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Let Y be a smooth, quasi-projective scheme of finite type over an algebraically closed field of characteristic zero. Let X be the quotient of Y by a finite group of automorphisms. Assume that the branch locus of Y over X is of codimension at least 3. In this note, it is shown that X is locally rigid in the following sense: the singular locus of X is stratified and, given a point on a stratum, it is shown that there exists a locally algebraic transverse section to the stratum at the point which is rigid. This result is then applied to the coarse moduli scheme for curves of genus g, where $g > 4$ (in characteristic zero).

1. Stratifying quotient schemes. Let k be an algebraically closed field. Let V' be a smooth, irreducible quasi-projective algebraic k-scheme. By a *quotient scheme,* we mean a scheme $V = V'/G$, where G is a finite group of automorphisms of V'. In [3], Popp defines a stratification of such schemes.

Given a point $P \in V$ and a point $P' \in V'$ lying over P, one may define the inertia group of P':

$$I(P') = \{\sigma \in G \mid \sigma x \equiv x \bmod \mathcal{M}_{P'}, \quad \text{for all} \quad x \in \mathcal{O}_{V',P'}\} .$$

If $P'' \in V'$ is another point lying over P, then $I(P')$ and $I(P'')$ are conjugate subgroups of G.

Let Z_P denote the closed subscheme of Spec $(\mathcal{O}_P)$ which is ramified in the covering $f: V' \to V$ and let $Z_{P'}$ be the inverse image of Z_P in Spec $(\mathcal{O}_{P'})$. Denote by $Z'_1, \cdots, Z'_s$ those irreducible components of $Z_{P'}$ of dimension $n-1$ (where $n = \dim V$). Let $H_1, \cdots, H_s$ denote the inertia groups of the generic points of $Z'_1, \cdots, Z'_s$ respectively and let $H(P')$ denote the subgroup of $I(P')$ generated by the H_i, $i = 1, 2, \cdots, s$. (If $s = 0$, put $H(P') = (1)$.) Let

$$\bar{I}(P') = I(P')/H(P')$$

and call this the *small inertia group* of P'. Under the assumption that V' is smooth, Popp shows that $\bar{I}(P')$ is independent of the cover; i.e., for any smooth cover $V'' \to V$, if $P'' \in V''$ is a point lying over P, then $\bar{I}(P'') \cong \bar{I}(P')$. Thus, we may write $\bar{I}(P)$ and speak of the small inertia group of P.

Let W be an irreducible subscheme of V and suppose $P \in W$. Then one says that V is *equisingular at P along W* if the following two conditions hold:

(1) P is a smooth point of W

(2) Suppose P' is a point lying over P and W' is the irreducible component of $f^{-1}(W)$ containing P'. Then the canonical homomorphism $\bar{I}(W') \to \bar{I}(P')$ is a (surjective) isomorphism.

Let

$$\mathrm{Eqs}(V/W) = \{P \in W \mid V \text{ is equisingular at } P \text{ along } W\}.$$

Popp shows, under the assumption that k is of characteristic 0, that this notion of equisingularity satisfies the axioms which any good notion should (cf. **[6]**).

In particular, given $Q \in V$, let M_Q denote the family of closed, irreducible subschemes W of V such that $Q \in \mathrm{Eqs}(V/W)$. Then the family $\{\mathrm{Eqs}(V/W) \mid W \in M_Q\}$, for fixed Q, has a greatest element called the *stratum* through Q.

Another important property is that if E is a stratum and $P \in E$, then there exists a neighborhood U of P in V and a minimal biholomorphic embedding $\psi : U \to \mathbf{C}^e$ (where $e = \dim \mathcal{M}_P/\mathcal{M}_P^2$) such that $\psi(U)$ is topologically isomorphic to the direct product of $\psi(U \cap E) = \mathscr{E}$ and a locally algebraic transverse section to $\mathscr{E}$ at $\psi(P)$ (see **[3]** for details).

