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Using the techniques of noncommutative integration theory, classical results of Hermann Weyl concerning the positive eigenvalues of the sum of two self-adjoint compact operators are extended to self-adjoint operators which are measurable with respect to a gage space. Let (H, A, m) be a gage space and let K and L be self-adjoint operators which are measurable with respect to (H, A, m). Let $P_K[\lambda, \infty)$ be the spectral projection of K for the interval $[\lambda, \infty)$ and let $\Lambda_K(x) = \sup\{\lambda \mid m(P_K[\lambda, \infty)) \geqq x\}$. Then $\Lambda_{K+L}(x + r) \leqq \Lambda_K(x) + \Lambda_L(r)$. If $K \leqq L$, then $\Lambda_K(x) \leqq \Lambda_L(x)$. If L is bounded, then $\Lambda_{LKL}(x) \leqq \| L \|^2 \Lambda_K(x)$ for $x \leqq m(P_K[0, \infty))$. If $q = m$(support (L)) and $q < \infty$, then $\Lambda_K(x + q) \leqq \Lambda_{K+L}(x)$; if $\mu = \Lambda_{|K|}(q)$, then $\| K + L \|_p \geqq \| KP_K(-\mu, \mu) \|_p$ for $1 \leqq p \leqq \infty$.

1. Notation. We specifically work in the context of a gage space. [See 5 for definitions and notation.] We will always require that an operator be measurable [5, Definition 2.1]. This is a technical consideration which is necessary to avoid the pathologies which can occur with unbounded operators. Any one of the following conditions implies that a self-adjoint operator T is measurable with respect to the gage space (H, A, m):

1. $T \in A$.

2. $T\eta A$ and m is a finite gage. ($T\eta A$ means that $UT = TU$ for every unitary operator U in the commutant of A.)

3. $T\eta A$ and m(support $(T)) < \infty$, where support (T) is the orthocomplement of the nullspace of T.

4. $T\eta A$ and A is abelian.

If P is a projection operator, P will be identified with the range of P. If T is an operator, $R(T)$ denotes the range of T and $\bar{R}(T)$ denotes the closure of $R(T)$. If T is self-adjoint, note that support $(T) = \bar{R}(T)$.

(H, A, m) is a gage space. If S and T are self-adjoint operators which are measurable with respect to (H, A, m), then $S + T (ST)$ will denote the strong sum (product) of S with T; this is the closure of the ordinary sum (product) and is self-adjoint and measurable [5, Corollary 5.2]. T has spectral decomposition $T = \int_{-\infty}^{\infty} \lambda dP_T(\lambda)$; the function $P_T(\lambda)$ is chosen to be continuous from the right. If $\mathscr{S}$ is an interval, $P_T(\mathscr{S})$ is the spectral projection of T for the interval $\mathscr{S}$. The function Λ_T is defined, for $x > 0$, by $\Lambda_T(x) = \sup\{\lambda \mid m(P_T[\lambda, \infty)) \geqq x\}$.

Note that $m(P_T[\Lambda_T(x), \infty)) < x$ is possible if $m(P_T(\Lambda_T(x) - \varepsilon, \Lambda_T(x))) = \infty$ for every $\varepsilon > 0$. $\Lambda_T(x)$ is a nonincreasing function of x and is continuous from the left. If $x > m(I)$, where I is the identity operator on H, then $\Lambda_T(x) = \sup(\phi) = -\infty$; we will not explicitly mention this pathology in order to avoid excessive technicality and splitting into cases.

The author wishes to thank the referee for finding an error in one of the proofs and for suggestions which make the paper more readable.

2. Statement of the results. Below we state the theorems and corollaries and prove the corollaries. The theorems are proved in the next section. For the remainder of this paper, K and L are self-adjoint operators which are measurable with respect to the gage space (H, A, m).

THEOREM 1. *Let* $q = m(\text{support } (L))$ *and assume* $q < \infty$. *Then* $\Lambda_K(x + q) \leqq \Lambda_{K+L}(x)$ *for* $x > 0$; *equivalently,* $\Lambda_K(x) \leqq \Lambda_{K+L}(x - q)$ *for* $x > q$.

