

Pacific Journal of Mathematics

**ON FUNCTIONAL EQUATIONS CONNECTED WITH
DIRECTED DIVERGENCE, INACCURACY AND GENERALIZED
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ON FUNCTIONAL EQUATIONS CONNECTED WITH DIRECTED DIVERGENCE, INACCURACY AND GENERALIZED DIRECTED DIVERGENCE

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The measures directed divergence, inaccuracy as well as generalized directed divergence occurring in information theory can be characterized by the symmetry, expansibility, branching, and additivity properties together with some regularity and initial conditions. In this paper some functional equations generalizing those implicit in these characterizations shall be treated.

1. Introduction. Let $\Delta_n = \{P = (p_1, p_2, \dots, p_n) \mid p_i \geq 0 \text{ and } \sum_{i=1}^n p_i = 1\}$ and $\Delta'_n = \{P = (p_1, p_2, \dots, p_n) \mid p_i > 0 \text{ and } \sum_{i=1}^n p_i \leq 1\}$ be the set of all finite complete and incomplete probability distributions respectively. In 1948 C. E. Shannon [16] introduced the following measure of information

$$(1.1) \quad H_n(P) = - \sum_{i=1}^n p_i \log p_i ,$$

on Δ_n which is now known as Shannon's entropy. This has been generalized to inaccuracy [10]. Inaccuracy and the related quantities directed divergence or information gain [11, 15] and generalized directed divergence [3] are given by

$$(1.2) \quad H_n(P \parallel Q) = - \sum_{i=1}^n p_i \log q_i , \quad (P \in \Delta_n, Q \in \Delta_n \text{ or } \Delta'_n) ,$$

$$(1.3) \quad I_n(P \parallel Q) = \sum_{i=1}^n p_i \log \frac{p_i}{q_i} , \quad (P \in \Delta_n, Q \in \Delta_n \text{ or } \Delta'_n) ,$$

and

$$(1.4) \quad D_n(P \parallel Q \mid R) = \sum_{i=1}^n p_i \log \frac{q_i}{r_i} , \quad (P \in \Delta_n, Q, R \in \Delta_n \text{ or } \Delta'_n)$$

respectively. While characterizing these measures we come across the following functional equations

$$(1.5) \quad \sum_{i=1}^n \sum_{j=1}^m F(p_i q_j) = \sum_{i=1}^n F(p_i) + \sum_{j=1}^m F(q_j) , \quad (P \in \Delta_n, Q \in \Delta_m) ,$$

$$(1.6) \quad \sum_{i=1}^n \sum_{j=1}^m F(p_i q_j, x_i y_j) = \sum_{i=1}^n F(p_i, x_i) + \sum_{j=1}^m F(q_j, y_j) ,$$

$(P \in \Delta_n, Q \in \Delta_m, X \in \Delta_n \text{ or } \Delta'_n, Y \in \Delta_m \text{ or } \Delta'_m)$

and

$$(1.7) \quad \sum_{i=1}^n \sum_{j=1}^m F(p_i q_j, x_i y_j, u_i v_j) = \sum_{i=1}^n F(p_i, x_i, u_i) + \sum_{j=1}^m F(q_j, y_j, v_j),$$

$$(P \in \mathcal{A}_n, Q \in \mathcal{A}_m, X, U \in \mathcal{A}_n \text{ or } \mathcal{A}'_n, Y, V \in \mathcal{A}_m \text{ or } \mathcal{A}'_m)$$

(cf. [2], [4], [5], [6], [7], [8], [9], [13]).

For the motivation to consider (1.6) and (1.7) and the application of this result, refer to the Remark at the end of this paper.

In this paper we consider the functional equation

$$(1.8) \quad \sum_{i=1}^2 \sum_{j=1}^3 F_{i,j}(p_i q_j, x_i y_j) = \sum_{i=1}^2 G_i(p_i, x_i) + \sum_{j=1}^3 H_j(q_j, y_j),$$

$$(P \in \mathcal{A}_2, Q \in \mathcal{A}_3, X \in \mathcal{A}'_2, Y \in \mathcal{A}'_3)$$

for unknown functions $F_{i,j}, G_i, H_j$. Then this gives the measurable solutions of (1.6) for all $P \in \mathcal{A}_2, Q \in \mathcal{A}_3, X \in \mathcal{A}'_2, Y \in \mathcal{A}'_3$ as a special case. The measurable solution of (1.7) for $P \in \mathcal{A}_2, Q \in \mathcal{A}_3, X, U \in \mathcal{A}'_2, Y, V \in \mathcal{A}'_3$ can also be obtained by a reduction to (1.8).

