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METRIZABILITY OF TOPOLOGICAL SPACES

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This paper is a study of conditions under which a topological space is metrizable or has a countable base. In § 2 we consider the metrizability of spaces having a weak base in the sense of Arhangel'skiĭ. In § 3 we extend earlier work of Bennett on quasi-developments by showing that every regular θ -refinable β -space with a quasi- G_δ -diagonal is semi-stratifiable. One consequence of this result is a generalization of the Borges-Okuyama theorem on the metrizability of a paracompact wA -space with a G_δ -diagonal. In § 4 we prove that a regular space has a countable base if it is hereditarily a CCC wA -space with a point-countable separating open cover. This result is motivated by the remarkable theorem of Arhangel'skiĭ which states that a regular space has a countable base if it is hereditarily a Lindelöf p -space. In § 5 we show that every regular p -space which a Baire space has a dense subset which is a paracompact p -space. This result, related to work of Šapirovskiĭ, is then used to obtain conditions under which a Baire space satisfying the CCC is separable or has a countable base. In § 6 we prove that every locally connected, locally peripherally separable meta-Lindelöf Moore space is metrizable. Finally, in § 7 we consider the metrizability of spaces which are the union of countably many metrizable subsets. The results obtained in this section extend earlier work of Čoban, Corson-Michael, Smirnov, and Stone.

1. Preliminaries. We begin with some definitions and known results which are used throughout this paper. Unless otherwise stated, no separation axioms are assumed; however, regular, normal, and collectionwise normal spaces are always T_1 and paracompact spaces are always Hausdorff. The set of natural numbers is denoted by N , and i, j, k, m, n, r, s , and t denote elements of N .

Let X be a set, let \mathcal{G} be a collection of subsets of X , let p be an element of X . The *star of p with respect to \mathcal{G}* , denoted $st(p, \mathcal{G})$, is the union of all elements of \mathcal{G} containing p . The *order of p with respect to \mathcal{G}* , denoted $ord(p, \mathcal{G})$, is the number of elements of \mathcal{G} containing p . The union of all elements of \mathcal{G} is denoted by \mathcal{G}^* . If \mathcal{G} covers X , then \mathcal{G} is said to be *separating* [43] if given any two distinct points p and q in X , there is some G in \mathcal{G} such that $p \in G$, $q \notin G$.

A topological space X is said to be *developable* if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open covers of X such that, for each p in X , $\{st(p, \mathcal{G}_n): n = 1, 2, \dots\}$ is a fundamental system of neighborhoods of

p . Such a sequence of open covers is called a *development* for X . A regular developable space is called a *Moore space*. Bing [12] proved that every paracompact Moore space is metrizable.

A *quasi-development* for a topological space X is a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open collections in X such that, given any point p in X and any neighborhood R of p , there is some n in N such that $\text{st}(p, \mathcal{G}_n) \neq \emptyset$ and $\text{st}(p, \mathcal{G}_n) \subseteq R$. Clearly every development is a quasi-development, and in [10] Bennett proved that a space is developable if it has a quasi-development and every closed subset is a G_δ .

The notion of a G_δ -diagonal plays an important role in metrization theory; see, for example, [13], [14], [18], [51]. In [18] Ceder proved that a space X has a G_δ -diagonal if and only if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open covers of X such that, given any two distinct points p and q in X , there is some n in N such that $q \notin \text{st}(p, \mathcal{G}_n)$. Such a sequence of open covers is called a *G_δ -diagonal sequence* for X .

A collection \mathcal{G} of subsets of a topological space X is called a *net* [6] if given any point p in X and any neighborhood R of p , there is some G in \mathcal{G} such that $p \in G \subseteq R$. A space with a σ -locally finite net is called a *σ -space* [50].

A topological space X is *θ -refinable* [64] if for each open cover \mathcal{U} of X , there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open refinements such that, for each p in X , there is some n in N such that $\text{ord}(p, \mathcal{G}_n)$ is finite. Such a sequence of open covers is called a *θ -refinement* of \mathcal{U} .

A topological space X is a *$w\mathcal{A}$ -space* [14] if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open covers of X such that, for each p in X , if $x_n \in \text{st}(p, \mathcal{G}_n)$ for $n = 1, 2, \dots$, then the sequence $\langle x_n \rangle$ has a cluster point. Clearly every countably compact space and every developable space is a $w\mathcal{A}$ -space. In [31] it is proved that every regular $w\mathcal{A}$ -space with a σ -point finite separating open cover is developable.

The p -spaces of Arhangel'skii [4], which generalize Čech complete spaces and completely regular developable spaces, are defined in terms of the Stone-Čech compactification. In this paper we adopt as the definition of a p -space an internal characterization given by Burke in [16]. A *plumbing* for a topological space X is a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open covers of X such that, for each p in X , if $p \in G_n \in \mathcal{G}_n$, $n = 1, 2, \dots$, then (a) $C(p) = \bigcap_{n=1}^{\infty} \bar{G}_n$ is compact; (b) $\{\bigcap_{i=1}^n \bar{G}_i : n = 1, 2, \dots\}$ is a "base" for $C(p)$ in the sense that given any open set R with $C(p) \subseteq R$, there is some n in N such that $\bigcap_{i=1}^n \bar{G}_i \subseteq R$. A topological space is a *p -space* if it has a plumbing.

Quasi-complete spaces were introduced by Creede in [22] as a generalization of p -spaces and $w\mathcal{A}$ -spaces. In this paper we adopt as the definition of quasi-completeness a characterization due independently to Sohn-Choi [61] and Gittings [25]. A topological space X is *quasi-*

complete if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open covers of X such that the following holds for each p in X : if $p \in G_n \in \mathcal{G}_n$, $n = 1, 2, \dots$, and $\langle x_n \rangle$ is a sequence with $x_n \in \bigcap_{i=1}^n G_i$ for all n , then $\langle x_n \rangle$ has a cluster point. Such a sequence of open covers is called a *quasi-complete sequence* for X . In [16] Burke proved that $w\Delta$ -spaces and p -spaces coincide for the class of completely regular θ -refinable spaces. Recently Gittings [25] extended this result by showing that p -spaces (as defined above), $w\Delta$ -spaces, and quasi-complete spaces coincide for the class of regular θ -refinable spaces.

