Pacific Journal of Mathematics

SINGULAR PERTURBATION OF A TIME-DEPENDENT CAUCHY PROBLEM IN A HILBERT SPACE

LARRY EUGENE BOBISUD AND JAMES CALVERT

Vol. 54, No. 1 May 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 54 No. 1, 1974

SINGULAR PERTURBATION OF A TIME-DEPENDENT CAUCHY PROBLEM IN A HILBERT SPACE

L. E. BOBISUD AND JAMES CALVERT

Let A be a self-adjoint operator, not necessarily bounded, in the Hilbert space H, with resolution of the identity E_λ. Define $h(t, A) = \int_{-\infty}^{\infty} h(t, \lambda) dE_\lambda$. It is shown that as $\varepsilon \to 0+$ the solution of the abstract problem $\varepsilon U_\varepsilon'' + bU_\varepsilon' + h(t, A)U_\varepsilon = 0$, $U_\varepsilon(0) = x_0$, $U_\varepsilon'(0) = x_1$ tends in the norm of H to the solution of $bU_0' + h(t, A)U_0 = 0$, $U_0(0) = x_0$ for data x_0, x_1 in a dense subset of H.

Let A be a (possibly unbounded) self-adjoint operator in a Hilbert space H, and let E_λ be the resolution of the identity for A, so that

$$Ax = \int_{-\infty}^{\infty} \lambda dE_\lambda x$$

for $x \in D(A) \subset H$. Let $h(t, \lambda)$ defined on $[0, \infty) \times (-\infty, \infty)$ be a Borel measurable function of λ for fixed t and a continuous function of t for fixed real λ. Then an operator $h(t, A)$ can be defined by

$$h(t, A)x = \int_{-\infty}^{\infty} h(t, \lambda) dE_\lambda x$$

for $x \in D(h(t, A))$, where

$$D(h(t, A)) = \left\{ x \in H: \int_{-\infty}^{\infty} |h(t, \lambda)|^2 d\|E_\lambda x\|^2 < \infty \right\} .$$

We shall be concerned with the behavior as $\varepsilon \to 0+$ of the solution of the problem

$$(1) \qquad \varepsilon U_\varepsilon'' + bU_\varepsilon' + h(t, A)U_\varepsilon = 0,\ U_\varepsilon(0) = x_0,\ U_\varepsilon'(0) = x_1 ,$$

where b is a positive constant. It seems reasonable to expect that $U_\varepsilon \to U_0$ as $\varepsilon \to 0$, where U_0 solves the problem

$$(2) \qquad bU_0' + h(t, A)U_0 = 0,\ U_0(0) = x_0 .$$

We prove this convergence, as well as $U_\varepsilon' \to U_0'$, in the norm of H for data x_0, x_1 restricted to a certain dense subset of H.

Several abstract singular perturbation problems of this nature have been considered before. Kisyński [5] considered the case $h(t, A) \equiv A$ where A is positive as well as self-adjoint; in addition, he considered the inhomogeneous problem. Smoller [9, 10], Latil [6], Friedman [4] and the authors [1] have extended his results to higher-

order equations and have removed the restriction that A be positive. The use of the resolution of the identity for A and estimates for the special case $H = L^2(-\infty, \infty)$, $A =$ a real parameter λ is central to all these treatments, as well as to the present study. Singular perturbations in Banach spaces have been studied by Bobisud and Hersh [2], Sova [11], and Schoene [8].

Time-dependent equations of the form

$$\varepsilon p(t) U_\varepsilon'' + q(t) U_\varepsilon' + A U_\varepsilon = 0$$

and higher-order generalizations have been considered by Friedman [4]. The only previous study of a nonfactorable time-variable operator $h(t, A)$ which is known to the authors is that of Nur [7], who considers the case $h(t, \lambda) = e^{\lambda t}$, so $h(t, A)$ is a semigroup with generator A. The result of Nur is contained in the theorems to follow.

