and if $x \leq t < y$ then $|g - f(t)| < \varepsilon$. In a similar manner, $f(y^+)$ may be defined.

The following theorems are established:

THEOREM I. If a > b, β is as in (ii), P is in D, and $\varepsilon > 0$ then there is a subdivision s of $\{a, b\}$ such that if t is a refinement of s then

$$|\prod_{s} [1 - V]^{-1}P - \prod_{t} [1 - V]^{-1}P| < \{4 + 2[eta(a) - eta(b)]\}arepsilon$$
 .

Let *M* be a function defined as follows: If $x \ge y$ and *P* is in *D* then $M(x, y)P = {}_{x}\prod^{y} [1 - V]^{-1}P$.

THEOREM II. If a > b then M(a, b) is a function from D to D and

(1) If each of P and Q is in D then $|M(a, b)P - M(a, b)Q| \leq |P - Q|$,

(2) If $x \ge y \ge z$ and P is in D then M(x, y)M(y, z)P = M(x, z)P,

(3) If P is in D, and $a \ge x \ge y \ge b$ then $|M(x, b)P - M(y, b)P| \le \alpha(x) - \alpha(y)$,

(4) If $a \geq b$, $\varepsilon > 0$, and P is in D then there is a positive number δ having the property that if Q is in D such that $|Q - P| < \delta$ and $a \geq x \geq y \geq b$ then $|[M(x, y) - 1]P - [M(x, y) - 1]Q| \leq [\beta(x) - \beta(y)]\varepsilon$.

THEOREM III. If P is in D and b is a member of S then the only function g which is of bounded variation on each finite interval of S and which satisfies the integral equation $g(x) = P + (L) \int_{x}^{b} V[g]$ for each $x \ge b$ is given by g(x) = M(x, b)P for $x \ge b$.

Proof of Theorem I.

LEMMA 1. If a > b, P is in D, and α is as in (iv), then (1) $\lim_{x \downarrow b} ([1 - V(x, b)]^{-1}P)$ exists and is $[1 - V(b^+, b)]^{-1}P$ and (2) If t is a subdivision of $\{a, b\}$ then $|\prod_t [1 - V]^{-1}P - [1 - V(b^+, b]^{-1}P| \le \alpha(a) - \alpha(b^+)$,

(3) $\lim_{x\uparrow a} ([1 - V(a, x)]^{-1}P)$ exists and is $[1 - V(a, a^{-})]^{-1}P$ and (4) If t is a subdivision of $\{a, b\}$ then $|\prod_{t} [1 - V]^{-1}P - [1 - V(a, a^{-})]^{-1}P| \leq \alpha(a^{-}) - \alpha(b).$

Indication of proof. Suppose that $x \ge y > b$. Then

$$egin{aligned} &|[1-V(x,\,b)]^{-1}P-[1-V(y,\,b)]^{-1}P\,| \ &\leq |V(x,\,b)[1-V(y,\,b)]^{-1}P-V(y,\,b)[1-V(y,\,b)]^{-1}P\,| \ &\leq lpha(x)-lpha(y) \;. \end{aligned}$$

The existence of $\lim_{x\downarrow b} \alpha(x)$, together with the fact that D is closed, implies the existence of $\lim_{x\downarrow b} ([1 - V(x, b)]^{-1}P)$ in D. Let Q be this

limit. Then $|[1 - V(x, b)]Q - P| \leq |Q - [1 - V(x, b)]^{-1}P| + |V(x, b)Q - V(x, b)[1 - V(x, b)]^{-1}P|$. Consequently, $P = \lim_{x \downarrow b} [1 - V(x, b)]Q = [1 - V(b^+, b)]Q$. That is, $Q = [1 - V(b^+, b)]^{-1}P$ so that (1) is established. In order to establish (2), suppose that $\{t_p\}_0^n$ is a subdivision of $\{a, b\}$. With Q as above,

$$egin{aligned} & \left|\prod_{p=1}^{n}\left[1-V(t_{p-1},\,t_{p})
ight]^{-1}P-Q
ight| \ & \leq \left|\prod_{p=1}^{n}\left[1-V(t_{p-1},\,t_{p})
ight]^{-1}P-\left[1-V(t_{n-1},\,t_{n})
ight]^{-1}P
ight|+lpha(t_{n-1})-lpha(b^{+}) \ & \leq \sum_{p=1}^{n-1}\left|V(t_{p-1},\,t_{p})[1-V(t_{n-1},\,t_{n})]^{-1}P
ight|+lpha(t_{n-1})-lpha(b^{+}) \ & \leq lpha(a)-lpha(b^{+}) \;. \end{aligned}$$

In a similar manner, one can establish (3) and (4).

