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CONTINUOUS OPERATORS ON PARANORMED SPACES AND MATRIX TRANSFORMATIONS

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The concept of a paranormed β -space is defined and some theorems of Banach-Steinhaus type are proved for sequences of continuous linear functionals on such a space. For example, necessary and sufficient conditions are given for a sequence $(A_n(x))$ of continuous linear functionals to be in the space of generalized entire sequences, for each x belonging to a paranormed β -space. The general theorems are then used to characterize matrix transformations between generalized l_p spaces and generalized entire sequences.

1. In § 2 we present theorems which generalize some results in [10]. These theorems are applied in § 3 to characterize some classes of matrix transformations. By N, R and C we denote respectively, the sets of natural numbers, real numbers, and complex numbers. By a sequence (x_k) we mean (x_1, x_2, \dots) , and by $\sum_{k=1}^{\infty} x_k$ we mean

X will denote a nontrivial complex linear space of elements x , with zero element θ and with paranorm g , i.e. $g: X \rightarrow R$ satisfies $g(\theta) = 0$, $g(x) = g(-x)$ on X , g is subadditive, and, for $\lambda \in C$ and $x \in X$, $\lambda \rightarrow \lambda_0$ and $g(x - x_0) \rightarrow 0$ imply $g(\lambda x - \lambda_0 x_0) \rightarrow 0$, where $\lambda_0 \in C$ and $x_0 \in X$.

Extending the definitions of Sargent in [8], we can define a paranormed β -space as follows. Let (X_n) be a sequence of subsets of X such that $\theta \in X_1$ and such that if $x, y \in X_n$ then $x \pm y \in X_{n+1}$ for $n \in N$; then (X_n) is called an α -sequence in X . If we can write $X = \bigcup_{n=1}^{\infty} X_n$, where (X_n) is an α -sequence in X and each X_n is nowhere dense in X , then X is called an α -space; otherwise X is a β -space. Clearly, every α -space is of the first category, whence we see that any complete paranormed space is a β -space.

If $Y \subset X$ then we denote the closure of Y in X by \bar{Y} . We write, for any $a \in X$ and $\delta > 0$, $S(a, \delta) = \{x: x \in X \text{ and } g(x - a) < \delta\}$. A subset G of X is called a fundamental set in X if $l.\text{hull}(G)$, the set of all finite linear combinations of elements of G , is dense in X . A sequence (b_k) of elements of X is said to be a basis in X if for each $x \in X$ there is a unique complex sequence (λ_k) such that $g(x - \sum_{k=1}^n \lambda_k b_k) \rightarrow 0 (n \rightarrow \infty)$. Thus any basis in X is also a fundamental set in X .

We denote the set of continuous linear functionals on X by X^* . A linear functional A on X is an element of X^* if and only if

$$\|A\|_M \equiv \sup \left\{ \|A(x)\| : \|x\| \leq \frac{1}{M} \right\} < \infty \text{ for some } M > 1.$$

If X is a space of complex sequences $x = (x_k)$, then we denote the generalized Köthe-Toeplitz dual of X by X^\dagger , i.e.

$$X^\dagger = \{(\alpha_k) : \sum_k \alpha_k x_k \text{ converges for every } x \in X\}.$$

We now list some sets of complex sequences due to Maddox [4]. If $p = (p_k)$ is a sequence of strictly positive real numbers, then

$$l_\infty(p) = \{x : \sup_k |x_k|^{p_k} < \infty\},$$

$$c_0(p) = \{x : \lim_k |x_k|^{p_k} = 0\},$$

$$c(p) = \{x : \lim_k |x_k - l|^{p_k} = 0 \text{ for some } l \in C\},$$

$$l(p) = \{x : \sum_k |x_k|^{p_k} < \infty\}.$$

We write $e^{(k)} = (0, 0, \dots, 1, 0, 0, \dots)$, the 1 occurring in the k^{th} place, for each $k \in N$, and $e = (1, 1, 1, \dots)$, and we write $l_\infty = l_\infty(e)$, $c_0 = c_0(e)$, $c = c(e)$, and $l_1 = l(e)$.

The case $p = (1/k)$ of $c_0(p)$ is of particular interest, since the function defined by $\sum_{k=0}^\infty \alpha_k z^k$, $z \in C$, is an entire function if and only if $(\alpha_k) \in c_0(1/k)$. Work on the space of entire functions has been carried out, by V. Ganapathy Iyer in [2] and in other papers, and by other authors, using this correspondence with $c_0(1/k)$. It is shown in [2] that $c_0(1/k)^\dagger = l_\infty(1/k)$.

Now we collect some known results which will be useful in what follows.

LEMMA 1. $l(p)$ is a linear space if and only if p is bounded. (See [4], Theorem 1, and [7], Theorem 1.)

LEMMA 2. If p is bounded with $H = \max(\sup_k p_k, 1)$, then $g(x) = (\sum_k |x_k|^{p_k})^{1/H}$ defines a paranorm on $l(p)$, $l(p)$ is complete under g , and $(e^{(k)})$ is a basis in $l(p)$. (See [5], Theorem 1 and Corollary 1, and [7].)