In solving (1.8) we make use of the following result of C. T. Ng [13]:

THEOREM 1.1. *The measurable solutions of the functional equation*

$$(1.9) \quad \sum_{i=1}^2 \sum_{j=1}^3 F_{i,j}(p_i q_j) = \sum_{i=1}^2 G_i(p_i) + \sum_{j=1}^3 H_j(q_j),$$

for all $P \in \mathcal{A}_2, Q \in \mathcal{A}_3$, are given by

$$(1.10) \quad \left\{ \begin{array}{l} H_1(q) = aq \log q + b_1 q + c_1, \quad H_2(q) = aq \log q + (b_1 + d)q + c_4, \\ H_3(q) = aq \log q + (b_1 + e)q + c_7, \quad F_{1,1}(p) = ap \log p + b_2 p + c_2, \\ F_{1,2}(p) = ap \log p + (b_2 + d)p + c_5, \\ F_{1,3}(p) = ap \log p + (b_2 + e)p + c_8, \\ F_{2,1}(p) = ap \log p + b_3 p + c_3, \quad F_{2,2}(p) = ap \log p + (b_3 + d)p + c_6, \\ F_{2,3}(p) = ap \log p + (b_3 + e)p + c_9, \quad G_1(p) = g(p), \\ G_2(p) = -g(1-p) + a[p \log p + (1-p) \log(1-p)] \\ \quad + (b_3 - b_2)p + (b_2 - b_1) - c_1 + c_2 + c_3 - c_4 + c_5 + c_6 \\ \quad - c_7 + c_8 + c_9, \end{array} \right.$$

where $a, b_1, b_2, b_3, c_1, c_2, \dots, c_9, d, e$ are arbitrary constants and g is an arbitrary measurable function.

2. Measurable solutions of the functional equations (1.6) and (1.8). We first suppose that equation (1.8) is to hold for all $P \in \mathcal{A}_2, Q \in \mathcal{A}_3, X \in \mathcal{A}'_2, Y \in \mathcal{A}'_3$, where $F_{i,j}, G_i, H_j: [0, 1] \times]0, 1[\rightarrow \mathbb{R}$ are functions measurable in their first variables.

For arbitrarily fixed x_i, y_j in $]0, 1[$ with $\sum_{i=1}^2 x_i \leq 1, \sum_{j=1}^3 y_j \leq 1$, equation (1.8) is of the form (1.9) in the p_i 's and q_j 's. Therefore, by Theorem 1.1 there exist 'constants' $a(x_1, x_2, y_1, y_2, y_3), b_i(x_1, x_2, y_1, y_2, y_3), i = 1, 2, 3, c_j(x_1, x_2, y_1, y_2, y_3), j = 1, 2, \dots, 9, d(x_1, x_2, y_1, y_2, y_3), e(x_1, x_2, y_1, y_2, y_3)$ and a measurable function $g(\cdot, x_1, x_2, y_1, y_2, y_3)$ such that

$$(2.1) \quad \left\{ \begin{aligned} H_1(q, y_1) &= a(x_1, x_2, y_1, y_2, y_3)q \log q + b_1(x_1, x_2, y_1, y_2, y_3)q \\ &\quad + c_1(x_1, x_2, y_1, y_2, y_3), \\ H_2(q, y_2) &= a(x_1, x_2, y_1, y_2, y_3)q \log q + (b_1 + d)(x_1, x_2, y_1, y_2, y_3)q \\ &\quad + c_4(x_1, x_2, y_1, y_2, y_3), \\ H_3(q, y_3) &= a(x_1, x_2, y_1, y_2, y_3)q \log q + (b_1 + e)(x_1, x_2, y_1, y_2, y_3)q \\ &\quad + c_7(x_1, x_2, y_1, y_2, y_3), \\ F_{1,1}(p, x_1 y_1) &= a(x_1, x_2, y_1, y_2, y_3)p \log p + b_2(x_1, x_2, y_1, y_2, y_3)p \\ &\quad + c_2(x_1, x_2, y_1, y_2, y_3), \\ F_{1,2}(p, x_1 y_2) &= a(x_1, x_2, y_1, y_2, y_3)p \log p + (b_2 + d)(x_1, x_2, y_1, y_2, y_3)p \\ &\quad + c_5(x_1, x_2, y_1, y_2, y_3), \\ F_{1,3}(p, x_1 y_3) &= a(x_1, x_2, y_1, y_2, y_3)p \log p + (b_2 + e)(x_1, x_2, y_1, y_2, y_3)p \\ &\quad + c_8(x_1, x_2, y_1, y_2, y_3), \\ F_{2,1}(p, x_2 y_1) &= a(x_1, x_2, y_1, y_2, y_3)p \log p + b_3(x_1, x_2, y_1, y_2, y_3)p \\ &\quad + c_3(x_1, x_2, y_1, y_2, y_3), \\ F_{2,2}(p, x_2 y_2) &= a(x_1, x_2, y_1, y_2, y_3)p \log p + (b_3 + d)(x_1, x_2, y_1, y_2, y_3)p \\ &\quad + c_6(x_1, x_2, y_1, y_2, y_3), \\ F_{2,3}(p, x_2 y_3) &= a(x_1, x_2, y_1, y_2, y_3)p \log p + (b_3 + e)(x_1, x_2, y_1, y_2, y_3)p \\ &\quad + c_9(x_1, x_2, y_1, y_2, y_3). \end{aligned} \right.$$

$$(2.2) \quad \left\{ \begin{aligned} G_1(p, x_1) &= g(p, x_1, x_2, y_1, y_2, y_3), \\ G_2(p, x_2) &= -g(1 - p, x_1, x_2, y_1, y_2, y_3) + a(x_1, x_2, y_1, y_2, y_3)[p \log p \\ &\quad + (1 - p) \log (1 - p)] + (b_3 - b_2)(x_1, x_2, y_1, y_2, y_3)p \\ &\quad + (b_2 - b_1 - c_1 + c_2 + c_3 - c_4 + c_5 + c_6 - c_7 + c_8 \\ &\quad + c_9)(x_1, x_2, y_1, y_2, y_3). \end{aligned} \right.$$