As mentioned above, we begin by examining in part 1 the special case $H = L^2(-\infty, \infty)$ and show that $u_\varepsilon(t, \lambda) \to u_0(t, \lambda)$, where

$$\varepsilon u_\varepsilon'' + b u_\varepsilon' + h(t, \lambda) u_\varepsilon = 0,\ u_\varepsilon(0, \lambda) = x_0,\ u_\varepsilon'(0, \lambda) = x_1 , \tag{3}$$

$$b u_0' + h(t, \lambda) u_0 = 0 , \quad u_0(0, \lambda) = x_0 . \tag{4}$$

We also establish for this case certain estimates to be used in treating the Hilbert space problem in part 2.

1. **The problem on the real line.** Since the problems (3), (4) are linear, we may write

$$u_\varepsilon(t, \lambda) = p_\varepsilon(t, \lambda) x_0 + q_\varepsilon(t, \lambda) x_1 ,$$
$$u_0(t, \lambda) = p_0(t, \lambda) x_0$$

for certain functions p_ε, q_ε, p_0. Regarding a solution of (3) for fixed ε as a solution of the equation $\varepsilon u_\varepsilon'' + b u_\varepsilon' = -h(t, \lambda) u_\varepsilon(t)$, we find that $u_\varepsilon(t, \lambda)$ satisfies the following integral equation:

$$u_\varepsilon(t, \lambda) = x_0 + \frac{\varepsilon}{b}[1 - e^{-(b/\varepsilon)t}] x_1 - \frac{1}{b} \int_0^t h(s, \lambda)[1 - e^{-(b/\varepsilon)(t-s)}] u_\varepsilon(s, \lambda) ds . \tag{5}$$

Similarly, for u_0 we obtain the integral equation

$$u_0(t, \lambda) = x_0 - \frac{1}{b} \int_0^t h(s, \lambda) u_0(s, \lambda) ds ; \tag{6}$$

thus for the difference we have

$$
\begin{aligned}
u_\varepsilon(t, \lambda) - u_0(t, \lambda) = {} & \frac{\varepsilon}{b}[1 - e^{-(b/\varepsilon)t}]x_1 \\
& - \frac{1}{b}\int_0^t h(s, \lambda)[u_\varepsilon(s, \lambda) - u_0(s, \lambda)]ds \\
& + \frac{1}{b}\int_0^t h(s, \lambda)e^{-(b/\varepsilon)(t-s)}u_\varepsilon(s, \lambda)ds \ .
\end{aligned} \tag{7}
$$

For convenience we define

$$
M_T(\lambda) = 1 + \frac{1}{b}e^{1/b\int_0^T |h(s,\lambda)|ds}\int_0^T |h(s, \lambda)|\, ds \ .
$$

Observe that $M_t(\lambda) \leqq 2\exp\left\{2\int_0^T |h(s, \lambda)|\, ds\right\}$ since $\alpha e^\alpha \leqq e^{2\alpha}$.

LEMMA 1. *For* $t \in [0, T]$

$$
\begin{aligned}
|p_\varepsilon(t, \lambda)| &\leqq M_T(\lambda) \ , \\
|q_\varepsilon(t, \lambda)| &\leqq \frac{\varepsilon}{b}M_T(\lambda) \ , \\
|p_0(t, \lambda)| &\leqq M_T(\lambda) \ .
\end{aligned}
$$

Proof. From (5) we obtain

$$
|u_\varepsilon(t, \lambda)| \leqq |x_0| + \frac{\varepsilon}{b}|x_1| + \frac{1}{b}\int_0^t |h(s, \lambda)|\, |u_\varepsilon(s, \lambda)|\, ds \ ;
$$

Gronwall's lemma [3, p. 37] then implies that

$$
|u_\varepsilon(t, \lambda)| \leqq \left[|x_0| + \frac{\varepsilon}{b}|x_1|\right]M_T(\lambda) \ . \tag{8}
$$

The first two statements of the lemma follow on setting $x_0 = 1$, $x_1 = 0$ and $x_0 = 0$, $x_1 = 1$, respectively. The last statement follows in the same manner from (6).