LEMMA 3. (i) If $1 < p_k \leq H$ and $p_k^{-1} + s_k^{-1} = 1$ for each $k \in N$, then

$$l(p)^\dagger = \{(\alpha_k) : \sum_k |\alpha_k|^{s_k} \cdot M^{-s_k} < \infty \text{ for some } M > 1\}.$$

(ii) If $0 < p_k \leq 1$ for all $k \in N$ then $l(p)^\dagger = l_\infty(p)$. (See [6], Theorem 1, and [9], Theorem 7.)

LEMMA 4. If either $1 < p_k \leq H$ for all k , or $0 < p_k \leq 1$ for all

k , then every $A \in l(p)^*$ may be written as $A(x) \equiv \sum_k \alpha_k x_k$ on $l(p)$ for some $(\alpha_k) \in l(p)^\dagger$, and conversely $A(x) \equiv \sum_k \alpha_k x_k$ defines an element of $l(p)^*$ for each $(\alpha_k) \in l(p)^\dagger$. (See [6], Theorem 2, and [9], Theorem 7.)

Given sets Y and Z of sequences and a matrix $A = (a_{n,k})$ of complex numbers $(n, k = 1, 2, \dots)$ we say that $A \in (Y, Z)$ if and only if $\sum_k a_{n,k} y_k$ converges for every $y = (y_k) \in Y$ and $n \in N$, and $(\sum_k a_{n,k} y_k) \in Z$ for every $y \in Y$.

We shall frequently use the following inequalities. Take $x, y \in C$; if $0 < p \leq 1$ then

$$|x|^p - |y|^p \leq |x + y|^p \leq |x|^p + |y|^p,$$

and if $p > 1$ and $p^{-1} + s^{-1} = 1$ then

$$|xy| \leq |x|^p + |y|^s.$$

2. For the remainder of this paper, $q = (q_n)$ will denote a sequence of strictly positive real numbers. If q is bounded with $H = \max(\sup q_n, 1)$ then it follows by Lemma 1 of [4] that $c_0(q) = c_0(H^{-1}q)$; similarly $l_\infty(q) = l_\infty(H^{-1}q)$ and $c(q) = c(H^{-1}q)$.

THEOREM 1. *Let X be a paranormed space and let (A_n) be a sequence of elements of X^* , and suppose q is bounded. Then*

$$(1) \quad \sup_n (\|A_n\|_M)^{q_n} < \infty \text{ for some } M > 1$$

implies

$$(2) \quad (A_n(x)) \in l_\infty(q) \text{ for every } x \in X,$$

and the converse is true if X is a β -space.

Proof. In view of the remarks at the beginning of this section, we may without loss of generality assume that $q_n \leq 1$ for all $n \in N$. First let (1) hold, and choose any $x \in X$. By the continuity of scalar multiplication in a paranormed space, there is a $K \geq 1$ such that $g(K^{-1}x) \leq 1/M$, where the M is that of (1). Then we have for any n , since $q_n \leq 1$,

$$\begin{aligned} |A_n(x)|^{q_n} &= |KA_n(K^{-1}x)|^{q_n} \leq K^{q_n} (\|A_n\|_M)^{q_n} \\ &\leq K \sup_n (\|A_n\|_M)^{q_n}, \end{aligned}$$

so that (2) holds.

Now let (2) hold, with X a β -space, and define for any $m \in N$,

$$X_m = \{x: x \in X \text{ and } |A_n(x)|^{q_n} \leq 2^m \text{ for all } n \in N\}.$$

Then (X_m) is an α -sequence in X , for obviously $\theta \in X_1$, and if for

some $m \geq 1$, $x, y \in X_m$ then, since $q_n \leq 1$ for every n ,

$$|A_n(x \pm y)|^{q_n} \leq |A_n(x)|^{q_n} + |A_n(y)|^{q_n} \leq 2^{m+1}$$

for any $n \in N$. Also $X = \bigcup_{m=1}^{\infty} X_m$, so since X is a β -space there exists a $B \in N$ such that X_B is not nowhere dense. Using the continuity of the A_n , it is not difficult to show that $\bar{X}_m = X_m$ for every m , whence there is a sphere $S(a, \delta) \subset X_B$. Thus if $g(x - a) < \delta$ we have $|A_n(x)|^{q_n} \leq 2^B$ for all n , so if $g(x) < \delta$ we have

$$|A_n(x)|^{q_n} \leq |A_n(x + a)|^{q_n} + |A_n(a)|^{q_n} \leq 2^{B+1} \text{ for all } n.$$

Taking $M > \delta^{-1}$ we obtain (1).