Recall that a topological space is called a *Baire space* if any countable intersection of open dense sets is dense, and is \aleph_1 -*compact* if every uncountable subset has a limit point. Also, a topological space is a *CCC space*, or *satisfies the CCC* (CCC = countable chain condition), if every mutually disjoint collection of nonempty open sets is countable. It is easy to check that a topological space is hereditarily \aleph_1 -compact if and only if it hereditarily satisfies the CCC.

Let (X, \mathcal{T}) be a topological space, let g be a function from $N \times X$ into \mathcal{T} such that $p \in \bigcap_{n=1}^{\infty} g(n, p)$ for each p in X . Consider the following conditions on g .

(A) If $g(n, p) \cap g(n, x_n) \neq \phi$ for all n , then p is a cluster point of $\langle x_n \rangle$.

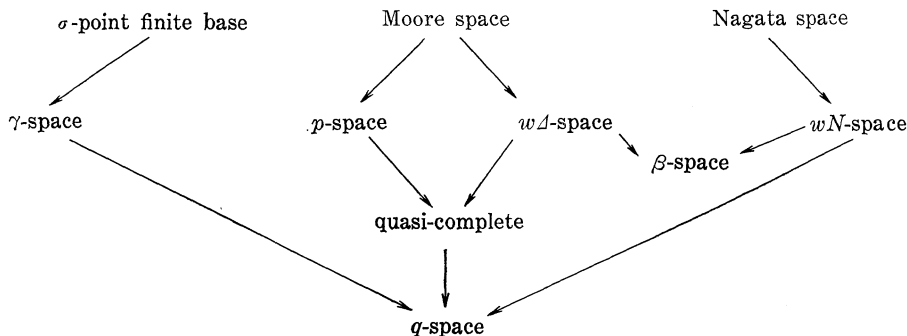
(B) If $g(n, p) \cap g(n, x_n) \neq \phi$ for all n , then $\langle x_n \rangle$ has a cluster point.

(C) If $y_n \in g(n, p)$ and $x_n \in g(n, y_n)$ for all n , then p is a cluster point of $\langle x_n \rangle$.

(D) If $x_n \in g(n, p)$ for all n , then $\langle x_n \rangle$ has a cluster point.

(E) If $p \in g(n, x_n)$ for all n , then $\langle x_n \rangle$ has a cluster point.

In [29] Heath characterized Nagata spaces in terms of a function g satisfying (A). (The T_0 separation axiom is assumed.) A function g satisfying (A) is called a *Nagata function*. A topological space X is, respectively, a wN -space, a γ -space, a q -space, a β -space if it has a function g satisfying, respectively, (B), (C), (D), (E). A function g satisfying (C) is called a γ -function, and one satisfying (E) is called



a β -function. See [31], [32], [41]. The idea of characterizing generalized metrizable spaces in the above fashion was first used by Heath in [30].

The relationship between some of the classes of spaces defined in this section is summarized in the preceding diagram.

2. Weak bases. The following natural technique for defining a topology on a set is due to Arhangel'skiĭ [5]. (Also, see [56].) Let X be a set, and suppose that for each p in X there is assigned a collection \mathcal{B}_p of subsets of X which is closed under finite intersections such that each element of \mathcal{B}_p contains p . Let \mathcal{T} be all subsets V of X satisfying this condition: for each p in V , there is some B_p in \mathcal{B}_p such that $B_p \subseteq V$. As is easily seen, the collection \mathcal{T} obtained in this fashion is a topology on X . This construction suggests the following definitions.

Let X be a topological space, and for each p in X let \mathcal{B}_p be a collection of subsets of X which is closed under finite intersections such that each element of \mathcal{B}_p contains p . The collection $\mathcal{B} = \{\mathcal{B}_p: p \text{ in } X\}$ is called a *weak base* for X [5] if the following condition holds: a subset V of X is open if and only if for each p in V , there is some B_p in \mathcal{B}_p such that $B_p \subseteq V$. Now let $\mathcal{G}_1, \mathcal{G}_2, \dots$ be a sequence of (not necessarily open) covers of a topological space X such that \mathcal{G}_{n+1} refines \mathcal{G}_n , $n = 1, 2, \dots$. Such a sequence of covers is said to be *semi-refined* [19] if $\{\mathcal{B}_p: p \text{ in } X\}$ is a weak base for X , where $\mathcal{B}_p = \{\text{st}(p, \mathcal{G}_n): n = 1, 2, \dots\}$. The two theorems in this section give conditions under which a weak base and a semi-refined sequence of covers yield metrizability. The first result is closely related to the general metrization theorem of Nagata [48], while the second result generalizes the metrization theorem of Alexandroff and Urysohn [1].

THEOREM 2.1. *A T_0 space (X, \mathcal{T}) is metrizable if and only if for each p in X there is a collection $\mathcal{B}_p = \{B(n, p): n = 1, 2, \dots\}$ of subsets of X satisfying these conditions:*

- (1) $p \in B(n, p)$ and $B(n+1, p) \subseteq B(n, p)$;
- (2) $q \in B(n+1, p)$ implies $B(n+1, q) \subseteq B(n, p)$;
- (3) $q \notin B(n, p)$ implies $B(n+1, p) \cap B(n+1, q) = \emptyset$;
- (4) $\{\mathcal{B}_p: p \text{ in } X\}$ is a weak base for X .