THEOREM 2. $\Lambda_{K+L}(x + r) \leqq \Lambda_K(x) + \Lambda_L(r)$, *for* $x > 0$, $r > 0$.

If (H, A, m) is the algebra of all bounded operators on H, m is the usual trace, and K is a compact operator and has (counting multiplicity) at least $j + 1$ positive eigenvalues, then $\Lambda_K(j)$ is the jth positive eigenvalue of K and $\Lambda_K(j + 1/2)$ is the $(j + 1)$st positive eigenvalue of K. If in addition L is compact and has at least $k + 1$ positive eigenvalues, then Theorem 2 implies $\Lambda_{K+L}(j + k + 1) \leqq \Lambda_K(j + 1/2) + \Lambda_L(k + 1/2) = \Lambda_K(j + 1) + \Lambda_L(k + 1)$, which is Weyl's result [6, Satz 1]. Similarly, Theorem 1 reduces to [6, Satz 2] if L has finite rank.

By $K \leqq L$ is meant $L - K \geqq 0$.

COROLLARY 1. *If* $K \leqq L$, *then* $\Lambda_K(x) \leqq \Lambda_L(x)$ *for* $x > 0$.

Proof of Corollary 1. $K = L + (K - L)$. Note that $K - L \leqq 0$. Let $x > 0$ and let ε be an arbitrary positive number with $\varepsilon < x$. Then $\Lambda_K(x) \leqq \Lambda_L(x - \varepsilon) + \Lambda_{K-L}(\varepsilon)$ by Theorem 2. Since $\Lambda_{K-L}(\varepsilon) \leqq 0$ and the function Λ_L is continuous from the left, $\Lambda_K(x) \leqq \Lambda_L(x)$.

COROLLARY 2. *If* $K \leqq L$ *and* f *is a nondecreasing real-valued function with domain* $(-\infty, \infty)$, *then* $m(f(K)) \leqq m(f(L))$, *provided these quantities are both defined.*

Proof of Corollary 2. $m(f(K)) = \int_{-\infty}^{\infty} f(\lambda) dm(P_K(\lambda))$ and $m(f(L)) =$

$\int_{-\infty}^{\infty} f(\lambda)dm(P_L(\lambda))$. The conclusion is immediate since by Corollary 1, $m(P_K[\lambda, \infty)) \leqq m(P_L[\lambda, \infty))$ and $m(P_L(-\infty, \lambda]) \leqq m(P_K(-\infty, \lambda])$ for all real numbers λ. Note that the hypotheses of Corollary 2 do not imply $f(K) \leqq f(L)$ [1].

THEOREM 3. *$\Lambda_{|K+L|}(x + r) \leqq \Lambda_{|K|}(x) + \Lambda_{|L|}(r)$ for $x > 0$ and $r > 0$, where $|K|$ is the absolute value of the operator K.*

COROLLARY 3. *Let $q = m(\text{support } (L))$ and assume $q < \infty$. Then $\Lambda_{|K|}(x + q) \leqq \Lambda_{|K+L|}(x)$ for $x > 0$; equivalently, $\Lambda_{|K|}(x) \leqq \Lambda_{|K+L|}(x - q)$ for $x > q$.*

Proof of Corollary 3. Let $x > 0$, and let ε be an arbitrary positive number with $\varepsilon < x$. Since $K = (K + L) - L$, by Theorem 3, $\Lambda_{|K|}(x + q) \leqq \Lambda_{|K+L|}(x - \varepsilon) + \Lambda_{|L|}(q + \varepsilon) = \Lambda_{|K+L|}(x - \varepsilon)$ since $\Lambda_{|L|}(q + \varepsilon) = 0$ or $-\infty$. Now apply the left continuity of the function $\Lambda_{|K+L|}$.