THEOREM 2. *Let X be a paranormed space and let (A_n) be a sequence of elements of X^* .*

(i) *If X has fundamental set G and if q is bounded, then the following propositions*

$$(3) \quad (A_n(b)) \in c_0(q) \text{ for every } b \in G,$$

$$(4) \quad \lim_M \limsup_n (||A_n||_M)^{q_n} = 0,$$

together imply

$$(5) \quad (A_n(x)) \in c_0(q) \text{ for every } x \in X.$$

(ii) *If $q_n \rightarrow 0 (n \rightarrow \infty)$ then (4) implies (5).*

(iii) *Let X be a β -space; then (5) implies (4) even if q is unbounded.*

Proof. (i) Again, we may without loss of generality assume that $q_n \leq 1$ for every $n \in N$. Let X have fundamental set G , and suppose (3) and (4) hold. Choose any $x \in X$ and any $\varepsilon > 0$. There exist $M > 1$ and n_0 such that $(||A_n||_M)^{q_n} < \varepsilon/2$ for all $n \geq n_0$, by (4). Since $l \cdot \text{hull}(G)$ is dense in X there exist $\lambda_1, \lambda_2, \dots, \lambda_m \in C$ and $b_1, b_2, \dots, b_m \in G$ such that $g(x - \sum_{k=1}^m \lambda_k b_k) < 1/M$, and we write $L = \max(|\lambda_1|, \dots, |\lambda_m|, 1)$. Then by (3) there is an $n_1 \geq n_0$ such that $|A_n(b_k)|^{q_n} < \varepsilon/(2Lm)$, $k = 1, 2, \dots, m$, if $n \geq n_1$, whence if $n \geq n_1$, we have

$$\begin{aligned} |A_n(x)|^{q_n} &= \left| A_n \left(x - \sum_{k=1}^m \lambda_k b_k \right) + \sum_{k=1}^m \lambda_k A_n(b_k) \right|^{q_n} \\ &\leq \left| A_n \left(x - \sum_{k=1}^m \lambda_k b_k \right) \right|^{q_n} + L \sum_{k=1}^m |A_n(b_k)|^{q_n} \\ &\leq (||A_n||_M)^{q_n} + mL \cdot \varepsilon/(2Lm) < \varepsilon; \end{aligned}$$

thus (5) holds.

(ii) Suppose (4) holds and $q \in c_0$, and choose any $x \in X$ and any $\varepsilon > 0$. There is an $M > 1$ and an n_0 such that $(\|A_n\|_M)^{q_n} < \varepsilon/2$ if $n \geq n_0$, and since scalar multiplication is continuous on X there is a $K \geq 1$ such that $g(K^{-1}x) \leq 1/M$. Then we can choose $n_1 \geq n_0$ such that $K^{q_n} \leq 2$ if $n \geq n_1$ whence if $n \geq n_1$

$$\|A_n(x)\|^{q_n} = K^{q_n} \|A_n(K^{-1}x)\|^{q_n} < \varepsilon,$$

so that (5) is true.

(iii) Let X be a β -space and suppose (5) is true. We define sequences $(B_n), (C_n)$ of elements of X^* and sequences $r = (r_n), s = (s_n)$ of strictly positive real numbers as follows. If $q_n \geq 1$ then define $B_n = A_n, C_n = 0, r_n = q_n$, and $s_n = 1$; if $q_n < 1$ write $B_n = 0, C_n = A_n, r_n = 1$, and $s_n = q_n$. Then $(B_n(x)) \in c_0(r)$ and $(C_n(x)) \in c_0(s)$ on X ; $\sup_n s_n \leq 1$, and $r_n \geq 1$ for all $n \in N$. Also, $(\|A_n\|_M)^{q_n} = (\|B_n\|_M)^{r_n} + (\|C_n\|_M)^{s_n}$ for all large enough $M, n = 1, 2, \dots$, whence

$$\begin{aligned} \lim_M \limsup_n (\|A_n\|_M)^{q_n} &\leq \lim_M \limsup_n (\|B_n\|_M)^{r_n} \\ &\quad + \lim_M \limsup_n (\|C_n\|_M)^{s_n}. \end{aligned}$$

Choose any $\varepsilon > 0$, and define for each $m \in N$

$$X_m = \{x: x \in X \text{ and } |2^{-m}C_n(x)|^{s_n} \leq \frac{\varepsilon}{2} \text{ for all } n \geq m\}.$$

Clearly $\theta \in X_1$, and if for some $m \in N$ we have $x, y \in X_m$ then for $n \geq m + 1$

$$\begin{aligned} |2^{-(m+1)}C_n(x \pm y)|^{s_n} &\leq (|2^{-(m+1)}C_n(x)| + |2^{-(m+1)}C_n(y)|)^{s_n} \\ &\leq (2 \max(|2^{-(m+1)}C_n(x)|, |2^{-(m+1)}C_n(y)|))^{s_n} \\ &= \max(|2^{-m}C_n(x)|^{s_n}, |2^{-m}C_n(y)|^{s_n}) \leq \frac{\varepsilon}{2}; \end{aligned}$$

thus (X_m) is an α -sequence in X . Also $X = \bigcup_{m=1}^{\infty} X_m$ and $X_m = \bar{X}_m$ for all $m \in N$ whence, since X is a β -space, some X_B contains a sphere $S(\alpha, \delta)$. Then if $g(x) < \delta$ we deduce that $|2^{-B}C_n(x)|^{s_n} \leq \varepsilon$ for $n \geq B$. Write $\rho = 2^{-B}\delta$ and choose $M > \rho^{-1}$; then by the subadditivity of g we have $g(2^Bx) < \delta$ if $g(x) < \rho$. Hence if $g(x) \leq 1/M$ we have

$$\|C_n(x)\|^{s_n} = |2^{-B}C_n(2^Bx)|^{s_n} \leq \varepsilon \text{ if } n \geq B,$$

and since $\varepsilon > 0$ was arbitrary we obtain $\lim_M \limsup_n (\|C_n\|_M)^{s_n} = 0$.