Proof. If X is metrizable, the existence of the collections is clear. Assume, then, that for each p in X there is a collection $\mathcal{B}_p = \{B(n, p): n = 1, 2, \dots\}$ of subsets of X satisfying (1)–(4). To prove X metrizable, it suffices to show that there is a function $g: N \times X \rightarrow \mathcal{T}$ (with $p \in \bigcap_{n=1}^{\infty} g(n, p)$ for all p) which is a γ -function and a Nagata function. (See [32].) For p in X , n in N let $g(n, p) = B(n, p)^0$. First

let us show that $p \in g(n, p)$. Let

$$V = \{x \text{ in } X: B(j_x, x) \subseteq B(n, p) \text{ for some positive integer } j_x\}.$$

It is clear that $p \in V \subseteq B(n, p)$, and so it remains to show that V is open. Suppose it is not. Then by (4), there is some q in V such that $B(k, q) \not\subseteq V$, $k = 1, 2, \dots$. Now $q \in V$, so $B(j, q) \subseteq B(n, p)$ for some j . But $B(j+1, q) \not\subseteq V$, so there exists $x \in B(j+1, q)$ such that $x \notin V$. Since $x \notin V$, $B(j+1, x) \not\subseteq B(n, p)$, so there exists $y \in B(j+1, x)$ such that $y \notin B(n, p)$. Now $x \in B(j+1, q)$, so by (2), $B(j+1, x) \subseteq B(j, q)$. Hence $y \in B(n, p)$, a contradiction. Thus V is an open set.

To see that g is a Nagata function, let $g(n, p) \cap g(n, x_n) \neq \phi$, $n = 1, 2, \dots$, and let us show that $x_n \rightarrow p$. Let V be an open neighborhood of p . By (4), $B(n_0, p) \subseteq V$ for some n_0 . Then for $n \geq n_0 + 1$, $x_n \in V$. Indeed, since $B(n, p) \cap B(n, x_n) \neq \phi$, it follows from (3) that $x_n \in B(n-1, p)$, and hence $x_n \in V$.

To see that g is a γ -function, let $y_n \in g(n, p)$, $x_n \in g(n, y_n)$, and let us show that $x_n \rightarrow p$. Let V be an open neighborhood of p . By (4), $B(n_0, p) \subseteq V$ for some n_0 . Then for $n \geq n_0 + 1$, $x_n \in V$. Indeed, since $y_n \in B(n, p)$, it follows from (2) that $B(n, y_n) \subseteq B(n-1, p)$. Since $x_n \in B(n, y_n)$ and $B(n-1, p) \subseteq B(n_0, p) \subseteq V$, it follows that $x_n \in V$.

THEOREM 2.2. *A T_0 space X is metrizable if and only if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of covers of X satisfying these conditions.*

(1) *The sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ is semi-refined;*

(2) *for all n , if G_1 and G_2 are elements of \mathcal{G}_{n+1} such that $G_1 \cap G_2 \neq \phi$, then $G_1 \cup G_2 \subseteq G$ for some G in \mathcal{G}_n .*

Proof. If X is metrizable, the existence of the covers is clear. Assume, then, that $\mathcal{G}_1, \mathcal{G}_2, \dots$ is a sequence of covers satisfying (1) and (2). For p in X , n in N let $B(n, p) = \text{st}(p, \mathcal{G}_n)$, and let $\mathcal{B}_p = \{B(n, p): n = 1, 2, \dots\}$. It is easy to check that the collections $\{\mathcal{B}_p: p \text{ in } X\}$ satisfy the conditions of Theorem 2.1, and hence X is metrizable.

3. Spaces with a quasi- G_δ -diagonal. A paracompact $w\mathcal{A}$ -space is metrizable if it has any one of the following: (1) a G_δ -diagonal (Borges [13], Okuyama [51]); (2) a point-countable base (Filippov [24]); (3) a point-countable separating open cover (Nagata [47]); (4) a quasi-development (Bennett [9]). The notion of a point-countable separating open cover was introduced by Nagata to achieve a unification of (1) and (2). The problem of obtaining a similar unification of (1) and (4) suggests the following property. A *quasi- G_δ -diagonal* for a topological space X is a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open collections in X such that,

given any two distinct points p and q in X , there is some n such that $\text{st}(p, \mathcal{G}_n) \neq \phi$ and $q \notin \text{st}(p, \mathcal{G}_n)$.

Recently Bennett and Berney [11] proved that every regular θ -refinable β -space with a quasi-development is developable. This result yields the following generalization of (4): every paracompact β -space with a quasi-development is metrizable.

In this section we modify Bennett's proof of (4) to obtain the result that every regular θ -refinable β -space with a quasi- G_δ -diagonal is semi-stratifiable. Consequences of this result include a unification of (1) and (4) and the above result of Bennett and Berney on developable spaces. As in Bennett's proof of (4), use is made of the following set-theoretical lemma. (See Theorem 114 on p. 47 of [45].)

LEMMA 3.1. *Let Q be a property, let $\mathcal{H}_1, \mathcal{H}_2, \dots$ be finite non-empty sets, and assume that for all n , each element of \mathcal{H}_{n+1} has property Q with respect to some element of \mathcal{H}_n . Then there is a sequence H_1, H_2, \dots such that for all n , $H_n \in \mathcal{H}_n$ and H_{n+1} has property Q with respect to H_n .*

THEOREM 3.2. *Every regular θ -refinable β -space with a quasi- G_δ -diagonal is semi-stratifiable.*

Proof. Let (X, \mathcal{T}) be a regular θ -refinable space, let $l: N \times X \rightarrow \mathcal{T}$ be a β -function for X (assume that $l(n+1, p) \subseteq l(n, p)$, $n = 1, 2, \dots$, and all p in X), and let $\mathcal{G}_1, \mathcal{G}_2, \dots$ be a quasi- G_δ -diagonal for X . Assume, for a moment, that every closed subset of X is a G_δ . It easily follows that X has a G_δ -diagonal. Indeed, for $n = 1, 2, \dots$ let $\mathcal{G}_n^* = \bigcup_{k=1}^\infty F_{nk}$, where each F_{nk} is a closed set, and let $\mathcal{H}_{nk} = \mathcal{G}_n \cup \{X - F_{nk}\}$. Then, as is easy to check, $\{\mathcal{H}_{nk}: n, k \text{ in } N\}$ is a G_δ -diagonal sequence for X . Since every regular θ -refinable β -space with a G_δ -diagonal is semi-stratifiable (see [31]), the proof is complete.