THEOREM 1. *For any* $T > 0$ *and any fixed* λ, $p_\varepsilon(t, \lambda) \to p_0(t, \lambda)$ *and* $q_\varepsilon(t, \lambda) \to 0$ *as* $\varepsilon \to 0+$, *uniformly in* $t \in [0, T]$.

Proof. That $q_\varepsilon(t, \lambda) \to 0$ is obvious from Lemma 1. Setting $x_1 = 0$, $x_0 = 1$ in (7) yields

$$
\begin{aligned}
|p_\varepsilon(t, \lambda) - p_0(t, \lambda)| &\leqq \frac{1}{b}\int_0^t |h(s, \lambda)|\, |p_\varepsilon(s, \lambda) - p_0(s, \lambda)|\, ds \\
&\quad + \frac{1}{b}M_T(\lambda)\int_0^t |h(s, \lambda)|\, e^{-(b/\varepsilon)(t-s)}ds \ .
\end{aligned}
$$

For any $\delta > 0$ we have

$$\int_0^t |h(s,\lambda)|e^{-(b/\varepsilon)(t-s)}ds \leqq \int_0^{\max(t-\delta,0)} |h(s,\lambda)|e^{-(b/\varepsilon)(t-s)}ds + \int_{\max(t-\delta,0)}^t |h(s,\lambda)|ds$$
$$\leqq e^{-(b/\varepsilon)\delta}\int_0^T |h(s,\lambda)|ds + \delta \sup_{0\leqq t\leqq T}|h(t,\lambda)|$$
$$\equiv \delta g(\lambda) + e^{-(b/\varepsilon)\delta}k(\lambda) .$$

Thus

$$|p_\varepsilon(t,\lambda) - p_0(t,\lambda)| \leqq \frac{1}{b}M_T(\lambda)[\delta g(\lambda) + e^{-(b/\varepsilon)\delta}k(\lambda)]$$
$$+ \frac{1}{b}\int_0^t |h(s,\lambda)|\,|p_\varepsilon(s,\lambda) - p_0(s,\lambda)|ds ;$$

application of Gronwall's lemma yields the inequality

$$|p_\varepsilon(t,\lambda) - p_0(t,\lambda)| \leqq \frac{1}{b}[\delta g(\lambda) + e^{-(b/\varepsilon)\delta}k(\lambda)]M_\lambda(T)^2 .$$

Here the right-hand side can be made arbitrarily small by first choosing $\delta > 0$ small and then requiring ε to be sufficiently small.

LEMMA 2. *For* $t \in [0, T]$,

$$|p_\varepsilon'(t,\lambda)| \leqq \frac{2b}{\varepsilon}e^{2\int_0^T |h(s,\lambda)|ds} ,$$
$$|q_\varepsilon'(t,\lambda)| \leqq 1 + e^{2\int_0^T |h(s,\lambda)|ds} .$$

Proof. Differentiating (5), taking absolute values, and using the estimate (8), we get

$$(9) \qquad |u_\varepsilon'(t,\lambda)| \leqq |x_1| + \frac{1}{\varepsilon}[|x_0| + \frac{\varepsilon}{b}|x_1|]M_T(\lambda)\int_0^T |h(s,\lambda)|ds ;$$

setting in turn $x_0 = 1$, $x_1 = 0$ and $x_0 = 0$, $x_1 = 1$, and using the inequality $\alpha(1 + \alpha e^\alpha) \leqq 2e^{2\alpha}$ for $\alpha > 0$, yields the result.