LEMMA 2. Suppose that a > b, β is as in (ii), ε is a positive number, and P is in D. There is a subdivision $\{s_p\}_0^m$ of $\{a, b\}$ such that if $\{t_p\}_0^n$ is a refinement of s and k is a sequence such that $t(k_p) = s_p$, $p = 0, 1, \dots, m$, then

$$\sum_{p=1}^m \sum_{q=1+k_{p-1}}^{k_p} \left| V(t_{q-1},\,t_q) \prod_{i=q}^{k_p} [1 - V(t_{i-1},\,t_i)]^{-1} \prod_{j=p+1}^m [1 - V(s_{j-1},\,s_j)]^{-1} P
ight.
onumber \ - V(t_{q-1},\,t_q) \prod_{i=1+k_{p-1}}^{k_p} [1 - V(t_{i-1},\,t_i)]^{-1} \prod_{j=p+1}^m [1 - V(s_{j-1},\,s_j)]^{-1} P \ < [4 + 2(eta(a) - eta(b))]arepsilon \; .$$

Proof. With the supposition of the lemma, let α be as in (iv). Define functions Δ , δ , and d as follows:

If R is in D then $\Delta(R)$ is the largest number e not exceeding 1 and having the property that if Q is in D, |Q - R| < e, and $a \ge x \ge y \ge b$ then $|V(x, y)Q - V(x, y)R| \le [\beta(x) - \beta(y)]\varepsilon$,

If $b \leq z < a$, R is in D, and $Q = \lim_{x \downarrow z} [1 - V(x, z)]^{-1}R$ then $\hat{o}(z, R)$ is defined as follows: If there is no point y such that z < y < a then $\hat{o}(z, R) = a$ and, otherwise, $\hat{o}(z, R)$ is the least upper bound of all x such that $z < x \leq a$ and such that if $z \leq y < x$ and t is a subdivision of $\{y, z\}$ then $|\prod_t [1 - V]^{-1}R - Q| < \Delta(Q)$, and

If $b \leq z < y \leq a$ and c is a positive number then let x be the greatest lower-bound of all w such that $z \leq w$ and such that if $w \leq u < y$ then $\alpha(y^-) - \alpha(u) < c$. If there is no point of S between x and y let d(y, z, c) be x. If there is, let d(y, z, c) be such a point. Note that if u is in S and $d(y, z, c) \leq u < y$ then $\alpha(y^-) - \alpha(u) < c$.

Define the sequence u as follows: $u_0 = b$, $u_2 = \delta(u_0, P)$, $u_1 = d(u_2, u_0, \varepsilon)$, and, if n is a positive integer,

$$u_{2n+2} = \delta \Big(u_{2n}, \prod_{q=1}^{2n} [1 - V(u_{2n-q+1}, u_{2n-q})]^{-1} P \Big)$$

and $u_{2n+1} = d(u_{2n+2}, u_{2n}, \varepsilon/2^n)$. Assume that u is an infinite sequence. Since u is nondecreasing and bounded, let u_{∞} be $\lim u_{p}$ and, for each positive integer *j*, let $R_j = \prod_{q=1}^{j} [1 - V(u_{j-q+1}, u_{j-q})]^{-1} P$. If m > nthen, as in [6, p. 250] $|R_m - R_n| \leq \alpha(u_m) - \alpha(u_n)$. Because $\lim_{x \neq u_m} \alpha(x)$ exists, $\{R_p\}_{p=1}^{\infty}$ converges. For each integer *n*, let $Q_n = \lim_{x \downarrow u_n} [1 - 1]$ $V(x, u_n)]^{-1}R_n$. The sequence $\{Q_p\}_{p=1}^{\infty}$ converges for suppose that γ is a positive number. Let $R_{\infty} = \lim R_{p}$ and let v be a member of S such that if $u_{\infty} > x \ge v$ then $\alpha(x^+) - \alpha(x) < \gamma/2$. Let N be a positive integer such that if n > N then $|R_{\infty} - R_n| < \gamma/2$ and $u_{\infty} > u_n \ge v$. Then $\lim Q_p = R_{\infty}$ for $|R_{\infty} - Q_n| < \alpha(u_n^+) - \alpha(u_n) + \gamma/2$. By [6, Lemma 2.1] there is a positive number ξ such that if n is a positive integer then $\varDelta(Q_n) > \xi$. Again, using the fact that $\lim_{x \uparrow u_{\infty}} \alpha(x)$ exists, there is an integer N such that if m > n > N then $\alpha(u_m)$ – $\alpha(u_n) < \xi$ and, in this case, if t is a subdivision of $\{u_m, u_n\}$ then $|\prod_t [1-V]^{-1}R_n - Q_n| < \alpha(u_n) - \alpha(u_n^+) < \xi \leq \Delta(Q_n). \text{ Hence, } \delta(u_n, R_n) \geq u_n.$ Because this holds for each integer m > n, $\delta(u_n, R_n) \ge u_{\infty}$. This is a contradiction to the assumption that u is an infinite sequence.

