COMMUTATIVITY PROPERTIES IN BANACH *-ALGEBRAS

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Let A be a Banach *-algebra which has a faithful *-representation as bounded linear operators on a Hilbert space. It follows from Fuglede's theorem concerning normal operators on a Hilbert space that $x^*y = yx^*$ for all x, y in A where $xx^* = x^*x$ and $xy = yx$. Other commutativity properties in suitable Banach *-algebras A involving elements not necessarily normal are considered.

1. **Introduction.** Let T_1, T_2, and U be bounded linear operators on a Hilbert space where T_1 and T_2 are normal. The well-known theorem of Fuglede [13] asserts that if $T_1U = UT_1$ then $T_1^*U = UT_1^*$. Putnam's generalization [13] states that if $T_1U = UT_2$ then $T_1^*U = UT_2^*$. With this in view Berberian [3] defined an FT-ring to be a ring with an involution $x \to x^*$ such that $x^*y = yx^*$ whenever x is normal and $xy = yx$. Likewise a PT-ring is one which gives $x_1^*y = yx_2^*$ for all x_1, x_2 normal and $x_1y = yx_2$. The usual examples of Banach *-algebras A [14] are FT and PT-algebras since they have faithful *-representations as bounded linear operators on a Hilbert space.

For our purposes we must demand somewhat more of A. We suppose that A is a semisimple hermitian *-algebra whose maximal commutative *-subalgebras are Shilov algebras and where $xx^* \in W$, W a minimal closed two-sided ideal implies that $x \in W$. These requirements may seem special, but are actually satisfied by all B^*-algebras, all H^*-algebras and all group algebras of compact groups. Suppose that $b \neq 0$ in A and $ba = ab = 0$ for some $a \neq 0$ in A. We show that there exist $c \neq 0$, $h \neq 0$, h self-adjoint, with $bc = cb = 0$ and $ch = hc = 0$ provided that either A has two closed two-sided ideals $I \neq (0)$, $J \neq (0)$ with $I \cap J = (0)$ or A has zero socle. Without such hypotheses the conclusion can fail, as it does for the algebra of all 2×2 matrices over the complex field.

2. **Notation and preliminaries.** As is customary, a Banach *-algebras A is called *hermitian* if the spectrum of each self-adjoint element is real. Suppose that A is hermitian and semisimple. Then so is the algebra obtained by adjoining an identity to A. Therefore, the theory expounded in [12] for hermitian Banach *-algebras with an identity applies here to show that A has a faithful *-representation as bounded linear operators on a Hilbert space. In particular, if $x \in A$ and $xx^* = 0$ then $x = 0$. Ptak's development [12] involves a pene-

trating study of the function $p(x) = r(x^*x)^{1/2}$ (where $r(y)$ is the spectral radius of y). It turns out that $p^{-1}(0) = 0$ and $p(x)$ is a B^*-norm (in general incomplete) for A. If h is self-adjoint and $r(h) = 0$, then $p(h) = 0$ and $h = 0$. It follows from this and [14, Theorem 4.1.3] that each maximal commutative *-subalgebra E of A is hermitian and semisimple. Also the involution on A is continuous [14, Theorem 4.1.15].

Next let B be a semisimple commutative Banach algebra with space $\mathfrak{M}$ of modular maximal ideals. As is customary we say that B is a *Shilov algebra* if, given $M_0 \in \mathfrak{M}$ and a closed set $\mathfrak{F}$ in $\mathfrak{M}$ not containing M_0, there exists $x \in B$ such that $\hat{x}(M_0) = 1$ and $\hat{x}(M) = 0$ for all $M \in \mathfrak{F}$. Here $\hat{x}(M)$ is the Gelfand transform of x.

Our interest in this paper is confined to the study of noncommutative Banach *-algebras where Shilov's concept enters in the following way.

DEFINITION. A Banach *-algebra A is called a *noncommutative Shilov *-algebra* if its maximal commutative *-subalgebras are Shilov algebras.