LEMMA 3. *For* $t \in [0, T]$,

$$|p_\varepsilon''(t,\lambda)| \leqq \frac{1}{\varepsilon^2}\Big(2b^2 + \varepsilon \max_{[0,T]}|h(t,\lambda)|\Big)e^{2\int_0^T |h(s,\lambda)|ds} ,$$
$$|q_\varepsilon''(t,\lambda)| \leqq \Big(\frac{2b}{\varepsilon} + \frac{2}{b}\max_{[0,T]}|h(t,\lambda)|\Big)e^{2\int_0^T |h(s,\lambda)|ds} ,$$
$$|p_0'(t,\lambda)| \leqq \frac{2}{b}\Big(\max_{[0,T]}|h(t,\lambda)|\Big)e^{2\int_0^T |h(s,\lambda)|ds} .$$

Proof. The proof follows easily by using the differential equations (3), (4) themselves and the estimates contained in (9) and Lemma 1.

THEOREM 2. *For any* $0 < \tau < T$, $p'_\varepsilon(t, \lambda) \to p'_0(t, \lambda)$ *uniformly for* $t \in [0, T]$ *and* $q'_\varepsilon(t, \lambda) \to 0$ *uniformly for* $t \in [\tau, T]$.

Proof. Differentiation of (5) and (6) yields the equations

$$p'_\varepsilon(t, \lambda) = -\frac{1}{\varepsilon}\int_0^t h(s, \lambda)e^{-(b/\varepsilon)(t-s)}p_\varepsilon(s, \lambda)ds ,$$

$$p'_0(t, \lambda) = -\frac{1}{b}h(t, \lambda)p_0(t, \lambda) ,$$

$$q'_\varepsilon(t, \lambda) = e^{-(b/\varepsilon)t} - \frac{1}{\varepsilon}\int_0^t h(s, \lambda)e^{-(b/\varepsilon)(t-s)}q_\varepsilon(s, \lambda)ds .$$

From the last of these equations we obtain, using Lemma 1,

$$|q'_\varepsilon(t, \lambda)| \leqq e^{-(b/\varepsilon)t} + \frac{1}{b}M_T(\lambda)\int_0^t |h(s, \lambda)|e^{-(b/\varepsilon)(t-s)}ds ;$$

in the proof of Theorem 1 this integral was shown to approach zero, uniformly in $t \in [0, T]$, as $\varepsilon \to 0$. Since $e^{-(b/\varepsilon)t} \to 0$ uniformly for $t \in [\tau, T]$, the second statement of theorem follows.

We turn to the first statement, writing for any fixed $\delta > 0$

$$\begin{aligned} p'_\varepsilon(t, \lambda) - p'_0(t, \lambda) = &-\frac{1}{\varepsilon}\int_0^{\max(t-\delta,0)} h(s, \lambda)e^{-(b/\varepsilon)(t-s)}p_\varepsilon(s, \lambda)ds \\ &-\frac{1}{\varepsilon}\int_{\max(t-\delta,0)}^t h(s, \lambda)e^{-(b/\varepsilon)(t-s)}p_\varepsilon(s, \lambda)ds \\ &+\frac{1}{b}h(t, \lambda)p_0(t, \lambda) \equiv J_1 + J_2 + J_3 . \end{aligned}$$

Since $\lambda e^{-\lambda} \leqq e^{-(1/2)\lambda}$, we have the estimate

$$\begin{aligned} |J_1| &\leqq \frac{1}{b\delta}\int_0^{\max(t-\delta,0)} \frac{b}{\varepsilon}(t-s)e^{-(b/\varepsilon)(t-s)}|h(s, \lambda)|\,|p_\varepsilon(s, \lambda)|ds \\ &\leqq \frac{1}{b\delta}M_T(\lambda)e^{-(b/2\varepsilon)\delta}\int_0^T |h(s, \lambda)|ds , \end{aligned}$$

where the final quantity tends to zero with ε. Also, since

$$\int_{t-\delta}^t \frac{b}{\varepsilon}e^{-(b/\varepsilon)(t-s)}ds = 1 - e^{-b\delta/\varepsilon} ,$$

we have that

$$\begin{aligned} |b(J_2 + J_3)| \leqq &\frac{b}{\varepsilon}\int_{t-\delta}^t e^{-(b/\varepsilon)(t-s)}|h(s, \lambda)|\,|p_\varepsilon(s, \lambda) - p_0(s, \lambda)|ds \\ &+ e^{-b\delta/\varepsilon}|h(t, \lambda)|\,|p_0(t, \lambda)| . \end{aligned}$$