From (2.1) we get

$$(2.3) \quad a(x_1, x_2, y_1, y_2, y_3) \equiv \text{constant} = a$$

and

$$(2.4) \quad \left\{ \begin{aligned} b_1(x, x_2, y_1, y_2, y_3) &\equiv \text{a function of } y_1 \text{ only} = b_1(y_1), \\ b_1(y_1) + d(x_1, x_2, y_1, y_2, y_3) &\equiv \text{a function of } y_2 = \theta_1(y_2), \\ b_1(y_1) + e(x_1, x_2, y_1, y_2, y_3) &\equiv \text{a function of } y_3 = \phi_1(y_3), \\ b_2(x_1, x_2, y_1, y_2, y_3) &\equiv \text{a function of } x_1 y_1 = b_2(x_1 y_1), \end{aligned} \right.$$

$$(2.4) \quad \begin{cases} b_2(x_1 y_1) + d(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } x_1 y_2 = \theta_2(x_1 y_2), \\ b_2(x_1 y_1) + e(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } x_1 y_3 = \phi_2(x_1 y_3), \\ b_3(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } x_2 y_1 = b_3(x_2 y_1), \\ b_3(x_2 y_1) + d(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } x_2 y_2 = \theta_2(x_2 y_2), \\ b_3(x_2 y_1) + e(x_1, x_2, y_1, y_2, y_3) \equiv \text{a function of } x_2 y_3 = \phi_3(x_2 y_3), \end{cases}$$

where x_i, y_j are in $]0, 1[$ with $\sum_{i=1}^2 x_i \leq 1$ and $\sum_{j=1}^3 y_j \leq 1$.
Similarly

$$(2.5) \quad \begin{cases} c_1(x_1, x_2, y_1, y_2, y_3) = c_1(y_1), \\ c_2(x_1, x_2, y_1, y_2, y_3) = c_2(x_1 y_1), \\ c_3(x_1, x_2, y_1, y_2, y_3) = c_3(x_2 y_1), \\ c_4(x_1, x_2, y_1, y_2, y_3) = c_4(y_2), \\ c_5(x_1, x_2, y_1, y_2, y_3) = c_5(x_1 y_2), \\ c_6(x_1, x_2, y_1, y_2, y_3) = c_6(x_2 y_2), \\ c_7(x_1, x_2, y_1, y_2, y_3) = c_7(y_3), \\ c_8(x_1, x_2, y_1, y_2, y_3) = c_8(x_1 y_3), \\ c_9(x_1, x_2, y_1, y_2, y_3) = c_9(x_2 y_3), \end{cases}$$

where x_i, y_j are in $]0, 1[$ with $\sum_{i=1}^2 x_i \leq 1$ and $\sum_{j=1}^3 y_j \leq 1$.

The simultaneous equations (2.4) are equivalent to

$$(2.6) \quad \begin{cases} d(x_1, x_2, y_1, y_2, y_3) = \theta_1(y_2) - b_1(y_1) = \theta_2(x_1 y_2) - b_2(x_1 y_1) \\ \quad = \theta_3(x_2 y_2) - b_3(x_2 y_2), \\ e(x_1, x_2, y_1, y_2, y_3) = \phi_1(y_3) - b_1(y_1) = \phi_2(x_1 y_3) - b_2(x_1 y_1) \\ \quad = \phi_3(x_2 y_3) - b_3(x_2 y_1), \end{cases}$$

where x_i, y_j are in $]0, 1[$ with $x_1 + x_2 \leq 1, y_1 + y_2 + y_3 \leq 1$.

We shall give the general solutions of equation (2.6) through the following lemma.

LEMMA 2.1. *The general solutions of the functional equation*

$$(2.7) \quad f(rs) - g(rt) = h(s) - k(t),$$

for all $r, s, t \in]0, 1[$ with $s + t \leq 1$, are given by

$$(2.8) \quad \begin{cases} f(x) = \psi(x) + A, \\ g(x) = \psi(x) + A + C, \\ h(x) = \psi(x) + B, \\ k(x) = \psi(x) + B + C, \end{cases}$$

for all $x \in]0, 1[$, where A, B, C are constants and $\psi:]0, \infty[\rightarrow \mathbf{R}$ (reals)

is a solution of the Cauchy equation,

$$(2.9) \quad \psi(rs) = \psi(r) + \psi(s) .$$

Proof. We rewrite equation (2.7) as

$$(2.10) \quad f(rs) - h(s) = g(rt) - k(t) ,$$

for all $r, s, t \in]0, 1[$ with $s + t \leq 1$. Thus $f(rs) - h(s)$ is a function of r only, say

