

# Pacific Journal of Mathematics

**EXISTENCE, UNIQUENESS AND LIMITING BEHAVIOR OF  
SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS IN  
BANACH SPACE**

JOHN LAGNESE

# EXISTENCE, UNIQUENESS AND LIMITING BEHAVIOR OF SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS IN BANACH SPACE

JOHN LAGNESE

Let  $X$  be a Banach space (real or complex) and  $A_n$  and  $B$  be linear operators in  $X$  with  $D(B) \subseteq D(A_n)$ ,  $n = 1, 2, \dots$ . The following note is concerned with existence and uniqueness of solutions of the problem

$$(1.1) \quad \frac{d}{dt} [(I - A_n)u(t)] - Bu(t) = 0, \quad (t > 0), \quad u(0) = u_0,$$

and the limiting behavior of these solutions as the operators  $A_n$  tend to zero in a sense to be specified. We will show that for a large class of operators the problem (1.1) is well posed and that its solutions tend to the solution of the problem

$$(1.2) \quad \frac{du(t)}{dt} - Bu(t) = 0, \quad (t > 0), \quad u(0) = u_0.$$

In particular, we obtain an extension to Banach spaces of a result of R. E. Showalter [5] to the effect that (1.1) is well posed when  $X$  is a Hilbert space and  $A_n$  and  $B$  are maximal dissipative operators in  $X$  which satisfy the algebraic condition

$$(1.3) \quad \operatorname{Re} ((I - A_n)x, Bx) \leq 0, \quad x \in D(B) \subseteq D(A_n).$$

In the next section we give sufficient conditions for (1.1) to be well posed. We note that these conditions do *not* guarantee that (1.2) is well posed. In §3 we show that if, in addition,  $\{A_n\}$  tends to zero in a certain sense, then (1.2) is well posed and the solutions  $u_n$  of (1.1) tend to the solution of (1.2). In particular, it will follow that if  $A$  and  $B$  are densely defined maximal dissipative operators in a Hilbert space and if (1.3) is satisfied with  $A_n = n^{-1}A$ , then

$$\frac{d}{dt} [(I - n^{-1}A)u_n(t)] - Bu_n(t) = 0, \quad (t > 0), \quad u_n(0) = u_n \in D(B),$$

is well posed and as  $n \rightarrow \infty$ ,  $u_n$  converges strongly to the unique solution of (1.2). Two examples are discussed in §4.

We emphasize that throughout this paper it is assumed that  $D(B) \subseteq D(A_n)$ . The question of limiting behavior of solutions of (1.1) when  $X$  is a Hilbert space,  $A_n = n^{-1}A$  and  $D(A) \subseteq D(B)$  has been considered previously [2], and it is interesting to compare the results of [2] with those of the present note in the case  $D(A) = D(B)$ . In [2] it was assumed that  $A$  and  $B$  were maximal dissipative operators

arising from certain densely defined, strongly coercive sesquilinear forms and that  $A$  was self-adjoint. On the other hand the algebraic condition (1.3) which is the most restrictive assumption of the present note, was not assumed in [2] and the convergence results are somewhat stronger than those obtained here. Thus while the results of [2] do not apply to perturbations of hyperbolic problems, they are in some respects more satisfactory as far as perturbations of parabolic problems are concerned when  $D(A) = D(B)$ . We note that the methods used here are completely different from those of [2].

2. Existence and uniqueness of solutions. A solution of the problem (1.1) is a function  $u: [0, \infty) \rightarrow D(B)$  such that  $(I - A_n)u \in C([0, \infty); X) \cap C'((0, \infty), X)$  and (1.1) is satisfied. The initial condition in (1.1) is supposed to hold in the sense that  $(I - A_n)u(t) \rightarrow (I - A_n)u_0$  strongly in  $X$  as  $t \rightarrow 0_+$ . While we will always assume that  $I - A_n$  is invertible, the inverse need not be bounded and so we do not know in general that  $u(t) \rightarrow u_0$  strongly in  $X$ .

THEOREM 2.1. *Let  $X$  be a Banach space and  $A_n$  and  $B$  linear operators in  $X$  which satisfy the following*

$$(2.1) \quad I - A_n \text{ is one-to-one.}$$

$$(2.2) \quad D(B) \subseteq D(A_n).$$

$$(2.3) \quad \|x - A_n x - \zeta Bx\| \geq \|x - A_n x\| \text{ for all } x \in D(B) \text{ and } \zeta > 0.$$

$$(2.4) \quad \text{For some } \zeta_n > 0, \operatorname{Rg}(I - A_n - \zeta_n B) = X.$$

*Then for any  $u_0 \in D(B)$  the problem (1.1) has a unique solution  $u(t)$  and*

$$(2.5) \quad \|(I - A_n)u(t)\| \leq \|(I - A_n)u_0\|, t \geq 0.$$

*Proof.* Set  $\tilde{A}_n = A_n|_{D(B)}$  and  $B_n = B(I - \tilde{A}_n)^{-1}$  with  $D(B_n) = \operatorname{Rg}(I - \tilde{A}_n)$ . A function  $u$  is a solution of (1.1) if and only if  $(I - A_n)u = v \in C([0, \infty); X) \cap C'((0, \infty); X)$  and