Let *m* be the least integer such that $u_{2m} = a$, and define s_p to be u_{2m-p} for $p = 1, 2, \dots, 2m$. Let $\{t_q\}_{q=0}^n$ be a refinement of *s* and *k* be an increasing sequence such that $k_0 = 0$, $k_{2m} = n$, and $t(k_p) = s_p$ for $p = 0, 1, \dots, 2m$. If *p* is an integer in [1, m] and *q* is an integer in $[1 + k_{2p-1}, k_{2p}]$ then $u_{2(m-p)+2} = \delta(u_{2(m-p)}, R_{2(m-p)})$. Hence

$$\left|\prod_{i=q}^{k^{2,p}} \left[1 - V(t_{i-1}, t_{i})\right]^{-1} R_{2(m-p)} - Q_{2(m-p)}\right| < \varDelta(Q_{2(m-p)})$$

and

$$\left| \begin{array}{l} V(t_{q-1}, t_q) \prod_{i=q}^{k_{2,p}} [1 - V(t_{i-1}, t_i)]^{-1} R_{2(m-p)} - V(t_{q-1}, t_q) Q_{2(m-p)} \right| \\ \\ \leq [\beta(t_{q-1}) - \beta(t_q)] \varepsilon \ . \end{array} \right.$$

If p is an integer in [1, m] and q is an integer in $[1 + k_{2p-2}, k_{2p-1}]$ then

$$V(t_{q-1}, t_q) \prod_{i=q}^{k_{2p-1}} [1 - V(t_{i-1}, t_i)]^{-1} \prod_{j=2p}^{2m} [1 - V(s_{j-1}, s_j)]^{-1} P$$

- $V(t_{q-1}, t_q) \prod_{i=1+k_{2p-2}}^{k_{2p-1}} [1 - V(t_{i-1}, t_i)]^{-1} \prod_{j=2p}^{2m} [1 - V(s_{j-1}, s_j)]^{-1} P \Big|$

is zero if $q = 1 + k_{2^{p-2}}$ and does not exceed $2[\alpha(t_{q-1}) - \alpha(t_q)]$ if $1 + k_{2^{p-2}} < q \leq k_{2^{p-1}}$. Furthermore, $\alpha(t_{k_{2^{p-2}}}) - \alpha(t_{k_{2^{p-1}}}) = \alpha(s_{2^{p-2}}) - \alpha(s_{2^{p-2}}) - \alpha(s_{2^{p-1}}) = \alpha(s_{2^{p-2}}) - \alpha(s_{2^{p-1}}) < \varepsilon/2^{m-p}$. It follows that