Now $(B_n(x)) \in c_0(r)$ on X implies $(B_n(x)) \in c_0$ on X . For suppose if possible that for some sequence $(n(i))$ of integers and some $x \in X$ $\inf |B_{n(i)}(x)| = \alpha > 0$; then $|B_{n(i)}(\alpha^{-1}x)|^{r_{n(i)}} \geq 1$ for all i , contrary to hypothesis. By the argument used above we deduce that

$$\lim_M \limsup_n \|B_n\|_M = 0,$$

whence since $r_n \geq 1$ for all n , $\lim_M \limsup_n (\|B_n\|_M)^{r_n} = 0$. By our earlier remarks, (4) now follows.

THEOREM 3. *Let X be a paranormed space and let (A_n) be a sequence of element of X^* and suppose q is bounded.*

(i) *If X has fundamental set G , and if there is an $l \in X^*$ such that $(A_n(b) - l(b)) \in c_0(q)$ for all $b \in G$ and*

$$(6) \quad \lim_M \limsup_n (\|A_n - l\|_M)^{q_n} = 0,$$

then

$$(7) \quad (A_n(x)) \in c(q) \text{ on } X.$$

(ii) *If $q_n \rightarrow 0 (n \rightarrow \infty)$ and if there is an $l \in X^*$ such that (6) holds, then (7) is true.*

(iii) *If X is a β -space and if (7) is true, then there is an $l \in X^*$ such that (6) holds.*

Proof. (i) If the hypotheses hold, then $A_n - l \in X^*$ for every $n \in N$ whence by part (i) of Theorem 2 $((A_n - l)(x)) \in c_0(q)$ on X ; thus (7) is true.

(ii) Follows similarly from Theorem 2(ii).

(iii) Suppose (7) holds; then for some l we have $|A_n(x) - l(x)|^{q_n} \rightarrow 0 (n \rightarrow \infty)$ on X . We deduce that $l(x) = \lim_n A_n(x)$ on X and $\sup_n |A_n(x)| < \infty$ on X . Then by Theorem 1 we have $\sup_n \|A_n\|_M < \infty$ for some $M > 1$, whence $\|l\|_M < \infty$. Clearly l must be linear, so that $l \in X^*$. Thus $A_n - l \in X^*$ for each $n \in N$, and by hypothesis $((A_n - l)(x)) \in c_0(q)$ on X , so by Theorem 2(iii), (6) must be true.

3. We now apply the theorems above in characterizing the classes $(l(p), l_\infty(q))$, $(l(p), c_0(q))$, and $(l(p), c(q))$ in the case when both p and q are bounded. Throughout, $A = (a_{n,k})$ will denote an infinite matrix of complex numbers. As a preliminary, we state Theorem 1 of [3]:

THEOREM 4. (i) *Let $1 < p_k \leq H < \infty$ and $p_k^{-1} + s_k^{-1} = 1$ for every k . Then $A \in (l(p), l_\infty)$ if and only if there exists an integer $B > 1$ such that $\sup_n \sum_k |a_{n,k}|^{s_k} \cdot B^{-s_k} < \infty$.*

(ii) *Let $0 < p_k \leq 1$ for every k . Then $A \in (l(p), l_\infty)$ if and only if $\sup_{n,k} |a_{n,k}|^{p_k} < \infty$.*

In the proofs of the following results, as in earlier ones, we may without loss of generality assume that $q_n \leq 1$ for all $n \in N$, and we shall do so when convenient.

We first consider the case when $0 < p_k \leq 1$ for all $k \in N$.

THEOREM 5. Suppose $0 < p_k \leq 1$ for all $k \in N$, and $q = (q_n)$ is bounded. Then,

(i) $A \in (l(p), l_\infty(q))$ if and only if

$$(8) \quad \sup_n (\sup_k |a_{n,k}| M^{-1/p_k})^{q_n} < \infty \text{ for some } M > 1.$$

(ii) $A \in (l(p), c_0(q))$ if and only if

$$(9) \quad |a_{n,k}|^{q_n} \rightarrow 0 (n \rightarrow \infty) \text{ for every } k \in N$$

and

$$(10) \quad \lim_M \limsup_n (\sup_k |a_{n,k}| M^{-1/p_k})^{q_n} = 0.$$

(iii) $A \in (l(p), c(q))$ if and only if $\sup_n \sup_k |a_{n,k}| M^{-1/p_k} < \infty$ for some $M > 1$ and there exist $\alpha_1, \alpha_2, \dots$ such that

$$(11) \quad |a_{n,k} - \alpha_k|^{q_n} \rightarrow 0 (n \rightarrow \infty) \text{ for each } k \in N$$

and

$$(12) \quad \lim_M \limsup_n (\sup_k |a_{n,k} - \alpha_k| M^{-1/p_k})^{q_n} = 0.$$