Note that any such A must be semisimple. For let J be the radical of A. $J = J^*$. If h is self-adjoint and $h \in J$, then $r(h) = 0$. Since h lies in a commutative Shilov algebras, $h = 0$. Therefore $J = (0)$.

As in [7] we say that A is a *CC algebra* if the mappings $x \to ax$ and $x \to xa$ are completely continuous on A.

PROPOSITION 2.1. *Let A be a semisimple CC Banach *-algebra where $x = 0$ if $xx^* = 0$. Then A is a hermitian noncommutative Shilov *-algebra. If W is a minimal closed two-sided ideal in A containing xx^* then $x \in W$.*

Proof. A result of Barnes [2, Theorem 7.2] asserts that A is a modular annihilator algebra. It follows from the arguments of [5, Theorem 3.8] that the involution is hermitian.

Let E be a maximal commutative *-subalgebra with $\mathfrak{M}$ as its space of modular maximal ideals. In this situation, as noted above, E is semisimple. Again using [2, Theorem 7.2] we see that if $M_0 \in \mathfrak{M}$ there exists $x \neq 0$, $x \in E$, such that $xM_0 = (0)$. Then $\hat{x}(M_0) \neq 0$ while $\hat{x}(M) = 0$, $M \neq M_0$. Therefore B is a Shilov algebra.

Take a minimal closed two-sided ideal W in A. By [7, Theorem 14], W is finite-dimensional. Let $x \to \alpha(x)$ be a faithful *-representation of A as a subalgebra of $B(H)$, all the bounded linear operators on a Hilbert space H. Since W is finite-dimensional, $\alpha(W)$ is a closed two-sided ideal in K, the closure of $\alpha(A)$ in $B(H)$. If $xx^* \in W$ then

$\alpha(x)(\alpha(x))^* \in \alpha(W)$. From this we see that $\alpha(x) \in \alpha(W)$ via [14, Corollary 4.9.3] so that $x \in W$.

Examples of algebras satisfying the hypotheses of Proposition 2.1 include the group algebra of a compact group G and, in addition, $C(G)$ with convolution multiplication, the sup norm and the involution $f^*(t) = \overline{f(t^{-1})}$. See [7]. These algebras have the following more specific property (P) than that given for the minimal closed two-sided ideals by Proposition 2.1.

(P) Let $x \in A$ and I be a closed two-sided ideal in A. If $xx^* \in I$ then $x \in I$.

To show this for $C(G)$ we use the natural inner product for $C(G)$ given by

$$(f, g) = \int_G f(t)\overline{g(t)}dt$$

where the integration is taken with respect to normalized Haar measure. We call on the following properties of $A = C(G)$: (a) $A^2 \subset I \oplus I^\perp$, (b) $x \in \overline{xA} \cap \overline{Ax}$ and (c) $I = I^{\perp\perp}$. Suppose that $xx^* \in I$ and $z \in A$. Then $(zx)(zx)^* \in I$. Let $zx = u + v$, $u \in I$, $v \in I^\perp$ and let $w \in I^\perp$. Then

$$0 = (zxx^*z^*, w) = (vv^*, w) \ .$$

Therefore, $vv^* \in I \cap I^\perp = (0)$ so that $v = 0$ and $zx \in I$. By (b) we see that $x \in I$.

That $L(G)$, G compact, has property (P) follows from the theory of closed two-sided ideals in $L(G)$ developed in [6, Chapter IX]. We refer, in particular, to [6, Theorem 3.8.7] (see also [6, Theorem 28.40]) but do not give details here.

PROPOSITION 2.2. *B*-algebras and H*-algebras are hermitian noncommutative Shilov*-algebras with property* (P).

Proof. For B^*-algebras see [14, Chapter IV]. That a maximal commutative*-subalgebra E of an H^*-algebra A is a Shilov algebra follows from the fact that E is a commutative H^*-algebra and [1, Corollary 4.1]. That A has property (P) follows from the same analysis used for $C(G)$ above.