$$\begin{split} &\sum_{p=1}^{2^{m}} \left\{ \sum_{q=1+k_{p-1}}^{k_{p}} \left| V(t_{q-1}, t_{q}) \prod_{i=q}^{k_{p}} \left[1 - V(t_{i-1}, t_{i}) \right]^{-1} \prod_{j=p+1}^{2^{m}} \left[1 - V(s_{j-1}, s_{j}) \right]^{-1} P \right. \\ &- V(t_{q-1}, t_{q}) \prod_{i=1+k_{p-1}}^{k_{p}} \left[1 - V(t_{i-1}, t_{i}) \right]^{-1} \prod_{j=p+1}^{2^{m}} \left[1 - V(s_{j-1}, s_{j}) \right]^{-1} P \right| \right\} \\ &= \sum_{p=1}^{m} \left\{ \sum_{q=1+k_{2p-2}}^{k_{2p-1}} \left| V(t_{q-1}, t_{q}) \prod_{i=q}^{k_{2p-1}} \left[1 - V(t_{i-1}, t_{i}) \right]^{-1} \prod_{j=2p}^{2^{m}} \left[1 - V(s_{j-1}, s_{j}) \right]^{-1} P \right. \right. \\ &- V(t_{q-1}, t_{q}) \prod_{i=1+k_{2p-2}}^{k_{2p-1}} \left[1 - V(t_{i-1}, t_{i}) \right]^{-1} \prod_{j=2p}^{2^{m}} \left[1 - V(s_{j-1}, s_{j}) \right]^{-1} P \\ &+ \sum_{q=2+k_{2p-1}}^{k_{2p}} \left| V(t_{q-1}, t_{q}) \prod_{i=q}^{k_{2p}} \left[1 - V(t_{i-1}, t_{i}) \right]^{-1} R_{2(m-p)} \\ &- V(t_{q-1}, t_{q}) \prod_{i=1+k_{2p-1}}^{k_{2p-1}} \left[1 - V(t_{i-1}, t_{i}) \right]^{-1} R_{2(m-p)} \right] \right\} \\ &\leq \sum_{p=1}^{m} \varepsilon / 2^{m-p} + \sum_{p=1}^{m} 2 \left[\beta(s_{2p-1} -) - \beta(s_{2p}) \right] \varepsilon < \left\{ 4 + 2 \left[\beta(a) - \beta(b) \right] \right\} \varepsilon \,. \end{split}$$

Indication of proof for Theorem I. The inequalities in the proof of Theorem 2.1 on pages 251 and 252 of [6] carry over almost without change by using the above Lemma 2.

The techniques above also provide the following

COROLLARY. If a > b, β is as in (ii), P is in D, and $\varepsilon > 0$ then there is a subdivision s of $\{a, b\}$ such that if $\{t_p\}_0^n$ is a refinement of s and p is an integer in [0, n] then $|M(t_p, b)P - \prod_{i=p+1}^n [1 - V(t_{i-1}, t_i)]^{-1}P| < \varepsilon$.

Proof of Theorem II. Parts (1) and (2) follow from the corresponding inequalities for the approximations to M; further details are indicated in Theorem 2.2 of [6]. To establish part 3 of Theorem II, suppose that $a \ge x \ge y \ge b$ and P is in D. Let α be as in (iv), and t and s be a subdivision of $\{x, y\}$ and $\{y, b\}$ respectively. Then

$$egin{aligned} & |M(x,\,b)P-M(y,\,b)P| \leq |M(x,\,b)P-\prod_t [1-V]^{-1}\prod_s [1-V]^{-1}P| \ & + |\{\prod_t [1-V]^{-1}-1\}\prod_s [1-V]^{-1}P| \ & + |\prod_s [1-V]^{-1}P-M(y,\,b)P| \ . \end{aligned}$$

Also,

$$|\{\prod_{t} [1 - V]^{-1} - 1\} \prod_{s} [1 - V]^{-1} P|$$

= $|\sum_{p=1}^{n} V(t_{p-1}, t_p) \prod_{i=p}^{n} [1 - V(t_{i-1}, t_i)]^{-1} \prod_{s} [1 - V]^{-1} P|$
 $\leq \alpha(t_0) - \alpha(t_n) .$

For part (4) of Theorem II, suppose that a > b, β is as in (iv), $\varepsilon > 0$, and P is in D. Since $M(\cdot, b)P$ is quasi continuous, M([b, a], b)Pis compact. Hence, there is a positive number δ such that if Q is in M([b, a], b)P, R is in D such that $|Q - R| < \delta$, and $a \ge x \ge y \ge b$ then $|V(x, y)Q - V(x, y)R| \leq [\beta(x) - \beta(y)] \cdot \varepsilon/3$. Suppose that Q is in D such that $|Q - P| < \delta$, $\{t_p\}_0^n$ is a subdivision of $\{x, y\}$ such that if R is P or Q and p is an integer in [1, n] then

$$\left|\prod_{i=p}^{n} [1 - V(t_{i-1}, t_i)]^{-1}R - M(t_{p-1}, b)R\right| < \delta \; .$$