Setting $\mu_\varepsilon(\lambda, T) = \sup_{t\in[0,T]} |p_\varepsilon(t, \lambda) - p_0(t, \lambda)|$, $\nu(\lambda, T) = \sup_{t\in[0,T]} |h(t, \lambda)|$, we get that

$$|b(J_2 + J_3)| \leqq e^{-b\delta/\varepsilon}\nu(\lambda, T) \sup_{t\in[0,T]} |p_0(t, \lambda)| + \mu_\varepsilon(\lambda, T)\nu(\lambda, T)(1 - e^{-b\delta/\varepsilon}) ;$$

the right-hand side tends to zero with ε by Theorem 1.

2. **The problem in a Hilbert space.** Let H be a Hilbert space, E_λ the resolution of the identity for the self-adjoint operator A, and define operators P_ε, P_0, Q_ε on H by

$$P_\varepsilon(t) = \int_{-\infty}^{\infty} p_\varepsilon(t, \lambda)dE_\lambda ,$$

$$P_0(t) = \int_{-\infty}^{\infty} p_0(t, \lambda)dE_\lambda ,$$

$$Q_\varepsilon(t) = \int_{-\infty}^{\infty} q_\varepsilon(t, \lambda)dE_\lambda ;$$

let $T > 0$ be a fixed number. Let D denote the (dense) domain of the operator

$$(I + A + \max_{[0,T]} |h(t, A)|) \exp\left\{2\int_0^T |h(s, A)|\,ds\right\}$$

defined as

$$\int_{-\infty}^{\infty}\left(1 + \lambda + \max_{[0,T]} |h(t, \lambda)|\right)e^{2\int_0^T |h(s,\lambda)|\,ds}dE_\lambda ;$$

then

$$D = \left\{x \in H: \int_{-\infty}^{\infty}\left(1 + \lambda + \max_{[0,T]} |h(t, \lambda)|\right)^2 e^{4\int_0^T |h(s,\lambda)|\,ds}d(E_\lambda x, x) < \infty\right\} .$$

D is contained in the domains of P_ε, P_0, Q_ε, because for $x \in D$

$$\|P_\varepsilon(t)x\|^2 = \int_{-\infty}^{\infty} |p_\varepsilon(t, \lambda)|^2 d(E_\lambda x, x) \leqq \int_{-\infty}^{\infty} [M_T(\lambda)]^2 d(E_\lambda x, x)$$
$$\leqq \int_{-\infty}^{\infty} 4e^{4\int_0^T |h(s,\lambda)|\,ds}d(E_\lambda x, x) < \infty ;$$

similar calculations are valid for P_0, Q_0. Also, if $x \in D$, then $P_\varepsilon x$, $P_0 x$, $Q_\varepsilon x \in D(A)$, as is shown by calculations like the following for $P_\varepsilon x$:

$$\int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda P_\varepsilon x\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\left\|E_\lambda \int_{-\infty}^{\infty} p_\varepsilon(t, \mu)dE_\mu x\right\|^2$$
$$= \int_{-\infty}^{\infty} \lambda^2 d\left\|\int_{-\infty}^{\lambda} p_\varepsilon(t, \mu)dE_\mu x\right\|^2 \leqq \int_{-\infty}^{\infty} \lambda^2 d\int_{-\infty}^{\lambda} 4e^{4\int_0^T |h(s,\mu)|\,ds}d\|E_\mu x\|^2$$
$$= 4\int_{-\infty}^{\infty} \lambda^2 e^{4\int_0^T |h(s,\lambda)|\,ds}d\|E_\lambda x\|^2 < \infty .$$

Defining $U_\varepsilon(t) = P_\varepsilon(t)x_0 + Q_\varepsilon(t)x_1$ for $\varepsilon > 0$ and $U_0(t) = P_0(t)x_0$, we have

LEMMA 4. *$U_\varepsilon(t)$, $U_0(t)$ solve the problems* (1), (2), *respectively, on* $[0, T]$.