Proof. Write, for each $x \in l(p)$ and each $n \in N$

$$(13) \quad A_n(x) \equiv \sum_k a_{n,k} x_k.$$

(i) Let $A \in (l(p), l_\infty(q))$; then for each n , $(a_{n,1}, a_{n,2}, \dots) \in l(p)^\dagger = l_\infty(p)$, by Lemma 3(ii). Also, by Lemma 4, $A_n \in l(p)^*$ for each $n \in N$. We show that for each n , $\|A_n\|_M = \sup_k |a_{n,k}| M^{-1/p_k}$ for all M such that $\|A_n\|_M$ is defined. Choose any $n \in N$. First, if M is such that, for some sequence $(k(i))$ of integers, $|a_{n,k(i)}| M^{-1/p_{k(i)}} \geq i$ for each $i \in N$, then by defining $x^{(k(i))} = (M^{-1/p_{k(i)}} \operatorname{sgn} a_{n,k(i)}) e^{(k(i))}$, $i = 1, 2, \dots$, we see that $\|A_n\|_M$ is undefined. Since $(a_{n,1}, a_{n,2}, \dots) \in l_\infty(p)$ there is an $M_n \geq 1$ such that $|a_{n,k}|^{p_k} \leq M_n$ for all k . Choose $M \geq M_n$. We have if $g(x) = \sum_k |x_k|^{p_k} \leq 1/M$, since $M^{1/p_k} |x_k| \leq 1$ for all k and since $\sup_k p_k \leq 1$,

$$\begin{aligned} |A_n(x)| &\leq \sum_k |a_{n,k}| M^{-1/p_k} \cdot M^{1/p_k} |x_k| \\ &\leq \sum_k |a_{n,k}| M^{-1/p_k} \cdot M |x_k|^{p_k} \\ &\leq M g(x) \sup_k |a_{n,k}| M^{-1/p_k}, \end{aligned}$$

whence $\|A_n\|_M \leq \sup_k |a_{n,k}| M^{-1/p_k}$. Given $\varepsilon > 0$ we can choose an m such that $|a_{n,m}| M^{-1/p_k} > \sup_k |a_{n,k}| M^{-1/p_k} - \varepsilon$. Defining $x = (M^{-1/p_k} \operatorname{sgn} a_{n,m}) e^{(m)}$ we have $g(x) \leq 1/M$ and $A_n(x) > \sup_k |a_{n,k}| M^{-1/p_k} - \varepsilon$, whence $\|A_n\|_M = \sup_k |a_{n,k}| M^{-1/p_k}$ as required. By Lemma 2, $l(p)$ is complete, so it is a β -space; thus by Theorem 1 we must have (8).

Conversely let (8) hold. Then as above it follows that for each n , $A_n \in l(p)^*$ with $\|A_n\|_M = \sup_k |a_{n,k}| M^{-1/p_k}$ for all M such that

$\|A_n\|_M$ is defined. Then using Theorem 1 we obtain $(A_n(x)) \in l_\infty(q)$ on $l(p)$, i.e. $A \in (l(p), l_\infty(q))$.

We remark that (8) reduces to $\sup_{n,k} |a_{n,k}|^{p_k} < \infty$ if $0 < \inf q_n \leq \sup q_n < \infty$, corresponding to the condition given for $A \in (l(p), l_\infty)$ in Theorem 4(ii).

(ii) If $A \in (l(p), c_0(q)) \subset (l(p), l_\infty(q))$ then as above we have $A_n \in X^*$ and $\|A_n\|_M = \sup_k |a_{n,k}| M^{-1/p_k}$ whenever $\|A_n\|_M$ is defined, for each $n \in N$. Then, by Theorem 2(iii), (10) must hold. Also taking $x = e^{(k)} \in l(p)$ ($k = 1, 2, \dots$) we obtain (9). Conversely if (9) and (10) hold we can show that $A_n \in l(p)^*$ with $\|A_n\|_M = \sup_k |a_{n,k}| M^{-1/p_k}$ whenever $\|A_n\|_M$ is defined, for each $n \in N$; also $(e^{(k)})$ is a basis in $l(p)$ by Lemma 2. Then by Theorem 2(i) we can deduce that $A \in (l(p), c_0(q))$.