It remains to show that every closed set is a G_δ . So let M be a closed subset of X , and assume that M has no isolated points. (Let I be the isolated points of M . If $M - I$ is a G_δ , it easily follows that M itself is a G_δ .) For each x in M , there is a strictly increasing sequence of positive integers $m(1, x) < m(2, x) < \dots < m(i, x) < \dots$ such that for all n , $x \in \mathcal{G}_n^*$ if and only if $n = m(i, x)$ for some $i \leq n$. For $i = 1, 2, \dots$ let $G(i, x)$ be some element of $\mathcal{G}_{m(i, x)}$ containing x .

For each x in M , let $H(1, x) = l(1, x) \cap G(1, x)$. Then $\{H(1, x): x \in M\}$, together with $(X - M)$, covers X . Let $\{\mathcal{W}_{1,k}: k = 1, 2, \dots\}$ be a θ -refinement of this cover. Note that if W is any element of $\mathcal{W}_{1,k}$ such that $W \cap M \neq \phi$, then there is some x in M such that $W \subseteq H(1, x)$.

Next, for each x in M , let $V(2, x)$ be an open neighborhood of x

such that $\overline{V(2, x)}$ is contained in some element of $\mathcal{W}_{1,1}$. Let $H(2, x) = V(2, x) \cap l(2, x) \cap G(1, x) \cap G(2, x)$. Then $\{H(2, x): x \text{ in } M\}$, together with $(X - M)$, covers X . Let $\{\mathcal{W}_{2,k}: k = 1, 2, \dots\}$ be a θ -refinement of this cover.

Continue this process. Then for each n we obtain a sequence $\mathcal{W}_{n,1}, \mathcal{W}_{n,2}, \dots$ of open covers of X , and a collection $\{H(n, x): x \in M\}$ of open sets, such that the following hold for all n :

- (1) if $W \in \mathcal{W}_{n,k}$ and $W \cap M \neq \phi$, then $W \subseteq H(n, x)$ for some x in M ;
- (2) for all x in M , $x \in H(n, x) \subseteq l(n, x) \cap (\bigcap_{i=1}^n G(i, x))$;
- (3) for all x in M and all $j \leq n$, $k \leq n$, $\overline{H(n+1, x)}$ is contained in some element of $\mathcal{W}_{j,k}$;
- (4) for all x in X , there is some k such that $\text{ord}(x, \mathcal{W}_{n,k})$ is finite.

Now let $V_{n,k} = \bigcup \{W \in \mathcal{W}_{n,k}: W \cap M \neq \phi\}$. Clearly each $V_{n,k}$ is open and $M \subseteq \bigcap_{n,k=1}^{\infty} V_{n,k}$. Let $p \in \bigcap_{n,k=1}^{\infty} V_{n,k}$, but suppose $p \notin M$. By (4), there is a positive integer $k(1)$ such that $\text{ord}(p, \mathcal{W}_{1,k(1)})$ is finite. Let $\{W_1, \dots, W_{t_1}\}$ be all elements of $\mathcal{W}_{1,k(1)}$ which contain p and intersect M . Such elements of $\mathcal{W}_{1,k(1)}$ exist since $p \in V_{1,k(1)}$. By (1), for $r = 1, \dots, t_1$, there is some x_r in M such that $W_r \subseteq H(1, x_r)$. Let $T_1 = \{x_1, \dots, x_{t_1}\}$ and let $j(1) = 1$.

Now let $j(2) = j(1) + k(1) + 1$, and by (4) pick a positive integer $k(2)$ such that $\text{ord}(p, \mathcal{W}_{j(2),k(2)})$ is finite. Let $\{W'_1, \dots, W'_{t_2}\}$ be all elements of $\mathcal{W}_{j(2),k(2)}$ which contain p and intersect M . Such sets exist since $p \in V_{j(2),k(2)}$. By (1), for $r = 1, \dots, t_2$, there is some x'_r in M such that $W'_r \subseteq H(j(2), x'_r)$. By (3), for $r = 1, \dots, t_2$, $\overline{H(j(2), x'_r)}$ is contained in some element of $\mathcal{W}_{1,k(1)}$. It easily follows that $\overline{H(j(2), x'_r)} \subseteq H(1, x_s)$ for some x_s in T_1 . Let $T_2 = \{x'_r: 1 \leq r \leq t_2\}$.

Continuing this process, we obtain a strictly increasing sequence of positive integers $1 = j(1) < j(2) < \dots < j(n) < \dots$, and for each n a nonempty finite subset T_n of M , such that the following hold for all n :

- (5) if $x \in T_n$, then $p \in H(j(n), x)$;
- (6) if $x \in T_{n+1}$, then $\overline{H(j(n+1), x)} \subseteq H(j(n), y)$ for some y in T_n .

For $n = 1, 2, \dots$ let $\mathcal{H}_n = \{H(j(n), x): x \text{ in } T_n\}$. Then by Lemma 3.1, there is a sequence $\langle x_n \rangle$ in M such that $p \in H(j(n), x_n)$ and $\overline{H(j(n+1), x_{n+1})} \subseteq H(j(n), x_n)$ for all n . Now by (2), $H(j(n), x_n) \subseteq l(j(n), x_n) \subseteq l(n, x_n)$, so $p \in l(n, x_n)$, $n = 1, 2, \dots$. Since l is a β -function, $\langle x_n \rangle$ has a cluster point, say q . Clearly $q \in M$, and it is easy to check that $q \in H(j(n), x_n)$, $n = 1, 2, \dots$. (Recall that $x_n \in H(j(n), x_n)$ by (2), and that $\overline{H(j(n+1), x_{n+1})} \subseteq H(j(n), x_n)$.) Now $p \notin M$, so $p \neq q$. Hence there is some n_0 such that $\text{st}(q, \mathcal{S}_{n_0}) \neq \phi$ and $p \notin \text{st}(q, \mathcal{S}_{n_0})$. Choose $n \geq n_0$ such that $x_n \in \text{st}(q, \mathcal{S}_{n_0})$. Since $x_n \in \mathcal{S}_{n_0}^*$, there is some

$i_0 \leq n_0$ such that $n_0 = m(i_0, x_n)$. Note that $G(i_0, x_n) \in \mathcal{G}_{m(i_0, x_n)} = \mathcal{G}_{n_0}$ and that $i_0 \leq n$. Now $\{p, q\} \subseteq H(j(n), x_n) \subseteq \bigcap_{i=1}^{j(n)} G(i, x_n)$, so both p and q belong to $G(i_0, x_n)$. Thus $p \in \text{st}(q, \mathcal{G}_{n_0})$, a contradiction. Hence $p \in M$, and so the proof that M is a G_δ is complete.