PROPOSITION 2.3. *If A is a noncommutative Shilov *-algebra so is every closed two-sided *ideal I in A.*

Proof. Let B be a maximal commutative *-subalgebra of I. Certainly B is contained in a maximal commutative *-subalgebra E of A. We show that B is an ideal in E. For $y \in B$ and $z \in E$ we have yz normal and permuting with each $x \in B$. Moreover $yz \in I$.

Then by the maximality of B, $yz \in B$. Since E is a Shilov algebra, so is B by [10, Proposition 9.2].

For a self-adjoint element h, we write $h \geqq 0$ in case its spectrum is contained in the set of nonnegative real numbers. By a *minimal idempotent* we mean an idempotent generator of a minimal one-sided ideal.

3. **Two-sided annihilation in Banach *-algebras.** We write $x \sharp y$ if $xy = yx = 0$. The involution $x \to x^*$ in a ring is called *proper* if $x^*x = 0$ implies that $x = 0$. The FT-property gives information on annihilation properties of normal elements which we put in the following form to point up what must be faced in the discussion below for nonnormal elements.

PROPOSITION 3.1. *Let A be an FT-ring with proper involution. Suppose $b \in A$ is normal, $b \neq 0$ and there exists $a \neq 0$ in A where $b \sharp a$. Then*
   (1) *there exists a self-adjoint element $h \neq 0$ such that $b \sharp h$ and*
   (2) *there exist $c \neq 0$ and $h \neq 0$, h self-adjoint, where $b \sharp c$ and $c \sharp h$.*

Proof. Since $ab = ba$ we get, from the FT-property that $ba^* = a^*b$. Therefore $0 = baa^* = aa^*b$. Then $b \sharp aa^*$ and $aa^* \neq 0$.

Note also that $b^*a = ab^*$. Then $ab^*b = b^*ba = 0$ and (2) is also verified.

Now we start to examine what can happen when b is not normal but otherwise satisfies all the hypotheses of Proposition 3.1. Let A be the algebra of all 2×2 matrices over the complex field and set

$$b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

One readily verified that $b \sharp b$ and that, for $a \in A$, $b \sharp a$ if and only if a is a scalar multiple of b. Therefore, both of the conclusions (1) and (2) of Proposition 3.1 fail to hold for the element b.

Next consider the case of the algebra A of all 3×3 matrices over the complex field. Consider

$$b = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{bmatrix}.$$

One verifies that $b \sharp a$, for $a \in A$, if and only if a is a scalar multiple of

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, b fails to satisfy the conclusion (1) of Proposition 3.1. Easy computations show that (2) is satisfied by b. Thus we can have (2) without (1).

Our treatment below of these questions makes essential use of ideas and techniques from Ono's interesting paper [11]. The next lemma is a modification to fit our needs of work in [11, pp. 155–156].

LEMMA 3.2. *Let A be a hermitian noncommutative Shilov *-algebra. Suppose that $h \neq 0$ is self-adjoint in A, $h \geqq 0$ and h not a scalar multiple of a minimal idempotent. Then there exist non-zero self-adjoint elements $u \geqq 0$, $v \geqq 0$ in A such that h, u, v commute pairwise, $hu \neq 0$, $hv \neq 0$, and $uv = 0$.*

Proof. Suppose first that h as at least two nonzero numbers in its spectrum. Let E be a maximal commutative *-subalgebra of A containing h with space $\mathfrak{M}$ of modular maximal ideals. There exist M_0 and M_1 in $\mathfrak{M}$ such that $0 < \hat{h}(M_1) < \hat{h}(M_0) = r(h)$. We choose positive numbers r_1, r_2, and r_3 such that

$$0 < r_1 < \hat{h}(M_0) < r_2 < r_3 < \hat{h}(M_0).$$

Next we consider the open sets in $\mathfrak{M}$ defined by

$$U(M_0) = \{M \in \mathfrak{M}: \hat{h}(M) > r_3\} \text{ and } V(M_1) = \{M \in \mathfrak{M}: r_1 < \hat{h}(M) < r_2\}.$$

Since E is a Shilov algebra there exist $u_1, v_1 \in E$ where $\hat{u}_1(M_0) = 1$, $\hat{u}_1(M) = 0$, $M \notin U(M_0)$, $\hat{v}_1(M_1) = 1$ and $\hat{v}_1(M) = 0$, $M \notin V(M_0)$. Since E is a hermitian *-algebra, $\hat{x}^*(M) = \overline{\hat{x}(M)}$, $x \in E$, $M \in \mathfrak{M}$. Then $u = u_1 u_1^*$, $v = v_1 v_1^*$ have the desired properties.