(iii) Let $A \in (l(p), c(q))$; then as in (i) and (ii) above we have for each n that $A_n \in X^*$. By Theorem 3(iii) there is an $l \in X^*$ such that $\lim_M \limsup_n (\|A_n - l\|_M)^{q_n} = 0$, and by Lemmas 3(ii) and 4 we can write $l(x) = \sum_k \alpha_k x_k$ on $l(p)$ for some $(\alpha_k) \in l_\infty(p)$. We deduce that $\|A_n - l\|_M = \sup_k |a_{n,k} - \alpha_k| M^{-1/p_k}$ for large enough M , $n = 1, 2, \dots$, whence (12) is true, and (11) must hold since $(A_n - l)(e^{(k)}) = a_{n,k} - \alpha_k$ for each n and k . Also $c(q) \subset l_\infty$ whence $(l(p), (c(q)) \subset (l(p), l_\infty)$; thus by (i) we must have $\sup_k |a_{n,k}| M^{-1/p_k} < \infty$ for some $M > 1$.

Finally, if $\sup_k |a_{n,k}| M^{-1/p_k} < \infty$ for some $M > 1$ then $A_n \in l(p)^*$ for all n . If in addition (11) and (12) hold then for any k we have, if n and M are large enough,

$$\begin{aligned} |\alpha_k| M^{-1/p_k} &\leq |a_k - a_{n,k}| M^{-1/p_k} + |a_{n,k}| M^{-1/p_k} \\ &\leq 1 + \sup_n (\sup_k |a_{n,k}| M^{-1/p_k}) = B \text{ say;} \end{aligned}$$

hence $|\alpha_k|^{p_k} \leq B^{p_k} \cdot M \leq BM$ for all k , i.e. $(\alpha_k) \in l_\infty(p) = l(p)^\dagger$. By Lemma 4, $l(x) \equiv \sum_k \alpha_k x_k$ defines an element of $l(p)^*$, and the result now follows if we employ the methods used above together with Theorem 3(i).

THEOREM 6. Suppose $0 < p_k \leq 1$ for all $k \in N$ and $q_n \rightarrow 0$ ($n \rightarrow \infty$). Then $A \in (l(p), c_0(q))$ if and only if (12) is true.

Proof. This follows from Theorem 2, parts (ii) and (iii), on using the methods of Theorem 5.

COROLLARY. (i) $A \in (l_1, c_0(1/n))$ if and only if $|a_{n,k}|^{1/n} \rightarrow 0$ uniformly in k as $n \rightarrow \infty$.

(ii) $A \in (l_1, l_\infty(1/n))$ if and only if $\sup_{n,k} |a_{n,k}|^{1/n} < \infty$.

Proof. These characterizations were given in Theorems 1 and 2 of [1], and follow readily on taking $p = e$ and $q = (1/n)$ in Theorems 5(i) and 6.

Now we consider the case when $1 < p_k \leq H < \infty$ for all k .

THEOREM 7. *Let $1 < p_k \leq H$ and $p_k^{-1} + s_k^{-1} = 1$ for each $k \in N$, and let q be bounded. Then $A \in (l(p), l_\infty(q))$ if and only if*

$$(14) \quad T(B) \equiv \sup_n \sum_k |a_{n,k}|^{s_k} \cdot B^{-s_k/q_n} < \infty \text{ for some } B > 1.$$

Proof. Define A_n by (13) on $l(p)$, for each $n \in N$. For the sufficiency, let (14) hold. Then if $x \in l(p)$ we have for each n , assuming $q_n \leq 1$ for all n ,

$$\begin{aligned} |A_n(x)|^{q_n} &\leq (\sum_k |a_{n,k} x_k|)^{q_n} = (\sum_k |a_{n,k}| B^{-1/q_n} \cdot B^{1/q_n} |x_k|)^{q_n} \\ &\leq (\sum_k |a_{n,k}|^{s_k} \cdot B^{-s_k/q_n} + \sum_k B^{p_k/q_n} |x_k|^{p_k})^{q_n} \\ &\leq (T(B)^{q_n} + B^H (g^H(x))^{q_n}) \\ &\leq T(B) + 1 + B^H (g^H(x) + 1) \end{aligned}$$

which implies $A \in (l(p), l_\infty(q))$.

Now let $A \in (l(p), l_\infty(q))$; then $(a_{n,1}, a_{n,2}, \dots) \in l(p)^\dagger$ for each n and so, by Lemmas 3(i) and 4, $A_n \in l(p)$ for all n . By Theorem 1 there exist $M > 1$ and $G \geq 1$ such that $|A_n(x)|^{q_n} \leq G$ for all n and all $x \in l(p)$ with $g(x) \leq 1/M$. Then $|\sum_k G^{-1/q_n} \cdot a_{n,k} x_k| \leq 1$, $n = 1, 2, \dots$, if $g(x) \leq 1/M$. Write $\Gamma = (G^{-1/q_n} a_{n,k})$, and choose any $x \in l(p)$. By the continuity of scalar multiplication on $l(p)$ there is a $K \geq 1$ such that $g(K^{-1}x) \leq 1/M$, whence $|\sum_k G^{-1/q_n} \cdot a_{n,k} x_k| \leq K$ for all n . Thus we see that $\Gamma \in (l(p), l_\infty)$ and so by Theorem 4(i) there is a $D > 1$ such that $\sup_n \sum_k |G^{-1/q_n} \cdot a_{n,k}|^{s_k} \cdot D^{-s_k} < \infty$. Writing $B = GD$ and using the fact that $D^{q_n} \leq D$ for all n , we obtain (14).