$$(2.6) \quad \frac{dv(t)}{dt} - B_n v(t) = 0, (t > 0), v(0) = v_0$$

where  $v_0 = (I - A_n)u_0 \in D(B_n)$ . From (2.3) we obtain

$$\|y - \zeta B_n y\| \geq \|y\|, y \in D(B_n), \zeta > 0,$$

which means that  $B_n$  is a dissipative operator in  $X$ , and from (2.4) we have  $\operatorname{Rg}(I - \zeta B_n) = X$  from some  $\zeta > 0$  (hence for all  $\zeta > 0$ ). From these facts it follows that  $D(B_n)$  is dense in  $X$  (Goldstein [1]; c.f. [4]). We may now apply the Lumer-Phillips theorem [3] to the

effect that  $B_n$  is the infinitesimal generator of a  $(C_0)$ -semigroup  $\{e^{tB_n}; t \geq 0\}$  of contractions on  $X$ . Thus for any  $v_0 \in D(B_n)$ , (2.6) has a unique solution given by  $v(t) = e^{tB_n}v_0$  and  $\|v(t)\| \leq \|v_0\|$ . The conclusions of the theorem now follow by setting

$$(2.7) \quad u(t) = (I - A_n)^{-1} e^{tB_n} (I - A_n) u_0.$$

**COROLLARY 2.1.** *Let  $X$  be a Hilbert space and  $A_n$  and  $B$  be densely defined, maximal dissipative linear operators in  $X$  such that  $D(B) \subseteq D(A_n)$  and which satisfy (1.3). Then the conclusions of Theorem 2.1 hold. Moreover,  $B \in C([0, \infty), X) \cap C'((0, \infty); X)$  and  $u(t) \rightarrow u_0$  strongly in  $X$  as  $t \rightarrow 0_+$ .*

*Proof.* Since  $A_n$  is densely defined and maximal dissipative,  $(I - A_n)$  is a bijection of  $D(A_n)$  onto  $X$  and  $\|(I - A_n)^{-1}\| \leq 1$ . Also, R. E. Showalter proved [5] that under the stated hypotheses,  $A_n + B$  is a densely defined, maximal dissipative operator in  $X$ . From this fact follows that  $\text{Rg}(I - A_n - B) = X$ . For a Hilbert space, conditions (1.3) and (2.3) are equivalent. The conclusions of the corollary now follow from (2.7) and Theorem 2.1.

**REMARK.** Suppose (2.1)–(2.4) hold and that in addition there is a constant  $C > 0$  such that

$$(2.8) \quad \|x - A_n x - \zeta Bx\| \geq C \|x - A_n x\|$$

for each  $x \in D(B)$  and all  $\zeta$  with  $\text{Re}(\zeta) > 0$ . Then the semigroup  $\{e^{tB_n}; t \geq 0\}$  has a strong holomorphic extension into some sector  $|\arg t| < \alpha$ , and therefore (2.6) (respectively, (1.1)) is uniquely solvable for any  $v_0 \in X$  (respectively,  $u_0 \in D(A_n)$ ). In fact, since  $B_n$  generates a  $(C_0)$ -semigroup of contractions, the open right half-plane lies in the resolvent set of  $B_n$  and from (2.8) we obtain  $\|(\lambda - B_n)^{-1}\| \leq (C|\lambda|)^{-1}$  whenever  $\text{Re} \lambda > 0$ , which implies the desired conclusion. When  $X$  is a Hilbert space, a sufficient condition for (2.8) is that all of the values of  $z = (x - A_n x, Bx)$  lie in some fixed sector

$$|\arg z - \pi| \leq \frac{\pi}{2} - \varepsilon, \quad \varepsilon > 0.$$

To prove this, write  $z = |z|e^{i\theta}$  and  $\zeta = |\zeta|e^{i\phi}$ . (2.8) is equivalent to

$$(1 - C^2)\|x - A_n x\|^2 - 2|\zeta||z|\cos(\phi - \theta) + |\zeta|^2\|Bx\|^2 \geq 0.$$

If  $|\theta - \pi| \leq \pi/2 - \varepsilon$ , there is a  $\delta > 0$  such that  $\cos(\phi - \theta) \leq 1 - \delta$  for all  $\phi \in (-\pi/2, \pi/2)$  and therefore

$$\begin{aligned} & (1 - \delta)^2\|x - A_n x\|^2 - 2|\zeta||z|\cos(\phi - \theta) + |\zeta|^2\|Bx\|^2 \\ & \geq (1 - \delta)^2\|x - A_n x\|^2 - 2|\zeta|(1 - \delta)\|x - A_n x\|\|Bx\| \\ & + |\zeta|^2\|Bx\|^2 = [(1 - \delta)\|x - A_n x\| - |\zeta|\|Bx\|]^2 \geq 0. \end{aligned}$$

Thus (2.8) holds with  $C^2 = 2\delta - \delta^2$ .

**3. Limiting behavior of solutions.** We first prove that if  $B$  is closed and  $A_n$  tends to zero in a certain way then (1.2) is well posed.

**THEOREM 3.1.** *Let  $X$  be a Banach space and  $A_n$  and  $B$  be linear operators in  $X$  which satisfy (2.1)–(2.4). Suppose in addition*

(3.1)  $B$  is closed .