Next suppose that the spectrum of h contains exactly one non-zero element (which we may take to be the number one without loss of generality). Then $\hat{h}(M)$ is either 0 or 1 for each $M \in \mathfrak{M}$. As E is semisimple, h is a self-adjoint idempotent. By hypothesis, h is not a minimal idempotent so that there exist $w \in A$ where hwh is not a scalar multiple of h. We can certainly, and so do, select w to be self-adjoint.

For $x \in hAh$, the nonzero spectrum of x is the same whether computed A or in hAh by [9, Lemma 3]. Then hAh is an hermitian Banach algebra with identity h. Moreover, as zero is the only self-adjoint element in hAh with spectrum solely zero, we see that hAh is semisimple. Next we can select $\lambda > 0$ so large that $\mathrm{sp}(z \,|\, hAh)$,

the spectrum of $z = \lambda h + hwh$ computed in hAh, is contained in the open set $(0, \infty)$. Therefore, z^{-1} exists in hAh. We rule out the possibility that $\operatorname{sp}(z \mid hAh)$ consists of just one number α. For in that case, $\alpha^{-1}z$ would be a invertible idempotent in hAh and therefore $\alpha^{-1}z = h$. From this we see that $hwh = (\alpha - \lambda)h$, contrary to the choice of w.

Therefore $\operatorname{sp}(z)$, computed in A, contains at least two nonzero numbers. By the first part of the proof of this lemma, working in a maximal commutative *-subalgebra E of A containing h and z, we see that there exist self-adjoint $u \geqq 0$, $v \geqq 0$, $u \neq 0$, $v \neq 0$ in E such that $zu \neq 0$, $zv \neq 0$, and $uv = 0$. Then $(\lambda h + hwh)u \neq 0$ so that $hu \neq 0$. Likewise $hv \neq 0$. Therefore, u and v have the desired properties.

The next lemma is also a modification and extension of work in [11].

As a preliminary we show that, in the algebra A under consideration, xx^* is a nonzero scalar multiple of a minimal idempotent if and only if x^*x also enjoys this property. For let $xx^* = \lambda e$ where $\lambda \neq 0$, $e = e^2 \neq 0$. Now $xx^* \geqq 0$ by the Shirali-Ford theorem [12, Theorem 5.9] and $\operatorname{sp}(e)$ consists of the numbers 0 and 1. Therefore $\lambda > 0$. Then setting $z = \lambda^{-1/2}x$, we get $zz^* = e$. By [16, Proposition 3] we see that z^*z is also a minimal idempotent. Note that e must be self-adjoint.

LEMMA 3.3. *Let A be a hermitian noncommutative Shilov *-algebra. Let $b \neq 0$ in A where $b \sharp a$ for some $a \neq 0$. Then either*

(α) $b^*b(c^*c)^2b^*b$ is a nonzero scalar multiple of a minimal idempotent, for each $c \neq 0$ in A where $b \sharp c$

or

(β) there exist $c \neq 0$, $h \neq 0$, $h = h^*$ in A such that $b \sharp c$ and $c \sharp h$.

Proof. Suppose $c \neq 0$, $b \sharp c$ and $b^*b(c^*c)^2b^*b = 0$. Then $b^*bc^*c = 0 = c^*cb^*(c^*cb^*)^*$. As the involution is proper, $c^*cb^* = 0$. But this gives $b(c^*c) = 0 = (c^*c)b$. We then have $b \sharp c^*c$ and $c^*c \sharp b^*b$.