COROLLARY 3.3. *Every regular θ -refinable Σ^* -space with a quasi- G_δ -diagonal is a σ -space.*

Proof. Let X be a regular θ -refinable Σ^* -space with a quasi- G_δ -diagonal. Now every Σ^* -space is a β -space (see [34]), and so X is semi-stratifiable by Theorem 3.2. Since every regular semi-stratifiable Σ^* -space is a σ -space (see [57]), the proof is complete.

COROLLARY 3.4. (Bennett and Berney [11].) *Every regular θ -refinable β -space with a quasi-development is developable.*

Proof. By Theorem 3.2, such a space is semi-stratifiable. Since every semi-stratifiable space with a quasi-development is developable (see [10]), the proof is complete.

COROLLARY 3.5. *Every regular θ -refinable $w\Delta$ -space with a quasi- G_δ -diagonal is developable.*

Proof. By Theorem 3.2, such a space is semi-stratifiable. Since every regular semi-stratifiable $w\Delta$ -space is developable (see [22]), the proof is complete.

COROLLARY 3.6. *Every paracompact $w\Delta$ -space with a quasi- G_δ -diagonal is metrizable.*

Proof. By Corollary 3.5, such a space is developable. Since every paracompact developable space is metrizable [12], the proof is complete.

COROLLARY 3.7. *Every paracompact $w\Delta$ -space with a θ -separating cover is metrizable.*

Proof. Let X be a paracompact $w\Delta$ -space, let $\mathcal{G}_1, \mathcal{G}_2, \dots$ be a θ -separating cover of X . (Recall that each \mathcal{G}_n is an open collection and that given any two distinct points p and q in X , there is some n such that $\text{ord}(p, \mathcal{G}_n)$ is finite and there is some G in \mathcal{G}_n such that $p \in G$, $q \notin G$. See [31].) For each pair of positive integers n and k let $\mathcal{H}_{nk} = \{H: H \neq \phi, H = \bigcap_{i=1}^k G_i, G_1, \dots, G_k \text{ distinct elements of } \mathcal{G}_n\}$. Then $\{\mathcal{H}_{nk}: n, k \text{ in } N\}$ is a quasi- G_δ -diagonal for X , so by Corollary 3.6 X is metrizable.

COROLLARY 3.8. (Bennett and Berney [11].) *Every paracompact β -space with a quasi-development is metrizable.*

4. Hereditarily CCC spaces. In [3] Arhangel'skiĭ proved that every regular space which is hereditarily a Lindelöf p -space has a countable base. In [33] the author obtained an analogue of Arhangel'skiĭ's theorem by showing that every regular space which is hereditarily a CCC strong Σ -space has a countable net. These two results suggest numerous questions concerning the metrizability of hereditarily CCC spaces. The following problem seems particularly interesting.

Problem 4.1. Does every regular space which is hereditarily a CCC $w\Delta$ -space have a countable base?

In this section we show that the answer to 4.1 is "yes" under the additional assumption that the space has a point-countable separating open cover. We begin with a lemma due to Arhangel'skiĭ [3]. Since it is not clear in Arhangel'skiĭ's paper that this lemma is independent of the continuum hypothesis, we sketch a proof here. The main idea of the proof, of course, is Arhangel'skiĭ's.

LEMMA 4.2. (Arhangel'skiĭ) *Let X be a regular CCC space which is hereditarily a q -space. Then X is first countable.*

Proof. Let p be a point of X , and let us show that p has a countable fundamental system of neighborhoods. Let \mathcal{V} be a maximal mutually disjoint collection of nonempty open sets in X such that $p \notin \bar{V}$ for all V in \mathcal{V} . Since X satisfies the CCC, \mathcal{V} is countable, say $\mathcal{V} = \{V_n : n = 1, 2, \dots\}$. Let $Z = \{p\} \cup (\bigcup_{n=1}^{\infty} V_n)$. By the maximality of \mathcal{V} , Z is a dense subset of X . Now Z is a regular q -space, and p is a G_δ in $Z(\{p\} = \bigcap_{n=1}^{\infty} (Z - \bar{V}_n))$, so by a result due to Lutzer [40] p has a countable fundamental system of neighborhoods in Z . Since Z is dense in X , it follows (see p. 10 of [39]) that p has a countable fundamental system of neighborhoods in X .

THEOREM 4.3. *A regular space has a countable base if and only if it has a point-countable separating open cover and is hereditarily a CCC $w\Delta$ -space.*

Proof. Let X be a regular space. If X has a countable base, the result is clear. Assume, then, that X has a point-countable separating open cover \mathcal{S} and is hereditarily a CCC $w\Delta$ -space. First we show that X is separable. (The technique used here is due to Ponomarev [53].) Suppose X is not separable. Then there is a subset

$Y = \{x_\sigma : 0 \leq \sigma < \omega_1\}$ of X such that $x_\sigma \notin \{x_\tau : \tau < \sigma\}^-$ for all $\sigma < \omega_1$. Now by Lemma 4.2, X is first countable, and so for each $\sigma < \omega_1$ there is a countable fundamental system of open neighborhoods $\{V(\sigma, n) : n = 1, 2, \dots\}$ of x_σ such that for all n , $V(\sigma, n) \cap \{x_\tau : \tau < \sigma\} = \emptyset$. Let $\mathcal{V} = \{V(\sigma, n) \cap Y : 0 \leq \sigma < \omega_1, n = 1, 2, \dots\}$, and note that \mathcal{V} is a point-countable base for Y . Since Y is \aleph_1 -compact, it follows from a result of Aquaro [2] that Y is Lindelöf. (Also, see Lemma 3.6 of [33].) Thus Y is a Lindelöf $w\Delta$ -space with a point-countable base and hence has a countable base [53]. This contradicts the fact that Y is not separable.