Looking at Theorem 4, one might except the necessary and sufficient condition for $A \in (l(p), l_\infty(q))$ to be

$$(15) \quad \sup_n (\sum_k |a_{n,k}|^{s_k} \cdot M^{-s_k})^{q_n} < \infty \text{ for some } M > 1.$$

Using the method above we can show that (15) implies $A \in (l(p), l_\infty(q))$. In fact it can be shown that (15) implies (14) directly. For let (15) hold; then for some $B > 1$, $(\sum_k |a_{n,k}|^{s_k} \cdot B^{-s_k})^{q_n} \leq H$ for all n , and we may suppose that $H > 1$. If $q_n \leq Q$ for all n then

$$(16) \quad (\sum_k |a_{n,k}|^{s_k} \cdot B^{-s_k} \cdot H^{-1/q_n})^{q_n/Q} \leq 1 \text{ for all } n.$$

Put $M = HB^Q$; then $M^{s_k} = H^{s_k} \cdot B^{Qs_k} \geq H \cdot B^{q_n s_k}$, whence $M^{s_k/q_n} \geq H^{1/q_n} \cdot B^{s_k}$ for all k and n . Thus by (16) we obtain

$$\begin{aligned} \sum_k |a_{n,k}|^{s_k} \cdot M^{-s_k/q_n} &\leq \sum_k |a_{n,k}|^{s_k} \cdot B^{-s_k} \cdot H^{-1/q_n} \\ &\leq (\sum_k |a_{n,k}|^{s_k} \cdot B^{-s_k} \cdot H^{-1/q_n})^{q_n/Q} \leq 1 \text{ for all } n, \end{aligned}$$

whence $T(M) \leq 1$, i.e. (14) holds.

Clearly, (14) implies (15) if $\inf_n q_n > 0$ or if $\inf_k p_k > 1$. However, (15) is not necessary for $A \in (l(p), l_\infty(q))$ if $\inf_n q_n = 0$ and $\inf_k p_k = 1$. For choose bounded p and q , with $p_k > 1$ for all k , and suppose there exist sequence $(n(i), (k(j)))$ of integers such that $q_{n(i)} \leq 1/i$, $i = 1, 2, \dots$, and $p_{k(j)} \leq 1 + 1/j$, $j = 1, 2, \dots$; then $s_{k(j)} \geq j + 1$ for each j . Define $a_{n(i), k(j)} = i$, $i, j = 1, 2, \dots$, and $a_{n,k} = 0$ for all other n and k . Then $A = (a_{n,k}) \in (l(p), l_\infty(q))$ since for all $i \in N$.

$$\Sigma_j |a_{n(i), k(j)}|^{s_{k(j)}} \cdot 2^{-s_{k(j)/q_{n(i)}}} \leq \Sigma_j (i/2^i)^{j+1} \leq 1,$$

but for any $M > 1$ we have if $i \geq M$,

$$(\Sigma_j |a_{n(i), k(j)}|^{s_{k(j)}} \cdot M^{-s_{k(j)}})^{q_{n(i)}} \geq (\Sigma_j |i/M|^{j+1})^{q_{n(i)}},$$

which diverges.

THEOREM 8. *Let q be bounded, and let $1 < p_k \leq H$ and $p_k^{-1} + s_k^{-1} = 1$ for all $k \in N$. Then $A \in (l(p), c_0(q))$ if and only if (9) holds and, for every $D \geq 1$,*

$$(17) \quad \lim_B \limsup_n (\Sigma_k |a_{n,k}|^{s_k} \cdot D^{s_k/q_n} \cdot B^{-s_k})^{q_n} = 0.$$

Proof. Again, define A_n on $l(p)$ by (13). First we prove the necessity: let $A \in (l(p), c_0(q))$. Obviously we must have (9), and as in Theorem 7 we see that $A_n \in l(p)^*$ for all n . If $A \in (l(p), c_0(q))$ then $(D^{1/q_n} \cdot a_{n,k}) \in (l(p), c_0(q))$ for all $D > 1$, so it is enough to show that (17) holds for $D = 1$. Since $c_0(q) \subset l_\infty$ and using Theorem 4(i) there is a $B > 1$ such that $T_n \equiv \Sigma_k |a_{n,k}|^{s_k} \cdot B^{-H s_k} \leq 1$ for every $n \in N$. Choose any n , and define $x_k^{(n)} = B^{-H s_k} |a_{n,k}|^{s_k-1} \operatorname{sgn} a_{n,k}$ for each k ; then

$$g^H(x^{(n)}) = \Sigma_k B^{-H s_k - H p_k} |a_{n,k}|^{s_k} \leq B^{-H} T_n \leq B^{-H}$$

and $A_n(x^{(n)}) = T_n$, whence $\|A_n\|_B \geq T_n$ for each n . By Theorem 2(iii) we must have $\lim_B \limsup_n (\|A_n\|_B)^{q_n} = 0$, whence (17) holds with $D = 1$.