THEOREM 4. *Assume L is bounded with norm $\|L\|$. Then $\Lambda_{LKL}(x) \leqq \|L\|^2 \Lambda_K(x)$ for $0 < x \leqq m(P_K[0, \infty))$. In particular, if P is a projection in A, then $\Lambda_{PKP}(x) \leqq \Lambda_K(x)$ for $0 < x \leqq m(P_K[0, \infty))$.*

COROLLARY 4. *Assume L is invertible and $aI \leqq |L| \leqq bI$ for some positive numbers a and b. Then $a^2\Lambda_K(x) \leqq \Lambda_{LKL}(x) \leqq b^2\Lambda_K(x)$ for $0 < x \leqq m(P_K[0, \infty))$, where I is the identity operator on H.*

Proof of Corollary 4. Clearly $|L| \leqq b$ and $|L^{-1}| \leqq 1/a$. Apply Theorem 4 to LKL and to $L^{-1}(LKL)L^{-1} = K$.

The p-norm of a self-adjoint measurable operator T is defined [3, Definition 3.1] by $|T|_p = (m(|T|^p))^{1/p}$ if $1 \leqq p < \infty$ and $\|T\|_\infty = \sup\{\lambda \mid m(P_{|T|}[\lambda, \infty)) > 0\}$. Note that $\|T\|_\infty$ equals the operator norm of T if the gage space is regular, that is, if the gage of every non-zero projection is positive.

THEOREM 5. *Let $q = m(\text{support } (L))$ and assume $q < \infty$. Let $\mu = \Lambda_{|K|}(q)$. Then $\|K + L\|_p \geqq \|KP_K(-\mu, \mu)\|_p$ for $1 \leqq p \leqq \infty$.*

3. Proof of the theorems.

LEMMA 1. *Let P and Q be projections in A. Let $Y = \{v \in H \mid Pv = v$ and $Qv = 0\}$. Then $Y \in A$ and $Y + \bar{R}(PQ) = P$.*

Proof of Lemma 1. Let $v \in Y$ and $w \in R(PQ), w = PQz$. Then

$\langle v, w\rangle = \langle v, PQz\rangle = \langle Pv, Qz\rangle = \langle v, Qz\rangle = \langle Qv, z\rangle = \langle 0, z\rangle = 0$, so that $Y \perp \bar{R}(PQ)$.

Now let $Py = y$. Let z be the projection of y on the subspace $\bar{R}(PQ)$. Then $y = (y - z) + z$. Clearly $z \in \bar{R}(PQ)$ and consequently, $P(y - z) = y - z$. In addition, $(y - z) \perp \bar{R}(PQ)$. Let w be any vector. Then $\langle Q(y - z), w\rangle = \langle QP(y - z), w\rangle = \langle y - z, PQw\rangle = 0$, so that $Q(y - z) = 0$.

LEMMA 2. *Let P and Q be projections in A. Let $Y = \{v \in H \mid Pv = v$ and $Qv = 0\}$. Then $m(Y) \geqq m(P) - m(Q)$. In particular, if $Pv = v$ implies $Qv \neq 0$, then $m(Q) \geqq m(P)$.*

Proof of Lemma 2. By additivity of the gage and Lemma 1, $m(Y) + m(\bar{R}(PQ)) = m(P)$. To prove the lemma, it suffices to show that $m(\bar{R}(PQ)) \leqq m(Q)$; this is well-known for factors but we know of no reference for the general case. For later use, this is proved under the assumption that P is self-adjoint but is not necessarily a projection.

The operator PQ has polar decomposition [2, pp. 323-324] $PQ = M(QP^2Q)^{1/2}$, where M is a partial isometry with initial domain support $(QP^2Q)^{1/2}$ = support $(QP^2Q) = \bar{R}(QP^2Q)$ and terminal domain support $(PQP)^{1/2}$ = support $(PQP) = \bar{R}(PQP)$. Consequently, $\bar{R}(PQ) = \bar{R}(PQP)$; also $m(\bar{R}(PQP)) = m(\bar{R}(QPQ))$ since the initial domain and the terminal domain of a partial isomerty have the same gage. Then $m(\bar{R}(PQ)) = m(\bar{R}(PQP)) = m(\bar{R}(QPQ)) \leqq m(Q)$ since $\bar{R}(QPQ)$ is a subspace of Q.

If H is finite dimensional, Lemma 2 states that the dimension of the solution space of a system of $m(Q)$ homogeneous linear equations in $m(P)$ unknowns is at least $m(P) - m(Q)$.