$$(2.11) \quad f(rs) - h(s) = l(r) ,$$

for all $r, s \in]0, 1[$. Thus by [11, p. 59] there exists $\psi:]0, \infty[\rightarrow \mathbf{R}$ satisfying

$$(2.9) \quad \psi(rs) = \psi(r) + \psi(s) ,$$

for all $r, s \in]0, \infty[$ such that it represents f, h , and l through the equations

$$(2.12) \quad \begin{cases} f(x) = \psi(x) + A , \\ h(x) = \psi(x) + B , \\ l(x) = \psi(x) + A - B , \end{cases}$$

for all $x \in]0, 1[$, where A and B are arbitrary constants. Similarly g and k are given by

$$(2.13) \quad \begin{cases} g(x) = \psi(x) + A + C , \\ k(x) = \psi(x) + B + C , \end{cases}$$

for all $x \in]0, 1[$ and where C is an arbitrary constant. This completes the proof of Lemma 2.1.

Thus the general solution of the equations (2.6) is given by

$$(2.14) \quad \begin{cases} b_i(x) = \psi(x) + A_i , & i = 1, 2, 3 \\ \theta_i(x) = \psi(x) + A_i + B , & i = 1, 2, 3 \\ \phi_i(x) = \psi(x) + A_i + C , & i = 1, 2, 3 \end{cases}$$

for all $x \in]0, 1[$, where A_i, B, C are constants and ψ is a solution of the Cauchy equation (2.9).

Now we shall determine the function g and the ‘constants’ c_i ’s in equation (2.2). We prepare our result by the following lemma.

LEMMA 2.2. *Let $k_i:]0, 1[\rightarrow \mathbf{R}$, $i = 1, 2, 3$ be functions satisfying the functional equation*

$$(2.15) \quad k_1(r) + k_2(rs) + k_3(rt) = T(s, t)$$

for all $r, s, t \in]0, 1[$ with $s + t \leq 1$. Then, and only then, there exist functions $\psi, \phi:]0, \infty[\rightarrow \mathbf{R}$ which are solutions of (2.9) and constants A, B, C such that

$$(2.16) \quad \begin{cases} k_1(x) = -\psi(x) - \phi(x) + C, \\ k_2(x) = \psi(x) + A, \\ k_3(x) = \phi(x) + B. \end{cases}$$

Proof. As the right side of (2.15) is independent of r , we have

$$(2.17) \quad k_1(r) + k_2(rs) + k_3(rt) = k_1(r') + k_2(r's) + k_3(r't),$$

for all $r, r', s, t \in]0, 1[$ with $s + t \leq 1$. For arbitrary $s, s' \in]0, 1[$ we can choose $t \in]0, 1[$ such that $s + t, s' + t \leq 1$ and thus from (2.17) we get

$$(2.18) \quad k_2(rs) - k_2(r's) = k_2(rs') - k_2(r's'),$$

for all $r, r', s, s' \in]0, 1[$. We can now fix r' and s' arbitrarily and then equation (2.18) reduces to

$$(2.19) \quad k_2(rs) = l_1(r) + l_2(s),$$

for all $r, s \in]0, 1[$, (for some functions l_i), which is an equation similar to (2.11). Thus there exists a function $\psi:]0, \infty[\rightarrow \mathbf{R}$ satisfying (2.9) such that

$$k_2(x) = \psi(x) + A,$$

for all $x \in]0, 1[$, where A is a constant. Similarly there exists $\phi:]0, \infty[\rightarrow \mathbf{R}$ satisfying (2.9) such that

$$k_3(x) = \phi(x) + B,$$

for all $x \in]0, 1[$. If we replace k_2, k_3 by ψ, ϕ respectively in equation (2.17) while fixing r' we get k_1 as is in (2.16). This proves our lemma.

From equation (2.2), we see that g is a function of p and x_1 only, say

$$(2.20) \quad g(p, x_1, x_2, y_1, y_2, y_3) = g(p, x_1).$$

Now, from equation (2.2), we see that $-c_1(y_1) + c_2(x_1y_1) + c_3(x_2y_1)$ is independent of y_1 and therefore by Lemma 2.2 we have

$$(2.21) \quad \begin{cases} c_1(x) = \psi_1(x) + \phi_1(x) + D_1, \\ c_2(x) = \psi_1(x) + E_1, \\ c_3(x) = \phi_1(x) + F_1, \end{cases}$$

for all $x \in]0, 1[$, where ψ_1 and ϕ_1 are solutions of the equation (2.9) and D_1, E_1, F_1 are arbitrary constants. Similarly we have

$$(2.22) \quad \begin{cases} c_4(x) = \psi_2(x) + \phi_2(x) + D_2, \\ c_5(x) = \psi_2(x) + E_2, \\ c_6(x) = \varphi_2(x) + F_2, \\ c_7(x) = \psi_3(x) + \phi_3(x) + D_3, \\ c_8(x) = \psi_3(x) + E_3, \\ c_9(x) = \phi_3(x) + F_3, \end{cases}$$