Since X is separable, the point-countable separating open cover \mathcal{S} is actually countable. Now every regular $w\Delta$ -space with a σ -point finite separating open cover is developable [31]. Thus X is a developable \aleph_1 -compact space and hence has a countable base [38].

REMARK 4.4. The above argument can be modified to prove the following result. *Every regular space which is hereditarily a quasi-complete CCC space is hereditarily separable.*

REMARK 4.5. Gittings [25] has recently proved that every regular quasi-complete space with a G_δ -diagonal is a p -space. His argument also shows that such a space is hereditarily a p -space. By Remark 4.4, then, we obtain this result. *Every regular quasi-complete space with a G_δ -diagonal which is hereditarily a CCC space is hereditarily separable.* It should be noted that Burke [17] has constructed a regular p -space with a G_δ -diagonal (hence a hereditarily p -space) which is not developable.

Problem 4.6. Does every regular $w\Delta$ -space with a G_δ -diagonal which hereditarily satisfies the CCC have a countable base?

5. Baire spaces satisfying the CCC. In [55] Šapirovskii showed that every Čech complete space has a dense subspace which is a paracompact Čech complete space. Using this result he then proved that every Čech complete space with a point-countable base which satisfies the CCC has a countable base. In this section we obtain similar results.

LEMMA 5.1. *Let X be a regular space, let $\mathcal{G}_1, \mathcal{G}_2, \dots$ be a plumbing for X such that each \mathcal{G}_n is a locally finite collection. Then X is paracompact.*

Proof. Let \mathcal{R} be an open cover of X , let \mathcal{R}' be all finite unions

of elements of \mathcal{R} . It suffices to show that \mathcal{R}' has a σ -locally finite open refinement [42]. For $n = 1, 2, \dots$ let $\mathcal{H}_n = \bigwedge_{i=1}^n \mathcal{G}_i$, and note that \mathcal{H}_n is a locally finite open cover of X . For $n = 1, 2, \dots$ let

$$\mathcal{L}_n = \{H \text{ in } \mathcal{H}_n : H \text{ is contained in some element of } \mathcal{R}'\}.$$

To complete the proof it suffices to show that $\bigcup_{n=1}^{\infty} \mathcal{L}_n$ covers X . Let $p \in X$, and for $n = 1, 2, \dots$ let $p \in G_n \in \mathcal{G}_n$. Now $C(p) = \bigcap_{n=1}^{\infty} \bar{G}_n$ is compact, so there is some R in \mathcal{R}' such that $C(p) \subseteq R$. Choose n in N such that $\bigcap_{i=1}^n \bar{G}_i \subseteq R$. Then $H = \bigcap_{i=1}^n G_i$ is an element of \mathcal{L}_n such that $p \in H$.

LEMMA 5.2. *Every regular p -space which is a Baire space has a dense subspace which is a paracompact p -space.*

Proof. Let X be a regular Baire space, and let $\mathcal{G}_1, \mathcal{G}_2, \dots$ be a plumbing for X . We may assume that each \mathcal{G}_n is a base for X . (See 4.2 in [36].) With the aid of the axiom of choice, construct a sequence $\mathcal{H}_1, \mathcal{H}_2, \dots$ of open collections satisfying these conditions.

- (1) \mathcal{H}_n is a mutually disjoint subcollection of \mathcal{G}_n ;
- (2) $V_n = \bigcup \{H : H \text{ in } \mathcal{H}_n\}$ is a dense subset of X ;
- (3) for all n , if $H \in \mathcal{H}_{n+1}$, then $\bar{H} \subseteq H'$ for some H' in \mathcal{H}_n .

The technique for constructing such a sequence is well known; see, e.g., [8], [44], or [54]. Let $D = \bigcap_{n=1}^{\infty} V_n$, and note that D is a dense subset of X since X is a Baire space. For $n = 1, 2, \dots$ let $\mathcal{L}_n = \{H \cap D : H \text{ in } \mathcal{H}_n\}$. It is clear that $\mathcal{L}_1, \mathcal{L}_2, \dots$ is a sequence of locally finite (in fact discrete) open covers of D , and so to complete the proof it suffices to show that $\mathcal{L}_1, \mathcal{L}_2, \dots$ is a plumbing for D . (Recall Lemma 5.1.)

Let $p \in D$, for $n = 1, 2, \dots$ let $p \in L_n = H_n \cap D$, where $H_n \in \mathcal{H}_n$, and let $C(p) = \bigcap_{n=1}^{\infty} cl_D(L_n)$. We must show that (a) $C(p)$ is compact; (b) $\{\bigcap_{i=1}^n cl_D(L_i) : n = 1, 2, \dots\}$ is a "base" for $C(p)$. Observe that $p \in H_n \in \mathcal{G}_n$, $n = 1, 2, \dots$ and so $C^*(p) = \bigcap_{n=1}^{\infty} \bar{H}_n$ is compact and $\{\bigcap_{i=1}^n \bar{H}_i : n = 1, 2, \dots\}$ is a "base" for $C^*(p)$. We are going to show that $C(p) = C^*(p)$, from which both (a) and (b) easily follow.

Clearly $C(p) \subseteq C^*(p)$; so let $q \in C^*(p)$, and let us show that $q \in C(p)$. First, $q \in D$, since for all n , $q \in \bar{H}_{n+1} \subseteq H_n \subseteq V_n$. (That $\bar{H}_{n+1} \subseteq H_n$ follows from (1), (3), and the fact that $p \in (H_n \cap H_{n+1})$.) To show that $q \in cl_D(L_n)$, consider an arbitrary open set R in X containing q . Now $q \in \bar{H}_n$, so $R \cap H_n \neq \emptyset$. Since D is dense, $(R \cap H_n) \cap D \neq \emptyset$. But $L_n = H_n \cap D$, so $R \cap L_n \neq \emptyset$.