Proof of Theorem 1. Let $x > 0$ and $\varepsilon > 0$ and $\mu = \Lambda_K(x + q)$. Then $m(P_K[\mu - \varepsilon, \infty)) \geqq x + q$. Apply Lemma 2 with $P = P_K[\mu - \varepsilon, \infty)$ and Q = support (L) to obtain $m\{v \in H \mid P_K[\mu - \varepsilon, \infty)v = v$ and $Lv = 0\} \geqq (x + q) - q = x$. If $P_K[\mu - \varepsilon, \infty)v = v$ and $Lv = 0$, then $\langle (K + L)v, v\rangle \geqq (\mu - \varepsilon) \| v \|^2$, so that $P_{K+L}[\mu - \varepsilon, \infty)v \neq 0$. By Lemma 2, $m(P_{K+L}[\mu - \varepsilon, \infty)) \geqq m\{v \in H \mid P_K[\mu - \varepsilon, \infty)v = v$ and $Lv = 0\} \geqq x$. Since ε is an arbitrary positive number, $\Lambda_{K+L}(x) \geqq \mu = \Lambda_K(x + q)$.

Proof of Theorem 2. Let $x > 0$ and $r > 0$, and assume $\Lambda_{K+L}(x + r) > \Lambda_K(x) + \Lambda_L(r)$. Let $4\delta = \Lambda_{K+L}(x + r) - \Lambda_K(x) - \Lambda_L(r)$. Let $P = P_{K+L}[\Lambda_{K+L}(x + r) - \delta, \infty)$; then $m(P) \geqq x + r$. Let Q be projection on the subspace of H spanned by $P_K[\Lambda_K(x) + \delta, \infty)$ and $P_L[\Lambda_L(r) + \delta, \infty)$; then $m(Q) < x + r$.

Let $Pv = v$ with $\| v \| = 1$. Then $\langle (K + L)v, v\rangle \geqq \Lambda_{K+L}(x + r) - \delta = \Lambda_K(x) + \Lambda_L(r) + 4\delta - \delta > (\Lambda_K(x) + \delta) + (\Lambda_L(r) + \delta)$, so that either

$\langle Kv, v\rangle > \Lambda_K(x) + \delta$ or $\langle Lv, v\rangle > \Lambda_L(r) + \delta$. Therefore $Qv \neq 0$. By Lemma 2, $m(Q) \geqq m(P)$, which is impossible since $m(P) \geqq x + r$ and $m(Q) < x + r$.

Proof of Theorem 3. Let $\lambda > 0$ and $\psi > 0$. Apply Lemma 2 with $P = P_{K+L}[\lambda + \psi, \infty)$ and $Q =$ projection on the subspace of H spanned by $P_K[\lambda, \infty)$ and $P_L[\psi, \infty)$ to obtain $m(P_{K+L}[\lambda + \psi, \infty)) \leqq m(Q) \leqq m(P_K[\lambda, \infty)) + m(P_L[\psi, \infty))$. Similarly, $m(P_{-(K+L)}[\lambda + \psi, \infty)) \leqq m(P_{-K}[\lambda, \infty)) + m(P_{-L}[\psi, \infty))$. Adding these inequalities yields $m(P_{|K+L|}[\lambda + \psi, \infty)) \leqq m(P_{|K|}[\lambda, \infty)) + m(P_{|L|}[\psi, \infty))$.

Let δ be a small positive number. Then $m(P_{|K+L|}[\Lambda_{|K|}(\lambda) + \delta + \Lambda_{|L|}(\psi) + \delta, \infty)) \leqq m(P_{|K|}[\Lambda_{|K|}(\lambda) + \delta, \infty)) + m(P_{|L|}[\Lambda_{|L|}(\psi) + \delta, \infty)) < \lambda + \psi$, so that $\Lambda_{|K+L|}(\lambda + \psi) < \Lambda_{|K|}(\lambda) + \Lambda_{|L|}(\psi) + 2\delta$.