where $\psi_2, \phi_2, \psi_3, \phi_3$ are solutions of (2.9) again. If we replace the c_i 's in the second equation of (2.2) by equations (2.20), (2.21), and (2.22) we see that $-g(1-p, x_1) - \psi(x_1)p + \psi(x_1) + \psi_1(x_1) + \psi_2(x_1) + \psi_3(x_1)$ is independent of x_1 , say

$$(2.23) \quad \begin{aligned} g(1-p, x_1) &= g(1-p) - \psi(x_1)p + \psi(x_1) \\ &\quad + \psi_1(x_1) + \psi_2(x_1) + \psi_3(x_1) \end{aligned}$$

for all $p \in [0, 1]$ and $x_1 \in]0, 1[$, where $g: [0, 1] \rightarrow \mathbf{R}$ is an arbitrary measurable function.

Combining equations (2.1), (2.2), (2.3), (2.4), (2.5), (2.14), (2.21), (2.22), and (2.23) we are ready to conclude the following theorem.

THEOREM 2.1. *Let $F_{ij}, G_i, H_j: [0, 1] \times]0, 1[\rightarrow \mathbf{R}$ ($i = 1, 2, j = 1, 2, 3$) be functions which are measurable in their first variables. Then these functions satisfy the functional equation (1.8) if and only if there exist $\psi, \psi_i, \phi_i:]0, \infty[\rightarrow \mathbf{R}$ all satisfy the Cauchy equation (2.9) such that*

$$(2.24) \quad \begin{cases} H_1(q, y) = aq \log q + [\psi(y) + A_1]q + \psi_1(y) + \phi_1(y) + D_1, \\ H_2(q, y) = aq \log q + [\psi(y) + A_1 + B]q + \psi_2(y) + \phi_2(y) + D_2, \\ H_3(q, y) = aq \log q + [\psi(y) + A_1 + c]q + \psi_3(y) + \phi_3(y) + D_3, \\ F_{1,1}(p, y) = ap \log p + [\psi(y) + A_2]p + \psi_1(y) + E_1, \\ F_{1,2}(p, y) = ap \log p + [\psi(y) + A_2 + B]p + \psi_2(y) + E_2, \\ F_{1,3}(p, y) = ap \log p + [\psi(y) + A_2 + c]p + \psi_3(y) + E_3, \\ F_{2,1}(p, y) = ap \log p + [\psi(y) + A_3]p + \phi_1(y) + F_1, \\ F_{2,2}(p, y) = ap \log p + [\psi(y) + A_3 + B]p + \phi_2(y) + F_2, \\ F_{2,3}(p, y) = ap \log p + [\psi(y) + A_3 + c]p + \phi_3(y) + F_3, \\ G_1(p, x) = g(p) + \psi(x)p + \psi_1(x) + \psi_2(x) + \psi_3(x), \\ G_2(p, x) = -g(1-p) + a[p \log p + (1-p) \log(1-p)] \\ \quad + [\psi(x) + A_3 - A_2]p + \phi_1(x) + \phi_2(x) + \phi_3(x) + A_2 \\ \quad - A_1 - D_1 - D_2 - D_3 + E_1 + E_2 + E_3 + F_1 + F_2 + F_3, \end{cases}$$

for all $p, q \in [0, 1]$, $x, y \in]0, 1[$, where $a, A_i, B, c, D_i, E_i, F_i, i = 1, 2, 3$, are all constants, and g is an arbitrary measurable function.

THEOREM 2.2. *If $F: [0, 1] \times]0, 1[\rightarrow \mathbf{R}$ is measurable in its first variable, then it satisfies the functional equation (1.6) for all $P \in \Delta_2$, $Q \in \Delta_3$, $X \in \Delta'_2$, $Y \in \Delta'_3$ if and only if F is of the form*

$$(2.25) \quad F(p, x) = ap \log p + [\psi(x) + A]p,$$

for all $p \in [0, 1]$, $x \in]0, 1[$, where ψ is a solution of the Cauchy equation (2.9) and a, A are constants.

3. On the measurable solutions of the functional equation (1.7). Let $F: [0, 1] \times]0, 1[\times]0, 1[\rightarrow \mathbf{R}$ be measurable in its first variable and satisfy the equation (1.7) for all $P \in \Delta_2$, $Q \in \Delta_3$, $X, U \in \Delta'_2$, $Y, V \in \Delta'_3$.