LEMMA 5.3. *Every regular quasi-complete space with a point-countable separating open cover is a p -space.*

Proof. Let X be a regular space with a point-countable separating open cover, let $\mathcal{G}_1, \mathcal{G}_2, \dots$ be a quasi-complete sequence for X . For $n = 1, 2, \dots$ let \mathcal{H}_n be an open cover of X such that $\{\bar{H} : H \text{ in } \mathcal{H}_n\}$ refines \mathcal{G}_n . Let $p \in H_n \in \mathcal{H}_n$, $n = 1, 2, \dots$. Using the fact that $\mathcal{G}_1, \mathcal{G}_2, \dots$ is a quasi-complete sequence, one can easily check that (a) $C(p) = \bigcap_{n=1}^{\infty} \bar{H}_n$ is countably compact and (b) $\{\bigcap_{i=1}^n \bar{H}_i : n = 1, 2, \dots\}$ is a “base” for $C(p)$. Since $C(p)$ is a regular countably compact space with a point-countable separating open cover, it is compact [7]. Hence $\mathcal{H}_1, \mathcal{H}_2, \dots$ is a pluming for X .

THEOREM 5.4. *Every regular $w\mathcal{A}$ -space with a point-countable separating open cover which is a Baire space and hereditarily satisfies the CCC has a countable base.*

Proof. Let X be a space satisfying the hypotheses of the theorem. By Lemma 5.3, X is a p -space. Hence, by Lemma 5.2, X has a dense subspace D which is a paracompact p -space. Now D also has a point-countable separating open cover, and so D is metrizable [47]. Using the denseness and the metrizability of D , together with the fact that X satisfies the CCC, one can show that X is separable. From this it follows that X has a countable separating open cover. Now every regular $w\mathcal{A}$ -space with a σ -point finite separating open cover is developable [31]. Thus X is a developable \aleph_1 -compact space and hence has a countable base [38].

REMARK 5.5. Comparing Theorems 4.3 and 5.4, we see that “hereditarily $w\mathcal{A}$ -space” in 4.3 can be weakened to “ $w\mathcal{A}$ -space”, provided the assumption “Baire space” is added. This suggests the following problem.

Problem 5.6. Does every regular $w\mathcal{A}$ -space with a point-countable separating open cover which hereditarily satisfies the CCC have a countable base?

THEOREM 5.7. *Let X be a regular quasi-complete Baire space which satisfies the CCC. If X has a G_δ -diagonal, or a point-countable separating open cover, then X is separable.*

Proof. If X has a G_δ -diagonal, then by a result of Gittings [25], X is a p -space. And, if X has a point-countable separating open cover, then X is a p -space by Lemma 5.3. In either case, then, by Lemma 5.2 there is a dense subspace D of X which is a paracompact p -space. Since D has a G_δ -diagonal or a point-countable separating open cover, D is metrizable (see [13], [51], [47]). Now D is dense in X and X

satisfies the CCC, so D also satisfies the CCC. Thus D is separable, from which it follows that X is separable.

REMARK 5.8. From Theorem 5.7 we obtain the following slight generalization of the result of Šapirovskii mentioned at the beginning of this section. *Every regular quasi-complete Baire space with a point-countable base which satisfies the CCC has a countable base.* It should be noted that the “Baire space” hypothesis in this result and Theorem 5.7 is not superfluous. Indeed, Pixley and Roy [52] have constructed a regular developable space with a σ -point finite base which satisfies the CCC but is not separable.

6. A metrization theorem for Moore spaces. In this section we extend two theorems of Traylor on the metrizability of Moore spaces. (See Theorems 4 and 5 of [63].) This new result should be compared with a theorem of Borges [14], which states that every locally connected, locally peripherally compact paracompact space with a G_δ -diagonal is metrizable.

THEOREM 6.1. *Let X be a locally connected, locally peripherally separable meta-Lindelöf Moore space. Then X is metrizable.*

Proof. It suffices to show that X is paracompact [12]. Let $\mathcal{C} = \{C_\alpha: \alpha \text{ in } A\}$ be the collection of all distinct components of X . Thus, each C_α is a closed, connected subset of X , and $\alpha \neq \beta$ implies $C_\alpha \cap C_\beta = \emptyset$. Since X is locally connected, each C_α is also an open subset of X . It is easy to check that each C_α is a connected, locally connected, locally peripherally separable meta-Lindelöf space, and so by a result of Grace and Heath [26], each C_α is Lindelöf. Now any regular space which is the union of a disjoint collection of open sets, each of which is Lindelöf, is paracompact. This completes the proof.

7. Metrizability of countable unions. Let X be a regular space, and suppose that $X = \bigcup_{n=1}^{\infty} M_n$, where each M_n is a metrizable subset of X . The following are known conditions under which X itself is metrizable:

- (1) X is locally countably compact and each M_n is separable (Smirnov [60]);
- (2) X is collectionwise normal and locally countably compact, and each M_n is closed (Stone [62]);
- (3) X is a collectionwise normal p -space, M_1 is separable, and M_n is closed for $n \geq 2$ (Čoban [20]);
- (4) each M_n is locally dense, and X is one of the following: (i) separable; (ii) locally compact; (iii) σ -compact (Corson-Michael [21]).

{Recall that a subset of X is *locally dense* if it is a dense subset of an open subset of X . This definition, due to Corson and Michael, is a clever device for considering subsets which are either open or dense.}

In this section we obtain generalizations of these results. In particular, Theorem 7.1 unifies and generalizes (1), (2), and (3), while Theorem 7.4 extends (i) and (iii) of (4).

THEOREM 7.1. *Let X be a collectionwise normal space which is locally a quasi-complete space, and suppose that $X = \bigcup_{n=1}^{\infty} M_n$, where each M_n is metrizable and either separable or closed. Then X is metrizable.*

Proof. If M_n is separable, then it has a countable base and hence a countable net. On the other hand, if M_n is closed, then there is a σ -locally finite collection in X which is a net for M_n . It easily follows that X has a σ -locally finite net, and hence is a σ -space. In particular, then, X is semi-stratifiable and subparacompact [22]. Now every regular semi-stratifiable quasi-complete space is developable [22], and so X is locally a developable space. Since every subparacompact locally developable space is developable [15], we see that X is a collectionwise normal developable space and thus metrizable [12].