We therefore may suppose that we have $a \neq 0$, $b \sharp a$ with $b^*b(a^*a)^2b^*b \neq 0$ and not a scalar multiple of a minimal idempotent. This entails the fact that $y = a^*a(b^*b)^2a^*a$ is not a nonzero scalar multiple of a minimal idempotent and $y \neq 0$. The Shirali-Ford theorem [12, Theorem 5.9] tells us that $y \geqq 0$. Lemma 3.2 provides $u \geqq 0$, $v \geqq 0$, different from zero, permuting with each other and y such that $uv = 0$ but $yu \neq 0$, $yv \neq 0$. In particular

(1) $a^*au \neq 0$, $(b^*b)^2a^*av \neq 0$.

On the basis of (1), we shall show

$$(2) \qquad\qquad aua^*a \neq 0 \ .$$

Consider the faithful *-representation $x \to \tau(x)$ of A as bounded linear operators on the Hilbert space H. In the algebra $B(H)$ of all bounded linear operators on H the element $\tau(u)$ has a positive self-adjoint square root W. If (2) is not valid then $a^*aua^*a = 0$ and

$$[\tau(a)]^*\tau(a)W^2[\tau(a)]^*\tau(a) = 0 \ ,$$

from which we derive $a^*au = 0$ contrary to (1).

We set

$$z = (b^*b)^2 a^* a v a^* a b^* b \ .$$

Clearly zb^*b is self-adjoint. Moreover $zb^*b \neq 0$. For suppose $zb^*b = 0$. Now $\tau(v)$ has the form W^2, W self-adjoint, in $B(H)$. For convenience, set $x = (b^*b)^2 a^* a$. Then $\tau(x)W^2\tau(x^*) = 0$. This makes $\tau(x)W^2 = 0$ and therefore $xv = 0$ contrary to (1).

Next observe that $b \sharp aua^*a$ and the latter is nonzero by (2). We complete the proof by showing that $aua^*a \sharp zb^*b$. Clearly $(zb^*b)(aua^*a) = 0$. On the other hand,

$$(aua^*a)zb^*b = auyva^*a(b^*b)^2 = 0$$

since u permutes with y and $uv = 0$.

In view of Lemma 3.3 we find it convenient to introduce the following notation. We set, for the algebra A,

$$Q = \{b \neq 0 \text{ in } A \colon b \sharp a \text{ for some } a \neq 0 \text{ in } A\}$$
$$R = \{b \neq 0 \text{ in } A \colon b \sharp c, c \sharp h \text{ where } c \neq 0, h \neq 0, h = h^* \text{ in } A\} \ .$$

Clearly $Q^* = Q$ and $R^* = R$.

THEOREM 3.4. *Let A be a hermitian noncommutative Shilov *-algebra with zero socle. Let I be any two-sided closed *-ideal in A. Then, in the algebra I, $Q \subset R$.*

Proof. By [14, Theorem 4.1.9] I is a hermitian *-algebra. That I has zero socle follows from [15, Lemma 3.10]. Proposition 2.3 and Lemma 3.3 give the desired result.

Consider a semisimple topological ring a algebra A. Let e be a minimal idempotent in A. It is readily shown that $I = AeA$ is a minimal two-sided ideal in A and that the closure of I is a minimal closed two-sided ideal in A. We say that I is *generated by the minimal idempotent* e. In our situation where A has an involution we need to consider the following property. The example of § 2 give ample motivation here.

DEFINITION. *A* has *property M* if, given a minimal closed two-sided ideal *I* generated by a minimal idempotent, $x \in I$ whenever $xx^* \in I$.

As usual *A* is considered to have property *M* if there are no such ideals *I*. Also note that, under property *M*, $I^* = I$.

THEOREM 3.5. *Let A be a hermitian noncommutative Shilov *-algebra with property M. If there exist closed two-sided ideals* $I_1 \neq (0)$, $I_2 \neq (0)$ *in A with* $I_1 \cap I_2 = (0)$ *then* $Q \subset R$.