(3.2) 
$$\lim_{n \rightarrow \infty} \sup_{\substack{x \in D(B) \\ x \neq 0}} \|A_n x\| / (\|Bx\| + \|x\|) = 0 .$$

*Then  $B$  is the infinitesimal generator of a  $(C_0)$ -semigroup of contractions on  $X$ .*

*Proof.* We have to show that  $B$  is a dissipative operator such that  $\text{Rg}(I - B) = X$ . From (2.3) and (3.2) we obtain, upon letting  $n \rightarrow \infty$ ,

(3.3) 
$$\|x - \zeta Bx\| \geq \|x\|, \quad x \in D(B), \quad \zeta > 0 ,$$

and so  $B$  is dissipative. For each  $n$  and  $\zeta > 0$ ,  $B_n$  is dissipative and  $\text{Rg}(I - \zeta B_n) = X$ . Let  $y \in X$  and  $x_n \in D(B)$  such that

$$x_n - A_n x_n - Bx_n = y, \quad n = 1, 2, \dots .$$

By (2.3),  $\|x_n - A_n x_n\| \leq \|y\|$  and therefore  $\{Bx_n\}$  is bounded. Let

$$C_n = \sup_{\substack{x \in D(B) \\ x \neq 0}} \|A_n x\| / (\|Bx\| + \|x\|) .$$

$C_n \rightarrow 0$  as  $n \rightarrow \infty$  according to (3.2). From (3.3)

$$\begin{aligned} \|x_n\| &\leq \|x_n - Bx_n\| = \|y + A_n x_n\| \\ &\leq \|y\| + C_n(\|Bx_n\| + \|x_n\|) \end{aligned}$$

so that

$$(1 - C_n)\|x_n\| \leq \|y\| + C_n\|Bx_n\| .$$

Hence  $\{x_n\}$  is also bounded. It follows from (3.2) that  $A_n x_n \rightarrow 0$  strongly in  $X$  as  $n \rightarrow \infty$ . Therefore

$$\begin{aligned} \|x_n - x_m\| &\leq \|(x_n - x_m) - B(x_n - x_m)\| \\ &\leq \|A_n x_n - A_m x_m\| \longrightarrow 0 \end{aligned}$$

as  $n, m \rightarrow \infty$ . Let  $x = \lim x_n$ . We have that  $x_n \rightarrow x$ ,  $Bx_n \rightarrow x - y$ . Since  $B$  is closed,  $x \in D(B)$  and  $x - Bx = y$ , that is,  $\text{Rg}(I - B) = X$ . This fact, together with the dissipativity of  $B$ , implies  $D(B)$  is dense

in  $X$ . The conclusion of the theorem now follows from the Lumer-Phillips theorem.

**THEOREM 3.2.** *Let  $X$  be a Banach space and  $A_n$  and  $B$  be linear operators in  $X$  which satisfy (2.1)–(2.4), (3.1) and (3.2). Then as  $n \rightarrow \infty$ ,  $e^{tB_n} \rightarrow e^{tB}$  strongly and uniformly on bounded subsets of  $[0, \infty)$ .*

*Proof.* We apply the Trotter convergence theorem [6]. To do this we show that for each  $\zeta > 0$ ,

$$(3.4) \quad \lim_{n \rightarrow \infty} (I - \zeta B_n)^{-1} = (I - \zeta B)^{-1}$$

in the uniform operator topology of  $\mathcal{L}(X)$  (= the linear space of bounded linear operators on  $X$ ).

We may write

$$(3.5) \quad \begin{aligned} (I - \zeta B_n)^{-1} &= (I - \zeta B(I - \tilde{A}_n)^{-1})^{-1} \\ &= (I - A_n)(I - \zeta B)^{-1}(I - A_n(I - \zeta B)^{-1})^{-1}. \end{aligned}$$

For each  $x \in X$ ,

$$(3.6) \quad \begin{aligned} \|A_n(I - \zeta B)^{-1}x\| &\leq C_n(\|B(I - \zeta B)^{-1}x\| + \|(I - \zeta B)^{-1}x\|) \\ &\leq C_n\left(1 + \frac{2}{\zeta}\right)\|x\|. \end{aligned}$$

Thus for all sufficiently large  $n$ ,

$$(I - A_n(I - \zeta B)^{-1})^{-1} = \sum_{k=0}^{\infty} (A_n(I - \zeta B)^{-1})^k$$

and

$$\|(I - A_n(I - \zeta B)^{-1})^{-1} - I\| \leq \sum_{k=0}^{\infty} C_n^{k+1} \left(1 + \frac{2}{\zeta}\right)^{k+1}$$

which tends to zero as  $n \rightarrow \infty$ . Therefore

$$(3.7) \quad \lim_{n \rightarrow \infty} (I - \zeta B)^{-1}(I - A_n(I - \zeta B)^{-1})^{-1} = (I - \zeta B)^{-1}$$

in the uniform operator topology of  $\mathcal{L}(X)$ . From (3.6) we have

$$(3.8) \quad \begin{aligned} \|A_n(I - \zeta B)^{-1}(I - A_n(I - \zeta B)^{-1})^{-1}\| &\leq C_n\left(1 + \frac{2}{\zeta}\right)\|(I - A_n(I - \zeta B)^{-1})^{-1}\| \\ &\leq \sum_{k=0}^{\infty} C_n^{k+1} \left(1 + \frac{2}{\zeta}\right)^{k+1} \longrightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . (3.4) now follows from (3.5), (3.7), and (3.8).