Proof. We shall prove only the statement concerning U_ε; that U_0 solves (2) is proved similarly. In view of the fact that $p_\varepsilon(t, \lambda)$ and $q_\varepsilon(t, \lambda)$ satisfy the differential equation (3), it is enough to show that the first and second derivatives of P_ε, Q_ε can be taken under the integral sign; that is, for $x \in D$,

$$\left(\frac{d}{dt}\right)^\nu P_\varepsilon(t)x = \int_{-\infty}^{\infty}\left(\frac{d}{dt}\right)^\nu p_\varepsilon(t, \lambda)dE_\lambda x,$$

$$\left(\frac{d}{dt}\right)^\nu Q_\varepsilon(t)x \sim = \int_{-\infty}^{\infty}\left(\frac{d}{dt}\right)^\nu q_\varepsilon(t, \lambda)dE_\lambda x \quad (\nu = 1, 2)\ ;$$

we present a proof of the statement for P_ε. By the mean value theorem we have that

$$\frac{1}{h}[P_\varepsilon(t + h)x - P_\varepsilon(t)x] = \int_{-\infty}^{\infty}\frac{d}{dt}p_\varepsilon(t', \lambda)dE_\lambda x$$

for some t' between t and $t + h$. Now

$$\left\|\int_{-\infty}^{\infty}\frac{d}{dt}p_\varepsilon(t', \lambda)dE_\lambda x - \int_{-\infty}^{\infty}\frac{d}{dt}p_\varepsilon(t, \lambda)dE_\lambda x\right\|^2$$

$$\leqq \int_{-\infty}^{\infty}\left|\frac{d}{dt}p_\varepsilon(t', \lambda) - \frac{d}{dt}p_\varepsilon(t, \lambda)\right|^2 d\|E_\lambda x\|^2\ ,$$

so the desired result is a consequence of the Lebesgue dominated convergence theorem if we show that the integrand of the last integral is bounded by a function integrable with respect to the measure $\|E_\lambda x\|^2$. To this end restrict h to be small enough that $0 \leqq t + h \leqq T$. Then from Lemma 2

$$\left|\frac{d}{dt}p_\varepsilon(t', \lambda)\right|^2, \frac{d}{dt}p_\varepsilon(t, \lambda)\Big|^2 \leqq \frac{4b^2}{\varepsilon^2}e^{4\int_0^T |h(s,\lambda)|ds}\ ,$$

which is integrable for each fixed $\varepsilon > 0$. The proof of the case $\nu = 2$ is similar but uses Lemma 3 instead of Lemma 2.

LEMMA 5. *The solutions of the problems* (1), (2) *are unique.*

Proof. We shall show that $P_\varepsilon(t)x_0$ is the only solution of (1) with $x_1 = 0$; the omitted cases are similar. Let $h_n(t, A)$ for any integer n be the bounded operator

$$h_n(t, A) \equiv \int_{-n}^{n} h(t, \lambda)dE_\lambda = (E_n - E_{-n})h(t, A) .$$

Suppose $R_\varepsilon(t)$ is a solution of (1) with $x_1 = 0$, and set $z_\varepsilon(t) = P_\varepsilon(t)x_0 - R_\varepsilon(t)$. Set $z_{\varepsilon,n}(t) = (E_n - E_{-n})z_\varepsilon(t)$; then

$$\begin{aligned} &\varepsilon z''_{\varepsilon,n} + bz'_{\varepsilon,n} + h_n(t, A)z_{\varepsilon,n} \\ &\quad = (E_n - E_{-n})(\varepsilon z''_\varepsilon + bz'_\varepsilon + h(t, A)z_\varepsilon) = 0 , \end{aligned}$$