Proof. Let $b \in Q$ where $b \sharp a$ for $a \neq 0$. We show $b \in R$. By Lemma 3.3 we may suppose that $b^*b(a^*a)^2 b^*b \neq 0$ is a positive scalar multiple of a minimal idempotent *e*, for otherwise $b \in R$. Let *W* denote the closure of *AeA*. Then *W* is a minimal closed ideal. Therefore, $W \cap I_j = I_j$ or $W \cap I_j = (0)$, $j = 1, 2$. For at least one of $j = 1, 2$ we must have $W \cap I_j = (0)$. Then, for that *j*, $WI_j = I_jW = (0)$. By [4, Theorem 7] the left and right annihilators of *W* in *A* coincide. Call this two-sided ideal *K*. Moreover, since $W = W^*$, it follows that $K = K^* \neq (0)$. Thus there exists a self-adjoint $h \neq 0$ so that $hW = Wh = (0)$.

By the definition of *e* we have $b^*ba^*a \neq 0$. Moreover

$$b^*ba^*a(b^*ba^*a)^* \in W .$$

This tells us that $b^*ba^*a \in W$. It follows that $ab^*ba^*a \neq 0$. For otherwise $a^*ab^*(a^*ab^*)^* = 0$ and $a^*ab^*b = 0 = b^*ba^*a$. But clearly $b \sharp ab^*ba^*a$. Inasmuch as $h \sharp ab^*ba^*a$ we see that $b \in R$.

THEOREM 3.6. *Let A be a hermitian noncommutative Shilov *-algebra with property M. Either* $Q \subset R$ *or there exists a unique minimal closed two-sided ideal W and W is generated by a minimal idempotent. If* $b \in Q$ *and* $b \notin R$ *then* $b \sharp c$ *for some* $c \neq 0$ *in W. Also* ba^* *and* a^*b *are different nonzero elements of W for all* $a \neq 0$ *such that* $b \sharp a$.

Proof. Suppose $Q \not\subset R$. Lemma 3.3 and Theorem 3.5 show there exists a unique *W* as asserted. The arguments of Theorem 3.4 provide us with the desired $c = ab^*ba^*a$ (where $b \sharp a$ and $a \neq 0$).

We continue discussing the setup $b \sharp a$, $a \neq 0$, using all the notation of the proof of Theorem 3.5. As in that proof $b^*ba^*a \in W$ and, obviously, $ba^* \neq 0$. We have $a^*ab^*(a^*ab^*)^* \in W$ and therefore $a^*ab^* \in W$. Since $W = W^*$ we have $ba^*a \in W$. Then $ba^*(ba^*)^* \in W$ so that $ba^* \in W$.

It remains to see that $a^*b \neq 0$, $a^*b \in W$, and $a^*b \neq ba^*$. Note that $b^* \in Q$, $b^* \notin R$, and $b^* \sharp a^*$. Via the uniqueness of the minimal closed

ideal W we see, from the above proof, that $b^*a \neq 0$ and $b^*a \in W$. If $ba^* = a^*b$ then $b \sharp a^*a$ and $a^*a \sharp b^*b$. This would put b in R.

In case A is an AW^*-algebra [8] we can obtain more. For if $c \neq h$ where $c \neq 0, h \neq 0$ and h is self-adjoint there is a nonzero projection p with $c \sharp p$. To see this consider a maximal commutative *-subalgebra E of A containing h. By [8, Lemma 2.1] there exists $y \in E$ such that $hy = p$ is a nonzero projection. It is easy to see that $c \sharp p$.

4. **On the FT and PT-properties.** Let A be a Banach *-algebra. For a maximal commutative subalgebra W let $N(W)$ denote the set of normal elements of A lying in W. These notions are intimately related to the FT-property.

PROPOSITION 4.1. *A Banach *-algebra is an FT-Banach algebra if and only if $N(W) = [N(W)]^*$ for each maximal commutative subalgebra W of A. In that case $N(W) = W \cap W^*$ and $N(W)$ is a closed *-subalgebra of A.*

Proof. Suppose that $N(W) = [N(W)]^*$ for each maximal commutative subalgebra W of A. Let x be normal in $A, y \in A$ and $xy = yx$. Let W be a maximal commutative subalgebra of A containing x and y. Since $x^* \in W$, we get $x^*y = yx^*$. Thus A has the FT-property.