**THEOREM 3.3.** *Let  $X$  be a Banach space and  $A_n$  and  $B$  be linear operators in  $X$  which satisfy (2.1)–(2.4), (3.1) and (3.2). Suppose in*

addition that  $\text{Rg}(I - A_n) = X$ ,  $n = 1, 2, \dots$ , and  $\sup_n \|(I - A_n)^{-1}\| < \infty$ . Let  $u_0 \in D(B)$  and  $u_n(t)$  be the unique solution of (1.1). Then as  $n \rightarrow \infty$ ,  $u_n(t) \rightarrow e^{tB}u_0$  strongly in  $X$ , uniformly on bounded subsets of  $[0, \infty)$ .

*Proof.* From (2.7) we obtain

$$\begin{aligned} \|u_n(t) - e^{tB}u_0\| &\leq \|(I - A_n)^{-1}(e^{tB_n} - e^{tB})u_0\| \\ &\quad + \|(I - A_n)^{-1}A_n e^{tB}u_0\| + \|(I - A_n)^{-1}e^{tB_n}A_n u_0\| \\ &\leq (\text{const.}) [\|e^{tB_n}u_0 - e^{tB}u_0\| + C_n(\|Bu_0\| + \|u_0\|)] \end{aligned}$$

and the right side tends to zero as  $n \rightarrow \infty$ , uniformly on bounded subsets of  $[0, \infty)$ .

**COROLLARY 3.1.** *Let  $X$  be a Hilbert space and  $A_n$  and  $B$  be densely defined, maximal dissipative operators in  $X$  such that  $D(B) \subseteq D(A_n)$  and which satisfy (1.3) and (3.2). Then for each  $u_0 \in D(B)$  the problem (1.1) has a unique solution  $u_n$  and  $u_n(t) \rightarrow e^{tB}u_0$  as  $n \rightarrow \infty$ , uniformly on bounded subsets of  $[0, \infty)$ .*

*Proof.* As noted in the proof of Corollary 2.1,  $A_n$  and  $B$  satisfy (2.1)–(2.4) and moreover,  $\text{Rg}(I - A_n) = X$  with  $\|(I - A_n)^{-1}\| \leq 1$ . In addition  $B$ , being a densely defined, maximal dissipative operator in a Hilbert space, is closed. The corollary now follows from Theorem 3.3.

**REMARK.** When  $A_n = n^{-1}A$ , (3.2) is automatically satisfied provided  $A$  and  $B$  are closed operators with  $D(B) \subseteq D(A)$ . Thus in this case hypothesis (3.2) may be omitted in Corollary 3.1. In fact, as a rather well-known consequence of the closed graph theorem we have

$$\|Ax\| \leq C(\|Bx\| + \|x\|), \quad x \in D(B)$$

where the constant  $C$  does not depend on  $x$ . Therefore

$$\sup_{\substack{u \in D(B) \\ u \neq 0}} \|A_n x\| / (\|Bx\| + \|x\|) \leq Cn^{-1}.$$

**4. Examples.** As a first example we consider the problem

$$\begin{aligned} (4.1) \quad \frac{\partial}{\partial t} \left( u - \alpha_n^1(x)u - \alpha_n(x) \frac{\partial u}{\partial x} \right) - \left( b^1(x)u + b(x) \frac{\partial u}{\partial x} \right) &= 0, \\ 0 < x < 1, \quad t > 0, \end{aligned}$$

$$(4.2) \quad u(x, 0) = u_0(x), \quad 0 < x < 1; \quad u(0, t) = cu(1, t), \quad t > 0,$$

where  $c$  is a complex constant satisfying certain conditions and the coefficients in (4.1) are real-valued and of class  $C'([0, 1])$ . Let  $X$  be the complex Hilbert space  $L_2(0, 1)$  and  $H_2^1(0, 1)$  be the subclass of

$L_2(0, 1)$  consisting of those functions whose first derivative in the sense of distributions is again in  $L_2(0, 1)$ . The norms in  $X$  and in  $H_2^1(0, 1)$  will be denoted by  $\|\cdot\|_0$  and  $\|\cdot\|_1$  respectively and the inner product in  $X$  by  $(\cdot, \cdot)$ ; we have

$$\|u\|_1 = \left( \|u\|_0^2 + \left\| \frac{du}{dx} \right\|_0^2 \right)^{1/2}, \quad u \in H_2^1(0, 1).$$

Each function in  $H_2^1(0, 1)$  is continuous, i.e., coincides with a function in  $C([0, 1])$  up to a set of Lebesgue measure zero, and the injection of  $H_2^1(0, 1)$  into  $C([0, 1])$  is continuous.

We define operators  $A_n$  and  $B$  in  $X$  as follows:

$$D(A_n) = D(B) = \{u \in H_2^1(0, 1): u(0) = cu(1)\}$$

and for  $u \in D(A_n) = D(B)$ ,

$$A_n u = a_n^1 u + a_n \frac{du}{dx}, \quad Bu = b^1 u + b \frac{du}{dx}.$$

From our preceding remarks it is easy to see that  $D(B)$  is a closed subspace of  $H_2^1(0, 1)$  and  $D(B)$  is dense in  $L_2(0, 1)$ .