Then

$$\begin{split} |\{\prod_{t} [1 - V]^{-1} - 1\}P - \{\prod_{t} [1 - V]^{-1} - 1\}Q | \\ &\leq \sum_{p=1}^{n} \left| V(t_{p-1}, t_{p}) \prod_{i=p}^{n} [1 - V(t_{i-1}, t_{i})]^{-1}P - V(t_{p-1}, t_{p}) \prod_{i=p}^{n} [1 - V(t_{i-1}, t_{i})]^{-1}Q \right| \\ &\leq [\beta(x) - \beta(y)]\varepsilon \;. \end{split}$$

Proof of Theorem III. This theorem established that the evolution operator M which was found in Theorem II provides a solution to the initial value problem indicated in Theorem III. Note that the integral used is the Cauchy-left integral: If f is a function from [b, a] with values in D then $(L) \int_{a}^{b} V[f]$ is approximated by $\sum_{p=1}^{n} V(t_{p-1}, t_p) f(t_{p-1})$ where t is a subdivision of $\{a, b\}$.

LEMMA 3. Suppose that a > b and f is a function from [b, a]to D which is of bounded variation. It follows that $(L) \int_{a}^{b} V[f]$ exists; in fact, if $\varepsilon > 0$ then there is a subdivision s of $\{a, b\}$ such that if $\{t_{p}\}_{p=0}^{n}$ is a refinement of s then

$$\sum_{p=1}^{n} \left| V(t_{p-1}, t_p) f(t_{p-1}) - (L) \int_{t_{p-1}}^{t_p} V[f] \right| < \varepsilon .$$

LEMMA 4. Suppose that b is in S, P is in D, each of f and g is of bounded variation, and, for each $x \ge b$, $f(x) = P + L \int_x^b V[f]$ and $g(x) = P + (L) \int_x^b V[g]$. It follows that if $x \ge b$ then f(x) = g(x).

Proof. With the supposition of the lemma, let x be in S such that $x \ge b$, ε be a positive number, and $\{t_p\}_{p=0}^n$ be a subdivision of $\{x, b\}$ such that

$$\sum_{p=1}^n \left\{ \left| \int_{t_{p-1}}^{t_p} V[f] - V(t_{p_{-1}}, t_p) f(t_{p_{-1}})
ight|
ight.
ight. + \left| \int_{p_{-1}}^{t_p} V[g] - V(t_{p_{-1}}, t_p) g(t_{p_{-1}})
ight|
ight\} < arepsilon \; .$$

Then

$$\begin{split} |f(x) - g(x)| &\leq |f(x) - g(x)| + \sum_{p=1}^{n} \{ |[1 - V(t_{p-1}, t_p)]f(t_{p-1}) \\ &- [1 - V(t_{p-1}, t_p)]g(t_{p-1})| - |f(t_{p-1}) - g(t_{p-1})| \} \\ &= \sum_{p=1}^{n} \{ -|f(t_p) - g(t_p)| + |[1 - V(t_{p-1}, t_p)]f(t_{p-1}) \\ &- [1 - V(t_{p-1}, t_p)]g(t_{p-1})| \} \leq \sum_{p=1}^{n} \{ \left| \int_{t_{p-1}}^{t_p} V[f] - V(t_{p-1}, t_p)f(t_{p-1}) \right| \\ &+ \left| - \int_{t_{p-1}}^{t_p} V]g \right| + V(t_{p-1}, t_p)g(t_{p-1}) \right| \} \leq \varepsilon \,. \end{split}$$

Thus

$$f(x) = g(x) \; .$$

Indication of proof for Theorem III. Suppose that a > b, P is in D, and s is a subdivision of $\{a, b\}$. Then

$$igg| \prod_{p=1}^n \left[1 - V(s_{p-1}, s_p)
ight]^{-1} P - P - \sum_{p=1}^n V(s_{p-1}, s_p) M(s_{p-1}, b) P igg| \ = igg| \sum_{p=1}^n V(s_{p-1}, s_p) \prod_{i=p}^n \left[1 - V(s_{i-1}, s_i)
ight]^{-1} P - V(s_{p-1}, s_p) M(s_{p-1}, b) P igg| \,.$$

Using the fact that M([b, a], b)P is compact, together with the above corollary, we get that $M(a, b)P - P - (L) \int_{a}^{b} VM(\cdot, b)P = 0$. Lemma 4 shows that this is the only solution to the Stieltjes integral equation.