For each fixed u_i, v_j equation (1.7) reduces to the form (1.8). Thus by Theorem 2.1 there exist in particular $\psi, \psi_1, \psi_2, \phi_1, \phi_2$ satisfying the Cauchy equation (2.9) in their first variables and $A_1, A_2, A_3, a, B, D_1, D_2, E_1, E_2, F_1$ such that

$$(3.1) \quad \left\{ \begin{aligned} F(q, y, v_1) &= a(u_1, u_2, v_1, v_2, v_3)q \log q + [\psi(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + A_1(u_1, u_2, v_1, v_2, v_3)]q + (\psi_1 + \phi_1)(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + D_1(u_1, u_2, v_1, v_2, v_3), \\ F(q, y, v_2) &= a(u_1, u_2, v_1, v_2, v_3)q \log q + [\psi(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + (A_1 + B)(u_1, u_2, v_1, v_2, v_3)]q + (\psi_2 + \phi_2) \\ &\quad (y, u_1, u_2, v_1, v_2, v_3) + D_2(u_1, u_2, v_1, v_2, v_3), \\ F(q, y, u_1 v_1) &= a(u_1, u_2, v_1, v_2, v_3)q \log q + [\psi(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + A_2(u_1, u_2, v_1, v_2, v_3)]q + \psi_1(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + E_1(u_1, u_2, v_1, v_2, v_3), \\ F(q, y, u_1 v_2) &= a(u_1, u_2, v_1, v_2, v_3)q \log q + [\psi(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + (A_2 + B)(u_1, u_2, v_1, v_2, v_3)]q \\ &\quad + \psi_2(y, u_1, u_2, v_1, v_2, v_3) + E_2(u_1, u_2, v_1, v_2, v_3), \\ F(q, y, u_2 v_1) &= a(u_1, u_2, v_1, v_2, v_3)q \log q + [\psi(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + A_3(u_1, u_2, v_1, v_2, v_3)]q + \phi_1(y, u_1, u_2, v_1, v_2, v_3) \\ &\quad + F_1(u_1, u_2, v_1, v_2, v_3). \end{aligned} \right.$$

$$(3.2) \quad a(u_1, u_2, v_1, v_2, v_3) \equiv a \text{ constant} = a.$$

Hence it follows that

$$(3.3) \quad \begin{aligned} &\psi(y, u_1, u_2, v_1, v_2, v_3) + A_1(u_1, u_2, v_1, v_2, v_3) \\ &\equiv a \text{ function of } y \text{ and } v_1 \text{ only} = \theta(y, v_1), \end{aligned}$$

$$(3.4) \quad \psi(y, u_1, u_2, v_1, v_2, v_3) + A_1(u_1, u_2, v_1, v_2, v_3) + B(u_1, u_2, v_1, v_2, v_3) = \theta(y, v_2),$$

$$(3.5) \quad \psi(y, u_1, u_2, v_1, v_2, v_3) + A_2(u_1, u_2, v_1, v_2, v_3) = \theta(y, u_1 v_1),$$

$$(3.6) \quad \psi(y, u_1, u_2, v_1, v_2, v_3) + A_2(u_1, u_2, v_1, v_2, v_3) + B(u_1, u_2, v_1, v_2, v_3) = \theta(y, u_1 v_2).$$

From equations (3.3) to (3.6) we have

$$(3.7) \quad \theta(y, v_2) - \theta(y, v_1) = \theta(y, u_1 v_2) - \theta(y, u_1 v_1)$$

and

$$(3.8) \quad A_2(u_1, u_2, v_1, v_2, v_3) - A_1(u_1, u_2, v_1, v_2, v_3) = \theta(y, u_1 v_1) - \theta(y, v_1).$$

For (3.7), by Lemma 2.1 there exists, for each fixed y , a function $\theta_1(\cdot, y)$ satisfying the Cauchy equation (2.9) and a constant $\theta_2(y)$ such that, we have

$$(3.9) \quad \theta(y, v) = \theta_1(v, y) + \theta_2(y).$$

Now equations (3.8) and (3.9) yield

$$(3.10) \quad \theta_1(v, y) \equiv \text{a function of } v \text{ alone} = \theta_1(v).$$

Thus we can rewrite the first equation of (3.1) as

$$(3.11) \quad F(q, y, v_1) = aq \log q + [\theta_1(v_1) + \theta_2(y)]q + (\psi_1 + \phi_1)(y, u_1, u_2, v_1, v_2, v_3) + D_1(u_1, u_2, v_1, v_2, v_3).$$

From (3.11) we see that $(\psi_1 + \phi_1)(y, u_1, u_2, v_1, v_2, v_3) + D_1(u_1, u_2, v_1, v_2, v_3)$ depends on y and v_1 only. Since ψ_1, ϕ_1 satisfy the Cauchy equation (2.9), $(\psi_1 + \phi_1)(y, u_1, u_2, v_1, v_2, v_3)$ and $D_1(u_1, u_2, v_1, v_2, v_3)$ depend on (y, v_1) and v_1 only respectively. Thus we can write (3.11) in the form

$$(3.12) \quad F(q, y, v) = aq \log q + [\theta_1(v) + \theta_2(y)]q + \alpha_1(y, v) + \alpha_2(v),$$

where θ_1 and $\alpha_1(\cdot, v)$ satisfy the Cauchy equation (2.9).