By a solution of (4.1), (4.2) we mean a solution of (1.1) in which  $A_n$  and  $B$  are the operators defined above. In order to apply the theory developed in §§ 2 and 3 to the problem (4.1) and (4.2) we shall have to verify in particular condition (1.3). Concerning this we have

LEMMA 4.1. Suppose  $a_n b \geq 0$  and that

$$(4.3) \quad b^1 - \frac{1}{2} \frac{db}{dx} - a_n^1 b^1 + \frac{1}{2} \frac{d}{dx} (a_n b^1 + a_n^1 b) \leq 0;$$

$$(4.4) \quad \alpha_n(1) - |c|^2 \alpha_n(0) \leq 0$$

where  $\alpha_n = b - a_n b^1 - a_n^1 b$ . Then (1.3) is satisfied.

*Proof.* For  $u \in D(B)$  we have

$$\begin{aligned} \operatorname{Re} (u - A_n u, Bu) &= \int_0^1 (b^1 - a_n^1 b^1) |u|^2 dx - \int_0^1 a_n b \left| \frac{du}{dx} \right|^2 dx \\ &\quad + \operatorname{Re} \int_0^1 (b - a_n^1 b - a_n b^1) \bar{u} \frac{du}{dx} dx. \end{aligned}$$

The following identity is easily obtained by an integration by parts: For  $u \in D(B)$  and  $f \in C'([0, 1])$ ,

$$\operatorname{Re} \int_0^1 f \bar{u} \frac{du}{dx} dx = \frac{1}{2} (f(1) - |c|^2 f(0)) |u(1)|^2 - \frac{1}{2} \int_0^1 \frac{df}{dx} |u|^2 dx.$$

From this identity we obtain for  $u \in D(B)$

$$\begin{aligned} \operatorname{Re}(u - A_n u, Bu) &= \frac{1}{2}(\alpha_n(1) - |c|^2 \alpha_n(0)) |u(1)|^2 - \int_0^1 a_n b \left| \frac{du}{dx} \right|^2 dx \\ &+ \int_0^1 \left[ b^1 - \frac{1}{2} \frac{db}{dx} - a_n^1 b^1 + \frac{1}{2} \frac{d}{dx} (a_n^1 b + a_n b^1) \right] |u|^2 dx \leq 0. \end{aligned}$$

REMARK. If  $\{a_n\}$  and  $\{a_n^1\}$  tend to zero in the topology of  $C'([0, 1])$  and if for some  $\varepsilon > 0$  we have

$$b^1 - \frac{1}{2} \frac{db}{dx} \leq -\varepsilon, \quad b(1) - |c|^2 b(0) \leq -\varepsilon,$$

then (4.3) and (4.4) are easily seen to be satisfied for all sufficiently large  $n$ .

THEOREM 4.1. Assume (4.3) and (4.4), that  $a_n b \geq 0$  and  $a_n^2 + b^2 > 0$ . In addition suppose

$$(4.5) \quad a_n(1) - |c|^2 a_n(0) \leq 0;$$

$$(4.6) \quad a_n^1 - \frac{1}{2} \frac{da_n}{dx} < 1, \quad 0 \leq x \leq 1;$$

$$(4.7) \quad \text{There exists } \zeta_n > 0 \text{ such that}$$

$$c \exp \left[ \int_0^1 \frac{1 - a_n^1(\xi) - \zeta_n b^1(\xi)}{a_n(\xi) + \zeta_n b(\xi)} d\xi \right] \neq 1.$$

Then the hypotheses of Theorem 2.1 are satisfied.

Thus, in particular, for each  $u_0 \in D(B)$  the problem (4.1), (4.2) has a unique solution.

THEOREM 4.2. Assume (4.3)–(4.5), that  $a_n b \geq 0$  and  $b \neq 0$ . In addition suppose

$$(4.8) \quad \{a_n\} \text{ and } \{a_n^1\} \text{ tend to zero in the topology of } C'([0, 1]) \text{ as } n \rightarrow \infty.$$

Then the hypotheses of Theorem 3.2 are satisfied for all sufficiently large  $n$ .

THEOREM 4.3. Assume (4.3)–(4.5), (4.8) and that  $a_n b > 0$ . Then the hypotheses of Theorem 3.3 are satisfied for all sufficiently large  $n$ .

Thus if  $u_n$  is the unique solution of (4.1), (4.2), as  $n \rightarrow \infty$   $u_n(t)$  converges in  $L_2(0, 1)$  to the unique solution of

$$\frac{\partial u}{\partial t} - b(x) \frac{\partial u}{\partial x} - b^1(x) u = 0, \quad 0 < x < 1, \quad t > 0,$$

$u(x, 0) = u_0(x)$ ,  $0 < x < 1$ ;  $u(0, t) = cu(1, t)$ ,  $t > 0$ , uniformly on bounded subsets of  $[0, \infty)$ .

*Proof of Theorem 4.1.* We have already verified (2.3). To check (2.1) consider the equation

$$u - A_n u = f \in X.$$

Multiplying by  $\bar{u}$  and integrating gives

$$\int_0^1 (1 - a_n^1) |u|^2 dx - \operatorname{Re} \int_0^1 a_n \bar{u} \frac{du}{dx} dx = \operatorname{Re} \int_0^1 \bar{u} f dx$$

and this may be written

$$(4.9) \quad \int_0^1 \left( 1 - a_n^1 + \frac{1}{2} \frac{da_n}{dx} \right) |u|^2 dx = \frac{1}{2} (a_n(1) - |c|^2 a_n(0)) |u(1)|^2 + \operatorname{Re} \int_0^1 \bar{u} f dx.$$

From (4.5), (4.6) and (4.9) follows that  $u = 0$  if  $f = 0$ .