and $z_{\varepsilon,n}(0) = 0$, $z'_{\varepsilon,n}(0) = 0$. If we show that $z_{\varepsilon,n} \equiv 0$, we will have $0 = \lim_{n\to\infty} z_{\varepsilon,n}(t) = z_\varepsilon(t)$, as desired. Now $z_{\varepsilon,n}$ satisfies the integral equation

$$z_{\varepsilon,n}(t) = -\frac{1}{b}\int_0^t [1 - e^{-(b/\varepsilon)(t-s)}]h_n(s, A)z_{\varepsilon,n}(s)ds ,$$

whence

$$||z_{\varepsilon,n}(t)|| \leqq \frac{1}{b}\int_0^t ||h_n(s, A)||\,||z_{\varepsilon,n}(s)||\,ds ,$$

and $z_{\varepsilon,n} \equiv 0$ follows in standard fashion.

THEOREM 3. *For $x_0, x_1 \in D$ we have*

$$\lim_{\varepsilon\to 0+} ||U_\varepsilon(t) - U_0(t)|| = 0$$

uniformly for $t \in [0, t]$, and for any $\tau > 0$, $\tau < T$,

$$\lim_{\varepsilon\to 0+} ||U'_\varepsilon(t) - U'_0(t)|| = 0$$

uniformly for $t \in [\tau, T]$.

Proof. It is necessary to show that

$$\begin{aligned} &\lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} |p_\varepsilon(t, \lambda) - p_0(t, \lambda)|^2 d||E_\lambda x_0||^2 = 0 , \\ &\lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} |q_\varepsilon(t, \lambda)|^2 d||E_\lambda x_1||^2 = 0 , \\ &\lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} |p'_\varepsilon(t, \lambda) - p'_0(t, \lambda)|^2 d||E_\lambda x_0||^2 = 0 , \\ &\lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} |q'_\varepsilon(t, \lambda)|^2 d||E_\lambda x_1||^2 = 0 . \end{aligned}$$

Since, by Lemma 1,

$$|p_\varepsilon(t, \lambda) - p_0(t, \lambda)|^2 \leqq 2|p_\varepsilon(t, \lambda)|^2 + 2|p_0(t, \lambda)|^2 \leqq 4M_T(\lambda) ,$$

the first result follows from Theorem 1 and the Lebesgue dominated convergence theorem. The remaining statements follow in a similar manner.

REFERENCES

1. L. Bobisud and J. Calvert, *Singularly perturbed differential equations in a Hilbert space*, J. Math. Anal. Appl., **30** (1970), 113-127.
2. L. Bobisud and R. Hersh, *Perturbation and approximation theory for higher-order abstract Cauchy problems*, Rocky Mountain J. Math., **2** (1972), 57-73.
3. E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, 1955.
4. A. Friedman, *Singular perturbations for the Cauchy problem and for boundary value problems*, J. Differential Equations, **5** (1969), 226-261.
5. J. Kisyński, *Sur les équations hyperboliques avec petit paramètre*, Colloq. Math., **10** (1963), 331-343.
6. B. Latil, *Singular perturbations of Cauchy's problem*, J. Math. Anal. Appl., **23** (1968), 683-698.
7. H. Nur, *Singular perturbations of differential equations in abstract spaces*, Pacific J. Math., **36** (1971), 775-780.
8. A. Schoene, *Semigroups and a class of singular perturbation problems*, Indiana Univ. Math. J., **20** (1970), 247-263.
9. J. Smoller, *Singular perturbations and a theorem of Kisyński*, J. Math. Anal. Appl., **12** (1965), 105-114.
10. ———, *Singular perturbations of Cauchy's problem*, Comm. Pure Appl. Math., **18** (1965), 665-677.
11. M. Sova, *Équations hyperboliques avec petit paramètre dans les espaces de Banach généraux*, Colloq. Math., **21** (1970), 303-320.

Received April 2, 1973.

UNIVERSITY OF IDAHO

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 54, No. 1 May, 1974