From the first, third, and fifth equations of (3.1) and (3.12) we have

$$\alpha_1(y, v_1) = \alpha_1(y, u_1 v_1) + \alpha_1(y, u_2 v_1),$$

for all $u_1, u_2, v_1 \in]0, 1[$ with $u_1 + u_2 \leq 1$. Hence α_1 is independent of the second variable and we may write the equation (3.12) as

$$(3.13) \quad F(q, y, v) = aq \log q + [\theta_1(v) + \theta_2(y)]q + \alpha_1(y) + \alpha_2(v),$$

for all $q \in [0, 1]$, $y, v \in]0, 1[$ where θ_1 and α_1 are solutions of the Cauchy

equation (2.9). If we interchange the roles of the second and the third arguments of F in the above procedure we see that θ_2, α_2 are also solutions of the Cauchy equation (2.9).

Substituting (3.13) into (1.7), taking into account that θ_i, α_i are solutions of the Cauchy equation (2.9) we get $\alpha_i \equiv 0$. Thus we have proved the following theorem.

THEOREM 3.1. *Let $F: [0, 1] \times]0, 1[\times]0, 1[\rightarrow \mathbf{R}$ be measurable in its first variable. Then F satisfies the functional equation (1.7) if and only if F has the form*

$$(3.14) \quad F(q, y, v) = aq \log q + [\theta_1(v) + \theta_2(y)]q,$$

where $\theta_1, \theta_2:]0, \infty[\rightarrow \mathbf{R}$ satisfy the Cauchy equation (2.9).

COROLLARY 3.1. *Let $F: ([0, 1] \times]0, 1[\times]0, 1[) \cup \{(0, 0, [0, 1])\} \cup \{(1, 1, [0, 1])\} \cup \{(0, [0, 1], 0)\} \cup \{(1, [0, 1], 1)\} \rightarrow \mathbf{R}$ be measurable in its first variable. Then it satisfies the equation (1.7) if and only if F has the form given by (3.14) on $[0, 1] \times]0, 1[\times]0, 1[$ and on the boundary $F(0, 0, \cdot) \equiv 0$, $F(1, 1, \cdot) = \theta_1(\cdot)$, $F(0, \cdot, 0) \equiv 0$ and $F(1, \cdot, 1) = \theta_2(\cdot)$.*

REMARK. The measures H_n, I_n, D_n in (1.2), (1.3), (1.4) possess in particular properties: (a) *Symmetry*: H_n, I_n, D_n are symmetric in the pairs $(p_i, q_i), (p_i, q_i), (p_i, q_i, r_i)$ respectively, (b) *Expansibility*: If $P = (p_1, p_2, \dots, p_n)$, $Q = (q_1, q_2, \dots, q_n)$, $R = (r_1, r_2, \dots, r_n)$ and $P' = (p_1, p_2, \dots, p_n, 0)$, $Q' = (q_1, q_2, \dots, q_n, 0)$, $R' = (r_1, r_2, \dots, r_n, 0)$, then $H_n(P \parallel Q) = H_{n+1}(P' \parallel Q')$, $I_n(P \parallel Q) = I_{n+1}(P' \parallel Q')$ and $D_n(P \parallel Q \parallel R) = D_{n+1}(P' \parallel Q' \parallel R')$, (c) *Branching*: If $P = (p_1, p_2, \dots, p_n)$, $Q = (q_1, q_2, \dots, q_n)$, $R = (r_1, r_2, \dots, r_n)$ and $P' = (p_1 + p_2, p_3, \dots, p_n)$, $Q' = (q_1 + q_2, q_3, \dots, q_n)$ and $R' = (r_1 + r_2, r_3, \dots, r_n)$, then $H_n(P \parallel Q) - H_{n-1}(P' \parallel Q')$, $I_n(P \parallel Q) - I_{n-1}(P' \parallel Q')$ and $D_n(P \parallel Q \parallel R) - D_{n-1}(P' \parallel Q' \parallel R)$ depend on (p_1, p_2, q_1, q_2) , (p_1, p_2, q_1, q_2) and $(p_1, p_2, q_1, q_2, r_1, r_2)$ respectively. It is shown by C. T. Ng [14] that these three properties are equivalent to the representability of H_n, I_n, D_n in the form $H_n(P \parallel Q) = \sum_{i=1}^n f(p_i, q_i)$, $I_n(P \parallel Q) = \sum_{i=1}^n g(p_i, q_i)$ and $D_n(P \parallel Q \parallel R) = \sum_{i=1}^n h(p_i, q_i, r_i)$ where f, g, h are any function satisfying $f(0, 0) = g(0, 0) = h(0, 0, 0) = 0$. From these representations, the additivity property of these measures motivates the study of the functional equations (1.6) and (1.7).

The Theorems 2.2 and 3.1 lead to a characterization of directed divergence and inaccuracy and of generalized directed divergence respectively. These three measures are determined by (a) Symmetry, (b) Expansibility, (c) Branching, (d) Additivity, and (e) Regularity conditions such as Lebesgue measurability and appropriate initial conditions.