Suppose, conversely, that A is an FT-algebra. Let W be a maximal commutative subalgebra. We have $x^*y = yx^*$ for all $x \in N(W)$ and $y \in W$. From the maximality of W we see that $x^* \in N(W)$. Suppose also that $y \in N(W)$. Then $(x + y)^*(x + y) = (x + y)(x + y)^*$ and $xy \in N(W)$. Clearly $N(W) \subset W \cap W^*$. If $z \in W \cap W^*$, z is normal and so lie in $N(W)$. Finally, inasmuch as W and W^* are maximal commutative subalgebras, we see that they are closed in A and, therefore, $N(W)$ is closed. Note that this does not require the involution to be continuous.

PROPOSITION 4.2. *Let W be a maximal commutative subalgebra of a B^*-algebra A with unit. Then $N(W)$ separates the maximal ideals of W if and only if $W = N(W) \oplus R$ where R is the radical of W.*

Proof. By [14, Theorem 4.8.11], A is an FT-algebra. Let $\mathfrak{M}$ be the space of maximal ideals of W and suppose that $N(W)$ separates every $M_1 \neq M_2$ in $\mathfrak{M}$. For $x \in W$, the spectrum, $\text{sp}(x)$ of x is, by [14, p. 35] the same as the set $\{\hat{x}(M): M \in \mathfrak{M}\}$ where $\tau: x \rightarrow \hat{x}(M)$ denotes the Gelfand transform of W into $C(\mathfrak{M})$. If $x \in N(W)$, $\|x\| =$

sup $(|\hat{x}(M)|: M \in \mathfrak{M})$. Therefore, τ is an isometry on $N(W)$. Since the spectrum of a self-adjoint element is real we see that $x^*(M) = \overline{\hat{x}(M)}$ for $x \in N(W)$ and $M \in \mathfrak{M}$. By Proposition 4.1, $N(W)$ is a B^*-algebra. The Stone-Weierstrass theorem insures that $\tau(N(W)) = C(\mathfrak{M})$. Therefore $\tau(A) = C(\mathfrak{M})$. The desired conclusion now follows from the semisimplicity of $N(W)$.

Maximal commutative subalgebra of this sort exist. Let A be the B^*-direct sum of $B = C[0, 1]$ and M_2 the B^*-algebra of all 2×2 matrices over the complex field. Let I be the identity 2×2 matrix and let T be the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

One checks that $W = B \oplus \{\lambda I + \mu T: \lambda, \mu \text{ complex}\}$ is a maximal commutative subalgebra of A, that $N(W) = W \cap W^* = B \oplus \{\lambda I: \lambda \text{ complex}\}$ and that $N(W)$ separates the maximal ideals of W.

THEOREM 4.3. *A noncommutative Shilov $*$-algebra is a PT-algebra if and only if it is hermitian.*

Proof. Suppose that A is an PT-algebra. Let h be a self-adjoint element, $h \neq 0$ and B be a maximal commutative $*$-subalgebra containing h, with $\mathfrak{M}$ as its space of modular maximal ideals. By [5, Theorem 2.2] there exists a homeomorphism σ of $\mathfrak{M}$ onto $\mathfrak{M}$ of period two such that $\hat{x}^*(M) = \overline{\hat{x}(\sigma(M))}$ for all $x \in B$, $M \in \mathfrak{M}$. Our task is to see that σ is the identity mapping. Suppose otherwise that $M_0 \neq \sigma(M_0)$ for some $M_0 \in \mathfrak{M}$. Let U and V be disjoint open sets with $M_0 \in U$ and $\sigma(M_0) \in V$. We select $x, y \in B$ such that $\hat{x}(M_0) = 1$, $\hat{x}(M) = 0$, $M \notin U$, $\hat{y}(\sigma(M_0)) = 1$ and $\hat{y}(M) = 0$, $M \notin V$. Then $xy = 0$. The PT-property yields $xy^* = 0$. Then $\hat{x}(M_0)\hat{y}(\overline{\sigma(M_0)}) = 0$ which is impossible.

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