For the sufficiency, let (9) be true and let (17) hold for all $D \geq 1$. It follows that $A_n \in l(p)^*$ for all $n \in N$. Since $(e^{(k)})$ is a basis in $l(p)$ and using Theorem 2(i) it is enough to show that $\lim_B \limsup_n (\|A_n\|_B)^{q_n} = 0$. Choose ε , $0 < \varepsilon \leq 1$, and $D > 2/\varepsilon$. There exist $B > 1$ and m such that $(\Sigma_k |a_{n,k}|^{s_k} \cdot D^{s_k/q_n} \cdot B^{-s_k})^{q_n} < \varepsilon/2$ if $n \geq m$. Then if $g(x) \leq 1/B$ and if $n \geq m$ we have

$$\begin{aligned} |A_n(x)|^{q_n} &\leq (\Sigma_k |a_{n,k}| D^{1/q_n} \cdot B^{-1} \cdot B D^{-1/q_n} |x_k|)^{q_n} \\ &\leq (\Sigma_k \{|a_{n,k}|^{s_k} D^{s_k/q_n} \cdot B^{-s_k} + D^{-p_k/q_n} \cdot B^{p_k} |x_k|^{p_k}\})^{q_n} \\ &< \varepsilon/2 + (D^{-1/q_n} \cdot B^H g^H(x))^{q_n} < \varepsilon, \end{aligned}$$

and this completes the proof.

One may show that if (9) is true and if (17) holds for $D = 1$, and if either $\inf_n q_n > 0$ or $\inf_k p_k > 1$, then $A \in (l(p), c_0(q))$, but that these conditions are not sufficient for $A \in (l(p), c_0(q))$ if $\inf_n q_n = 0$ and $\inf_k p_k = 1$.

THEOREM 9. *Let q be bounded, and let $1 < p_k \leq H$ and $p_k^{-1} + s_k^{-1} = 1$ for all $k \in N$. Then $A \in (l(p), c(q))$ if and only if $\sup_n \Sigma_k \times |a_{n,k}|^{s_k} B^{-s_k} < \infty$ for some $B > 1$ and there exist $\alpha_1, \alpha_2, \dots$ such that (11) holds and $\lim_B \limsup_n (\Sigma_k |a_{n,k} - \alpha_k|^{s_k} \cdot D^{s_k/q_n} \cdot B^{-s_k})^{q_n} = 0$ for all $D \geq 1$.*

Proof. As usual, define A_n on $l(p)$ by (13) for each $n \in N$. First let $A \in (l(p)), c(q) \subset (l(p), l_\infty)$; then $\sup_n \Sigma_k |a_{n,k}|^{s_k} \cdot B^{-s_k} < \infty$ for some $B > 1$. Also by Theorem 3 there is an $l \in l(p)^*$ such that $|A_n(e^{(k)}) - l(e^{(k)})|^{q_n} \rightarrow 0$ ($n \rightarrow \infty$) for each k and such that $\lim_B \limsup_n (\|A_n - l\|_B)^{q_n} = 0$. By Lemma 4 we can write $l(x) = \Sigma_k \alpha_k x_k$ on $l(p)$ for some sequence $(\alpha_k) \in l(p)^\dagger$, and the necessity now follows using the method of Theorem 8.

For the sufficiency, we show that the conditions of this theorem imply $\Sigma_k |\alpha_k|^{s_k} \cdot M^{-s_k} < \infty$ for some $M > 1$; then $l(x) \equiv \Sigma_k \alpha_k x_k$ defines an element of $l(p)^*$. We have for suitably large B and n

$$\begin{aligned} \Sigma_k |\alpha_k|^{s_k} (2B)^{-s_k} &= \Sigma_k |\alpha_k - a_{n,k} + a_{n,k}|^{s_k} \cdot (2B)^{-s_k} \\ &\leq \Sigma_k \max(|a_{n,k} - \alpha_k|, |a_{n,k}|)^{s_k} \cdot B^{-s_k} \\ &\leq \Sigma_k |a_{n,k} - \alpha_k|^{s_k} \cdot B^{-s_k} + \Sigma_k |a_{n,k}|^{s_k} \cdot B^{-s_k} \\ &\leq 1 + \sup_n \Sigma_k |a_{n,k}|^{s_k} \cdot B^{-s_k} < \infty. \end{aligned}$$

Then by Theorem 8, $(a_{n,k} - \alpha_k) \in (l(p), c_0(q))$ whence $|A_n(x) - l(x)|^{q_n} \rightarrow 0$ ($n \rightarrow \infty$) on $l(p)$, and the proof is complete.

We note that $(l(p), c)$ was characterized, for bounded p , in the corollary to Theorem 1 of [3].

The conditions for $A \in (l(p), l_\infty(q), (l(p), c_0(q))$ or $(l(p), c(q))$ in the general case $0 < p_k \leq \sup p_k < \infty$ and q bounded may be obtained by combining the separate cases $0 < p_k \leq 1$ and $1 < p_k \leq H$ above.

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