REFERENCES

1. J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966.
2. J. Aczél and Z. Daróczy, *Charakterisierung der Entropien positiver Ordnung und der Shannonschen Entropie*, Acta Math. Acad. Sci. Hungar., **14** (1963), 95-121.
3. J. Aczél and P. Nath, *Axiomatic characterization of some measures of divergence in information*, Z. Wahrscheinlichkeitstheorie verw. Geb., **21** (1972), 215-224.
4. T. W. Chaundy and J. B. McLeod, *On a functional equation*, Edinburgh Math. Notes, **43** (1962), 7-8.
5. Z. Daróczy, *On the measurable solutions of a functional equation*, Acta Math. Acad. Sci. Hungar., **22** (1971), 11-14.
6. Pl. Kannappan, *On Shannon's entropy, directed divergence and inaccuracy*, Z. Wahrscheinlichkeitstheorie verw. Geb., **22** (1972), 95-100.
7. ———, *On directed divergence and inaccuracy*, Z. Wahrscheinlichkeitstheorie verw. Geb., **25** (1972), 49-55.
8. ———, *On generalized directed divergence*, Funk. Ekvacioj, **16** (1973), 71-77.
9. ———, *On a functional equation connected with generalized directed divergence*, Aequationes Math., (to appear).
10. D. F. Kerridge, *Inaccuracy and inference*, J. Royal Statist. Soc. Ser. B, **23** (1961), 184-194.
11. S. Kullback, *Information Theory and Statistics*, J. Wiley & Sons, New York, 1959.
12. J. P. Mokanski, *Extensions of functions satisfying Cauchy and Pexider type equations*, Ph.D. Thesis, University of Waterloo, 1971.
13. C. T. Ng, *On the measurable solutions of the functional equation*

$$\sum_{i=1}^2 \sum_{j=1}^3 F_{i,j}(p_i q_j) = \sum_{i=1}^2 G_i(p_i) + \sum_{j=1}^3 H_j(q_j),$$
Acta Math. Acad. Sci. Hungar., **25** (1974).
14. ———, *Representation of measures of information with the branching property*, Information and Control, **25** (1974), 45-56.
15. A. Rényi, *On measures of entropy and information*, Proc. Fourth Berkeley Sympos. Math. Statistics, Probab., (1960), 547-561.
16. C. E. Shannon, *A mathematical theory of communication*, Bell System Tech. J., **27** (1948), 378-423 and 623-656.

Received June 15, 1973 and in revised form January 16, 1974. This research is partially supported by NRC of Canada grants.

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Ralph K Amayo, <i>Engel Lie rings with chain conditions</i>	1
Bernd Anger and Jörn Lembcke, <i>Hahn-Banach type theorems for hypolinear functionals on preordered topological vector spaces</i>	13
Gregory Frank Bachelis and Samuel Ebenstein, <i>On $\Lambda(p)$ sets</i>	35
Harvey Isaac Blau, <i>Indecomposable modules for direct products of finite groups</i>	39
Larry Eugene Bobisud and James Calvert, <i>Singular perturbation of a time-dependent Cauchy problem in a Hilbert space</i>	45
Walter D. Burgess and Robert Raphael, <i>Abian's order relation and orthogonal completions for reduced rings</i>	55
James Diederich, <i>Representation of superharmonic functions mean continuous at the boundary of the unit ball</i>	65
Aad Dijkma and Hendrik S. V. de Snoo, <i>Self-adjoint extensions of symmetric subspaces</i>	71
Gustave Adam Efroymson, <i>A Nullstellensatz for Nash rings</i>	101
John D. Elwin and Donald R. Short, <i>Branched immersions onto compact orientable surfaces</i>	113
John Douglas Faires, <i>Comparison of the states of closed linear transformations</i>	123
Joe Wayne Fisher and Robert L. Snider, <i>On the von Neumann regularity of rings with regular prime factor rings</i>	135
Franklin Takashi Iha, <i>A unified approach to boundary value problems on compact intervals</i>	145
Palaniappan L. Kannappan and Che Tat Ng, <i>On functional equations connected with directed divergence, inaccuracy and generalized directed divergence</i>	157
Samir A. Khabbaz and Elias Hanna Toubassi, <i>The module structure of $\text{Ext}(F, T)$ over the endomorphism ring of T</i>	169
Garo K. Kiremidjian, <i>On deformations of complex compact manifolds with boundary</i>	177
Dimitri Koutroufiotis, <i>Mappings by parallel normals preserving principal directions</i>	191
W. K. Nicholson, <i>Semiperfect rings with abelian adjoint group</i>	201
Norman R. Reilly, <i>Extension of congruences and homomorphisms to translational hulls</i>	209
Sadahiro Saeki, <i>Symmetric maximal ideals in $M(G)$</i>	229
Brian Kirkwood Schmidt, <i>On the homotopy invariance of certain functors</i>	245
H. J. Shyr and T. M. Viswanathan, <i>On the radicals of lattice-ordered rings</i>	257
Indranand Sinha, <i>Certain representations of infinite group algebras</i>	261
David Smallen, <i>The group of self-equivalences of certain complexes</i>	269
Kalathoor Varadarajan, <i>On a certain problem of realization in homotopy theory</i>	277
James Edward West, <i>Sums of Hilbert cube factors</i>	293
Chi Song Wong, <i>Fixed points and characterizations of certain maps</i>	305