EXAMPLE. Suppose that g is an increasing, number valued function, A is a function with values in a Banach space G, and that Ahas the following properties: (Compare [6, p. 258].)

(a) If t is a number then $A(t, \cdot)$ has domain all of G,

(b) If P is in G then $A(\cdot, P)$ is continuous,

(c) If a > b, P is in G, and $\varepsilon > 0$ then there is a positive number δ having the property that if $a \ge u \ge b$ and Q is in G such that $|Q - P| < \delta$ then $|A(u, Q) - A(u, P)| < \varepsilon$,

(d) If a > b and B is a bounded subset of G then A is bounded on $[b, a] \times B$, and

(e) If t is a number, P and Q are in G, and c > 0 then

$$|[P - cA(t, P)] - [Q - cA(t, Q)]| \ge |P - Q|.$$

Also, as in [6, p. 258] let $V(x, y)P = (L)\int_{y}^{x} dgA(, P)$ for $x \ge y$ and P in G.

Then V is in OA and if c is a number and P is in G then the preceeding provides the only function f such that

$$f(x) = P - (L) \int_x^c dg A(-, f) .$$

References

1. Carl W. Bitzer, Stieltjes-Volterra integral equations, Illinois J. Math., 14 (1970), 434-451.

2. W. P. Davis and J. A. Chatfield, Concerning product integrals and exponentials, Proc. Amer. Math. Soc., **25** (1970), 743-747.

3. Burrell W. Helton, Integral equations and product integrals, Pacific J. Math., 16 (1966), 297-322.

4. James V. Herod, Multiplicative inverses of solutions for Volterra-Stieltjes integral equations, Proc. Amer. Math. Soc., 22 (1969), 650-656.

5. _____, A Gronwall inequality for Linear Stieltjes integrals, Proc. Amer. Math. Soc., 23 (1969), 34-36.

6. _____, A pairing of a class of evolution systems with a class of generators, Trans. Amer. Math. Soc., 157 (1971), 247-260.

7. _____, Coalescence of solutions for nonlinear Stieltjes equations, J. Reine Angew. Mathematik, **238** (1972), 100-104.

8. T. H. Hildebrandt, On systems of linear differentio-Stieltjes-integral equations, Illinois J. Math., **3** (1959), 352-373.

9. D. B. Hinton, A Stieltjes-Volterra integral equation theory, Canad. J. Math., 18 (1966), 314-331.

10. David Lowell Lovelady, A variation-of-parameters inequality, Proc. Amer. Math. Soc., **26** (1970), 598-602.

11. _____, Perturbations of solutions of Stieltjes integral equations, Trans. Amer. Math. Soc., **155** (1971), 175-187.

12. _____, Bounded solutions of Stieltjes integral equations, Proc. Amer. Math. Soc., 28 (1971), 127-133.

13. _____, Multiplicative integration of infinite products, Canad. J. Math., 23 (1971), 692-698.

14. _____, Addition in a class of nonlinear Stieltjes integrators, Israel J. Math., **10** (1971), 391-396.

15. J. S. Mac Nerney, Integral equations and semigroups, Illinois J. Math., 7 (1963), 148-173.

16. ____, *A linear initial-value problem*, Bull. Amer. Math. Soc., **69** (1963), 314-329.

17. ____, A nonlinear integral operation, Illinois J. Math., 8 (1964), 621-638.

18. G. W. Marrah and T. G. Proctor, Solutions of some periodic Stieltjes integral equations, Proc. Amer. Math. Soc., **34** (1972), 121-127.

19. R. H. Martin, Jr., A bound for solutions of Volterra-Stieltjes integral equations, Proc. Amer. Math. Soc., 23 (1969), 506-512.

20. ____, Product integral approximations of solutions to linear operator equation, Proc. Amer. Math. Soc., **41** (1973), 506-512.

21. James A. Reneke, A product integral solution of a Stieltjes-Volterra integral equation, Proc. Amer. Math. Soc., 24 (1970), 621-626.

22. _____, Continuity for Stieltjes-Volterra integral equation, Technical Report, No. 41, Clemson University.

23. ____, Product integral solutions for hereditary systems, unpublished manuscript.

24. W. W. Schmaedeke and G. R. Sell, The Gronwall inequality for modified Stieltjes integrals, Proc. Amer. Math. Soc., **19** (1968), 1217-1222.

25. H. S. Wall, Concerning harmonic matrices, Arch. Math., 5 (1954), 160-167.

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