We next verify (2.4). Let  $f \in L_2(0, 1)$ . We have to solve

$$(4.10) \quad u - A_n u - \zeta_n B u = (1 - a_n^1 - \zeta_n b^1) u - (a_n + \zeta_n b) \frac{du}{dx} = f$$

where  $\zeta_n > 0$  is to be determined.

Since  $a_n b \geq 0$  and  $a_n^2 + b^2 > 0$ , we have  $a_n + \zeta b \neq 0$  for every  $\zeta > 0$  and therefore (4.10) is equivalent to

$$u(x) = k_n \exp \int_0^x K_n(\xi) d\xi + \int_0^x \tilde{K}_n(x, \xi) f(\xi) d\xi$$

where

$$K_n(\xi) = (1 - a_n^1(\xi) - \zeta_n b^1(\xi)) / (a_n(\xi) + \zeta_n b(\xi)),$$

$$\tilde{K}_n(x, \xi) = - \left[ \exp \int_\xi^x K_n(\eta) d\eta \right] / (a_n(\xi) + \zeta_n b(\xi)).$$

The constant  $k_n$  must be such that  $u(0) = cu(1)$ . This condition leads to

$$k_n = k_n c \exp \int_0^1 K_n(\xi) d\xi + c \int_0^1 \tilde{K}_n(1, \xi) f(\xi) d\xi$$

and this equation is solvable for  $k_n$  for arbitrary  $f \in L_2(0, 1)$  if and only if

$$c \exp \int_0^1 K_n(\xi) d\xi \neq 1.$$

This last condition is satisfied provided  $\zeta_n$  is chosen according to condition (4.7). Thus (2.4) is satisfied.

*Proof of Theorem 4.2.* We first note that (4.8) implies (4.6) for all sufficiently large  $n$ . Moreover, (4.7) is also satisfied for all large  $n$  if  $\{\zeta_n\}$  is any sequence of positive numbers which tends to zero. Thus conditions (2.1)–(2.4) are satisfied. That (3.1) and (3.2) also hold is a consequence of the inequality

$$(4.11) \quad \|u\|_1 \leq K(\|Bu\|_0 + \|u\|_0), \quad u \in D(B)$$

where the constant  $K$  is independent of  $u$ . In fact, suppose (4.11) holds and  $\{u_n\} \subset D(B)$ ,  $u_n \rightarrow u$ ,  $Bu_n \rightarrow v$  in  $L_2(0, 1)$ . By (4.11),  $\{u_n\}$  converges in  $H_2^1(0, 1)$ . Since  $D(B)$  is a closed subspace of  $H_2^1(0, 1)$  and  $\|Bu_n\|_0 \leq (\text{const.}) \|u_n\|_1$  it follows that  $u \in D(B)$  and  $Bu = v$ , i.e.,  $B$  is closed. Moreover, we have

$$\|A_n u\|_0 \leq \sup_{0 \leq x \leq 1} (|\alpha_n(x)| + |\alpha_n^1(x)|) \|u\|_1$$

and therefore

$$\sup_{\substack{u \in D(B) \\ u \neq 0}} \|A_n u\|_0 / (\|Bu\|_0 + \|u\|_0) \leq K \sup_{0 \leq x \leq 1} (|\alpha_n(x)| + |\alpha_n^1(x)|)$$

which tends to zero as  $n \rightarrow \infty$ . Thus (3.2) is satisfied.

It only remains to prove (4.11). We have

$$\|Bu\|_0^2 = \int_0^1 \left( b \frac{du}{dx} + b^1 u \right) \left( b \frac{d\bar{u}}{dx} + b^1 \bar{u} \right) dx.$$

Using the inequality

$$2|yz| \leq \delta |y|^2 + \frac{1}{\delta} |z|^2, \quad \delta > 0$$

we obtain

$$\|Bu\|_0^2 \geq \inf_{0 \leq x \leq 1} |b(x)|^2 \left\| \frac{du}{dx} \right\|_0^2 - \varepsilon \left\| \frac{du}{dx} \right\|_0^2 - K_\varepsilon \|u\|_0^2.$$

Choosing  $\varepsilon = 1/2 \inf_{0 \leq x \leq 1} |b(x)|^2$  leads to (4.11).

*Proof of Theorem 4.3.* We have only to verify that  $\text{Rg}(I - A_n) = L_2(0, 1)$ ,  $n \geq N$ , and

$$\sup_{n \geq N} \|(I - A_n)^{-1}\| < \infty.$$

From (4.9) and the present hypotheses it follows that for all sufficiently large  $n$ ,

$$\frac{1}{2} \|u\|_0^2 \leq \|u - A_n u\|, \quad u \in D(A_n).$$

Let  $f \in L_2(0, 1)$ . Since  $\alpha_n \neq 0$ , the equation

$$u - A_n u = (1 - a_n^1)u - a_n \frac{du}{dx} = f$$

is equivalent to

$$u(x) = k_n \exp \int_0^x \frac{1 - a_n^1(\xi)}{a_n} d\xi + F_n(x)$$

where  $F_n(x)$  is a known function and the constant  $k_n$  must be such that  $u(0) = cu(1)$ . This is possible for arbitrary  $f \in L_2(0, 1)$  if and only if

$$c \exp \int_0^1 \frac{1 - a_n^1(\xi)}{a_n(\xi)} d\xi \neq 1$$

and this last condition is obviously satisfied for all sufficiently large  $n$  in view of (4.8). Thus  $\text{Rg}(I - A_n) = X$ ,  $n \geq N$ , and the proof is complete.