Proof of Theorem 4. Without loss of generality assume $\|L\| = 1$. We will show that $m(P_{LKL}[\lambda, \infty)) \leqq m(P_K[\lambda, \infty))$ for $\lambda > 0$. Let $\lambda > 0$ and let $P = P_{LKL}[\lambda, \infty)$. Let $v \in H$, $\|v\| = 1$, and $Pv = v$. Then $\langle LKLv, v\rangle \geqq \lambda$, so that $\langle KLv, Lv\rangle \geqq \lambda$. Since $\|Lv\| \leqq 1$, $P_K[\lambda, \infty)Lv \neq 0$ and $LP_K[\lambda, \infty)Lv \neq 0$. Let Q be projection on $\bar{R}(LP_K[\lambda, \infty)L)$; then $Qv \neq 0$. By Lemma 2, $m(P) \leqq m(Q)$. But $m(Q) \leqq m(\bar{R}(LP_K[\lambda, \infty))$ because of set inclusion, and $m(\bar{R}(LP_K[\lambda, \infty)) \leqq m(P_K[\lambda, \infty))$; this is proved in the last paragraph of the proof of Lemma 2.

Proof of Theorem 5. If $p = \infty$, $\|K + L\|_\infty = \lim_{\varepsilon \to 0+} \Lambda_{|K+L|}(\varepsilon) \geqq \lim_{\varepsilon \to 0+} \Lambda_{|K|}(q + \varepsilon)$ by Corollary 3. If $m(P_{|K|}[\mu, \infty)) > q$, then $\lim_{\varepsilon \to 0+} \Lambda_{|K|}(q + \varepsilon) = \mu \geqq \|KP_K(-\mu, \mu)\|_\infty$. If $m(P_{|K|}[\mu, \infty)) = q$, then $\Lambda_{|K|}(q + \varepsilon) = \Lambda_{|KP_K(-\mu,\mu)|}(\varepsilon)$ for all $\varepsilon > 0$ and the result is immediate. If $m(P_{|K|}[\mu, \infty)) < q$, then $m(P_{|K|}(\mu - \varepsilon, \mu)) = \infty$ for every $\varepsilon > 0$, so that $\lim_{\varepsilon \to 0+} \Lambda_{|K|}(q + \varepsilon) = \mu = \|KP_K(-\mu, \mu)\|_\infty$.

Now let $1 \leqq p < \infty$. Since the theorem is trivial for $\mu = 0$, assume $\mu > 0$. If $m(P_{|K+L|}[\lambda, \infty)) = \infty$ for some $\lambda > 0$, then $\|K + L\|_p = \infty$ and the theorem holds trivially. Therefore, assume $m(P_{|K+L|}[\lambda, \infty)) < \infty$ for all $\lambda > 0$. Fix λ, $0 < \lambda < \mu$. Let $\gamma = m(P_{|K+L|}[\lambda, \infty))$ and let $\varepsilon > 0$. Note that if $\lambda = \Lambda_{|K+L|}(\gamma + \varepsilon)$, then $m(P_{|K+L|}(\lambda - \varepsilon_1, \lambda)) = \infty$ for every $\varepsilon_1 > 0$. Therefore, we can assume $\lambda > \Lambda_{|K+L|}(\gamma + \varepsilon)$. By Corollary 3, $\Lambda_{|K+L|}(\gamma + \varepsilon) \geqq \Lambda_{|K|}(\gamma + q + \varepsilon)$. Therefore, $m(P_{|K|}[\lambda, \infty)) \leqq \gamma + q + \varepsilon$. Since ε is arbitrary, $m(P_{|K+L|}[\lambda, \infty)) \geqq m(P_{|K|}[\lambda, \infty)) - q \geqq m(P_{|K|}[\lambda, \mu))$ for $\lambda < \mu$. Now $\|K + L\|_p^p = \int_{-\infty}^{\infty} |\lambda|^p dm(P_{K+L}(\lambda)) = \int_0^\infty \lambda^p dm(P_{|K+L|}(\lambda))$ and

$$\|KP_K(-\mu, \mu)\|_p^p = \int_{-\mu < \lambda < \mu} |\lambda|^p dm(P_K(\lambda)) = \int_{0 \leqq \lambda < \mu} \lambda^p dm(P_{|K|}(\lambda)) .$$

The conclusion follows immediately.

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