REMARK 7.2. It is clear that (2) and (3) follow from Theorem 7.1. To see that (1) also follows from 7.1, note that a countable union of separable metrizable spaces is Lindelöf and that every regular Lindelöf space is collectionwise normal. As the following example shows, the hypothesis that X is locally a quasi-complete space is not superfluous.

EXAMPLE 7.3. *A nonmetrizable, paracompact semi-metric space which is a countable union of closed separable metrizable subsets. The space S described by Heath in [27] has the required properties.*

THEOREM 7.4. *Let X be a regular space such that $X = \bigcup_{n=1}^{\infty} M_n$, where each M_n is a locally dense, metrizable subset of X . Then X is metrizable if it satisfies any of the following:*

- (a) X is locally a CCC space;
- (b) X is a wN -space;
- (c) X is a normal β -space;
- (d) X is a normal countably metacompact p -space.

Proof. First we show that X has a σ -disjoint base. For $n = 1, 2, \dots$ let R_n be an open subset of X such that M_n is a dense subset of R_n . Since M_n is metrizable, it has a σ -disjoint base $\mathcal{B}_n =$

$\bigcup_{k=1}^{\infty} \mathcal{B}_{n,k}$, where $\mathcal{B}_{n,k} = \{B_{\alpha} : \alpha \text{ in } A_{n,k}\}$ is a disjoint open collection in M_n . For each α in $A_{n,k}$ let G_{α} be an open subset of R_n such that $G_{\alpha} \cap M_n = B_{\alpha}$, and let $\mathcal{G}_{n,k} = \{G_{\alpha} : \alpha \text{ in } A_{n,k}\}$. It is easy to check that $\mathcal{G}_{n,k}$ is a disjoint open collection in X . To see that $\{\mathcal{G}_{n,k} : n, k \text{ in } N\}$ is a base for X , let $p \in X$ and let U be an open neighborhood of p . Let V be an open neighborhood of p such that $\bar{V} \subseteq U$, and choose n in N such that $p \in M_n$. For some α in $A_{n,k}$, $p \in B_{\alpha} \subseteq V \cap M_n$. Now $G_{\alpha} \subseteq \bar{B}_{\alpha}$ since M_n is dense in R_n , so $p \in G_{\alpha} \subseteq U$.

As consequences of the fact that X has a σ -disjoint base, we note that X is screenable and is a γ -space.

To prove (a), assume that X is locally a CCC space. Since X has a σ -disjoint base, it easily follows that X is locally metrizable and locally separable. Now it is easy to check that every open cover of a locally separable screenable (in fact meta-Lindelöf) space has a star-countable open refinement. It follows that X is paracompact. (See [58] or p. 229 of [23].) Since every paracompact, locally metrizable space is metrizable [59], the proof of (a) is complete.

To prove (b), assume that X is a wN -space. Since X has been observed to be a γ -space, it follows that X is metrizable [32].

To prove (c), assume that X is a normal, β -space. The metrizability of X follows from the fact that every T_1 , β , γ -space is developable [32] and that every normal screenable developable space is metrizable [12].

To prove (d), assume that X is a normal, countably metacompact p -space. By (c), it suffices to show that X is a β -space. This follows from the fact that every countably metacompact screenable space is metacompact and that every metacompact p -space is a β -space [16].

REMARK 7.5. Note that (i) of (4) follows from (a) of Theorem 7.4, while (iii) of (4) follows from (c) of 7.4. In addition, (b) of 7.4 generalizes Theorem 5.1 in [37]. As the following example shows, the assumptions in (a), (b), (c), and (d) are not superfluous.

EXAMPLE 7.6. *A nonmetrizable completely regular developable space which is the union of two open metrizable subsets.* Consider the space S described by Heath in [28, Example 2]. The points of S are all points of the plane on or above the x -axis, and S (equipped with the topology described in [28]) is a completely regular nonnormal screenable developable space. Let U be all rational points on the x -axis together with all points of the plane above the x -axis, and let V be all irrational points on the x -axis together with all points of the plane above the x -axis. Then $S = U \cup V$, and U and V are open metrizable subsets of S .

In [21] Corson and Michael prove that every normal Čech complete space which is the union of *two* locally dense metrizable subsets is metrizable. They also ask if the restriction to just two sets is really necessary. We now show that their result holds for any finite union, and that “Čech complete” can be weakened to “ p -space”.

LEMMA 7.7. *Let M be a locally dense, countably paracompact subset of a topological space X , let $\{U_n: n = 1, 2, \dots\}$ be a countable open collection in X which covers M . Then there is a point-finite open collection $\{W_n: n = 1, 2, \dots\}$ in X which covers M such that $W_n \subseteq U_n$, $n = 1, 2, \dots$.*

Proof. Let R be an open set in X such that M is a dense subset of R . Since M is countably paracompact, there is a cover $\{V_n: n = 1, 2, \dots\}$ of M , locally finite and open in M , such that $V_n \subseteq U_n \cap M$, $n = 1, 2, \dots$. For each n let G_n be an open subset of R such that $G_n \cap M = V_n$, and let

$$L = \{p \in R: \{G_n: n = 1, 2, \dots\} \text{ is locally finite at } p\}.$$

Clearly L is open, and $M \subseteq L$ since M is dense in R . For $n = 1, 2, \dots$ let $W_n = L \cap G_n \cap U_n$. Then $\{W_n: n = 1, 2, \dots\}$ is the desired point-finite open collection.

THEOREM 7.8. *Let X be a normal p -space which is covered by a finite number of locally dense, metrizable subsets. Then X is metrizable.*

Proof. By Theorem 7.4, part (d), it suffices to show that X is countably metacompact. But this follows immediately from Lemma 7.7.

Added in proof. The notion of a quasi- G_δ -diagonal has been studied by other topologists. In particular, Lutzer [Dissertationes Math. 89 (1971)] has proved that a LOTS is quasi-developable if and only if it has a quasi- G_δ -diagonal, and Shiraki [Proc. Japan Acad. 47 (1971), suppl. II, 1036-1041] has proved that every regular, metacompact, $w\mathcal{A}$ -space with a quasi- G_δ -diagonal has a uniform base. A. Okuyama, in a private communication, has answered Problem 5.6 affirmatively under the assumption of the continuum hypothesis.

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