EXAMPLE 2. We consider, for  $n = 1, 2, \dots$ , the problem

$$(4.12) \quad \frac{\partial}{\partial t} \left( u - \frac{1}{n} \frac{\partial u}{\partial x} \right) - \left( b(x)u + \frac{\partial^2 u}{\partial x^2} \right) = 0, \quad 0 < x < 1, \quad t > 0,$$

$$(4.13) \quad u(x, 0) = u_0(x), \quad 0 < x < 1,$$

$$(4.14) \quad u(0, t) = cu(1, t), \quad \bar{c} \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t), \quad t > 0.$$

The function  $b$  is real-valued and of class  $C'([0, 1])$  and  $c$  is a complex constant. Let  $X = L_2(0, 1)$ ,  $D(A_n)$  be as in the first example and  $A_n = 1/n d/dx$ . Let  $H_2^2(0, 1)$  be the set of functions in  $L_2(0, 1)$  whose first and second distributional derivatives are in  $L_2(0, 1)$  and set

$$D(B) = \left\{ u \in H_2^2(0, 1) : u(0) = cu(1), \bar{c} \frac{du}{dx}(0) = \frac{du}{dx}(1) \right\},$$

$$Bu = bu + \frac{d^2 u}{dx^2}, \quad u \in D(B).$$

The norm in  $H_2^2(0, 1)$  is denoted by  $\|\cdot\|_2$  and defined by

$$\|u\|_2 = \left( \|u\|_0^2 + \left\| \frac{du}{dx} \right\|_0^2 + \left\| \frac{d^2 u}{dx^2} \right\|_0^2 \right)^{1/2}.$$

Each function in  $H_2^2(0, 1)$  is of class  $C'([0, 1])$  and the injection of  $H_2^2(0, 1)$  into  $C'([0, 1])$  is continuous. It follows that  $D(B)$  is a closed subspace of  $H_2^2(0, 1)$ , is dense in  $L_2(0, 1)$  and as in the first example it is not difficult to verify that

$$(4.15) \quad \|u\|_2 \leq K(\|Bu\|_0 + \|u\|_0), \quad u \in D(B).$$

Let  $B^*$  be the adjoint of  $B$ . As is well-known,  $D(B^*) \subset H_2^2(0, 1)$

and, since  $b + d^2/dx^2$  is a formally self-adjoint differential operator,

$$B^*v = bv + \frac{d^2v}{dx^2}, \quad v \in D(B^*).$$

We show that  $B^* = B$ . If  $v \in D(B^*)$  then for all  $u \in D(B)$  we have

$$\begin{aligned} (Bu, v) &= \int_0^1 \left( bu + \frac{d^2u}{dx^2} \right) \bar{v} dx = \frac{du}{dx}(0)(\bar{c}\bar{v}(1) - \bar{v}(0)) \\ &\quad - u(1) \left( \frac{d\bar{v}}{dx}(1) - c \frac{d\bar{v}}{dx}(0) \right) + (u, B^*v). \end{aligned}$$

Since the first two terms on the right must vanish for all  $u \in D(B)$  we have  $v(0) = cv(1)$ ,  $\bar{c}(dv/dx)(0) = (dv/dx)(1)$ , that is,  $v \in D(B)$ . Thus  $B^* \subseteq B$ . On the other hand,  $(Bu, v) = (u, Bv)$  for all  $u$  and  $v$  in  $D(B)$  so that  $B$  is symmetric. Hence  $B$  is self-adjoint.

**THEOREM 4.4.** *Suppose  $b \leq 0$ ,  $db/dx \leq 0$  and*

$$b(1) - |c|^2 b(0) \geq 0.$$

*Then the hypotheses of Corollary 3.1 are satisfied.*

Thus for each  $n$  and  $u_0 \in D(B)$  the problem (4.12)–(4.14) has a unique solution  $u_n$  and, as  $n \rightarrow \infty$ ,  $u_n(t)$  converges in  $L_2(0, 1)$  to the solution of

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - b(x)u = 0, \quad 0 < x < 1, \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad 0 < x < 1,$$

$$u(0, t) = cu(1, t), \quad \bar{c} \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t), \quad t > 0,$$

uniformly on bounded subsets of  $[0, \infty)$ .

*Proof of Theorem 4.4.* We have for  $u \in D(B)$

$$(Bu, u) = \int_0^1 b |u|^2 dx - \int_0^1 \left| \frac{du}{dx} \right|^2 dx \leq 0.$$

$B$  is therefore a self-adjoint, dissipative operator and, consequently, maximal dissipative. We also have

$$n \cdot \operatorname{Re} (A_n u, u) = \frac{1}{2} (1 - |c|^2) |u(1)|^2, \quad u \in D(A_n).$$

Since  $b \leq 0$  and  $db/dx \leq 0$  we have  $b(1) \leq b(0) \leq 0$ . Since also  $b(1) \geq$

$|c|^2 b(0)$  it follows that  $|c|^2 \geq 1$ . Thus  $A_n$  is dissipative and one easily proves as in the first example that  $Rg(I - A_n) = X$ .

Next we verify (1.3). We have for  $u \in D(B)$

$$\begin{aligned} n \cdot \operatorname{Re}(A_n u, Bu) &= \operatorname{Re} \int_0^1 \frac{du}{dx} \left( b\bar{u} + \frac{d^2 \bar{u}}{dx^2} \right) dx \\ &= \frac{1}{2} (b(1) - |c|^2 b(0)) |u(1)|^2 - \frac{1}{2} \int_0^1 \frac{db}{dx} |u|^2 dx \\ &\quad + \frac{1}{2} \left| \frac{du}{dx}(0) \right|^2 (|\bar{c}|^2 - 1) \geq 0. \end{aligned}$$

(1.3) follows from this inequality and the fact that  $B$  is dissipative. Finally, (3.2) is an immediate consequence of (4.15).

### REFERENCES

1. J. A. Goldstein, *Lectures on Semigroups of Nonlinear Operators*, Tulane University Lecture Notes, 1972.
2. J. Lagnese, *Approximation of solutions of differential equations in Hilbert space*, J. Math. Soc. Japan, **25** (1973), 132-143.
3. G. Lumer and R. S. Phillips, *Dissipative operators in a Banach space*, Pacific J. Math., **11** (1961), 679-698.
4. R. S. Phillips, *Dissipative operators and hyperbolic systems of partial differential equations*, Trans. Amer. Math. Soc., **90** (1959), 193-254.
5. R. E. Showalter, *Equations with operators forming a right angle*, Pacific J. Math., **5** (1973), 357-362.
6. H. F. Trotter, *Approximation of semigroups of operators*, Pacific J. Math., **8** (1958), 887-919.

Received May 30, 1972 and in revised form February 20, 1973.

GEORGETOWN UNIVERSITY



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)  
University of California  
Los Angeles, California 90024

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT  
University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM  
Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1973 by Pacific Journal of Mathematics  
Manufactured and first issued in Japan

Kenneth Abernethy, <i>On characterizing certain classes of first countable spaces by open mappings</i> .....	319
Ross A. Beaumont and Donald Lawver, <i>Strongly semisimple abelian groups</i> .....	327
Gerald A. Beer, <i>The index of convexity and parallel bodies</i> .....	337
Victor P. Camillo and Kent Ralph Fuller, <i>On Loewy length of rings</i> .....	347
Stephen LaVern Campbell, <i>Linear operators for which <math>T^*T</math> and <math>TT^*</math> commute.</i> <i>II</i> .....	355
Charles Kam-Tai Chui and Philip Wesley Smith, <i>Characterization of a function by certain infinite series it generates</i> .....	363
Allan L. Edelson, <i>Conjugations on stably almost complex manifolds</i> .....	373
Patrick John Fleury, <i>Hollow modules and local endomorphism rings</i> .....	379
Jack Tilden Goodykoontz, Jr., <i>Connectedness in kleinen and local connectedness in <math>2^X</math> and <math>C(X)</math></i> .....	387
Robert Edward Jamison, II, <i>Functional representation of algebraic intervals</i> .....	399
Athanassios G. Kartsatos, <i>Nonzero solutions to boundary value problems for nonlinear systems</i> .....	425
Soon-Kyu Kim, Dennis McGavran and Jingyal Pak, <i>Torus group actions on simply connected manifolds</i> .....	435
David Anthony Klarner and R. Rado, <i>Arithmetic properties of certain recursively defined sets</i> .....	445
Ray Alden Kunze, <i>On the Frobenius reciprocity theorem for square-integrable representations</i> .....	465
John Lagnese, <i>Existence, uniqueness and limiting behavior of solutions of a class of differential equations in Banach space</i> .....	473
Teck Cheong Lim, <i>A fixed point theorem for families on nonexpansive mappings</i> ...	487
Lewis Lum, <i>A quasi order characterization of smooth continua</i> .....	495
Andy R. Magid, <i>Principal homogeneous spaces and Galois extensions</i> .....	501
Charles Alan McCarthy, <i>The norm of a certain derivation</i> .....	515
Louise Elizabeth Moser, <i>On the impossibility of obtaining <math>S^2 \times S^1</math> by elementary surgery along a knot</i> .....	519
Gordon L. Nipp, <i>Quaternion orders associated with ternary lattices</i> .....	525
Anthony G. O'Farrell, <i>Equiconvergence of derivations</i> .....	539
Dorte Olesen, <i>Derivations of <math>AW^*</math>-algebras are inner</i> .....	555
Dorte Olesen and Gert Kjærgaard Pedersen, <i>Derivations of <math>C^*</math>-algebras have semi-continuous generators</i> .....	563
Duane O'Neill, <i>On conjugation cobordism</i> .....	573
Chull Park and S. R. Paranjape, <i>Probabilities of Wiener paths crossing differentiable curves</i> .....	579
Edward Ralph Rozema, <i>Almost Chebyshev subspaces of <math>L^1(\mu; E)</math></i> .....	585
Lesley Millman Sibner and Robert Jules Sibner, <i>A note on the Atiyah-Bott fixed point formula</i> .....	605
Betty Salzberg Stark, <i>Irreducible subgroups of orthogonal groups generated by groups of root type 1</i> .....	611
N. Stavrakas, <i>A note on starshaped sets, (k)-extreme points and the half ray property</i> .....	627
Carl E. Swenson, <i>Direct sum subset decompositions of <math>Z</math></i> .....	629
Stephen Tefteller, <i>A two-point boundary problem for nonhomogeneous second order differential equations</i> .....	635
Robert S. Wilson, <i>Representations of finite rings</i> .....	643