The above straification, in characteristic 0, is really quite neat: if E is a stratum and $P \in E$, then $E = \{Q \mid Q \text{ is analytically isomorphic to } P\}$.

2. The local rigidity of certain quotient schemes.

DEFINITION. Let V be a quotient scheme in characteristic 0. Stratify V as in §1. Then we will say V is *locally rigid* if given a point P on a stratum E, then there is a locally algebraic transverse section to E at P which is rigid.

PROPOSITION 1. *Let a finite group I act by holomorphic automorphisms of $\mathbf{C}^m$, leaving the origin fixed. If I acts freely outside some I-invariant complex subspace W'* (through the origin) *of codimension $\geqq 3$, then $X = \mathbf{C}^m/I$ is rigid.*

Proof. As is noted in **[5]**, this is a valid generalization of Theorem 3 of **[4]**.

THEOREM 1. *Suppose k is an algebraically closed field of characteristic* 0. *Let Y be a smooth, quasi-projective algebraic k-scheme and let G be a finite group of automorphisms of Y. Let $X = Y/G$. If the branch locus of Y over X is of codimension at least* 3, *then X is locally rigid.*

Proof. Suppose x is a point of X. Let I denote the inertia group of x. Note that since there is no ramification in codimension 1, we have $I = \bar{I}$. In a neighborhood of x, we can linearize the action of I (cf. **[1]**, **[3]**) so that X at x is locally analytically isomorphic to $\mathbf{C}^n/I$ at the point Q which is the image of the origin under the canonical map $\mathbf{C}^n \to \mathbf{C}^n/I$.

Choose coordinates $z_1, \cdots, z_n$ in $\mathbf{C}^n$ such that $z_1, \cdots, z_r$ span the fixed space of I (we may do this since the fixed space is linear). Then

$$\mathbf{C}^n/I \cong \operatorname{Spec}(\mathbf{C}[z_1, \cdots, z_r] \otimes \mathbf{C}[z_{r+1}, \cdots, z_n]^I).$$

The stratum on which Q lies is

$$E = \operatorname{Spec}(\mathbf{C}[z_1, \cdots, z_r])$$

and the transverse section we desire is

$$S = \operatorname{Spec}(\mathbf{C}[z_{r+1}, \cdots, z_n]^I).$$

Locally at x, the space X is isomorphic to $E \times S$, not just topologically, but analytically as well. It follows from this and our hypotheses that the branch locus of the map $\operatorname{Spec}(\mathbf{C}[z_{r+1}, \cdots, z_n]) \to S$ has codimension at least 3. Hence, applying Proposition 1, we may conclude that S is rigid.

We may apply this theorem to M_g, the coarse moduli scheme for curves of genus g, in characteristic zero. M_g is the quotient of the smooth, higher-level moduli scheme $J_{g,n}$, for n sufficiently large, by the group $GL(2g, \mathbf{Z}/n)$ **[2]**. In **[2]**, Popp computes the dimension of ramification points of the map $J_{g,n} \to M_g$. An inspection of his computations shows that, for $g > 4$, the branch locus of this map has codimension at least 3. Applying our theorem then yields:

PROPOSITION 2. *M_g, the coarse moduli scheme for curves of genus g in characteristic* 0, *is locally rigid if $g > 4$.*

REFERENCES

1. H. Cartan, *Quotient d'un espace analytique par un groupe d'automorphismes*, in *Algebraic*

Geometry and Topology, Princeton University Press (Princeton, 1957).
2. H. Popp, *The singularities of the moduli scheme of curves*, J. Number Theory, **1** (1969), 90–107.
3. ———, *Stratifikation von Quotientmannigfaltigkeiten...*, J. reine angew. Math., **250** (1971), 12–41.
4. M. Schlessinger, *Rigidity of quotient singularities*, Invent. Math., **14** (1971), 17–26.
5. ———, *On rigid singularities*, Rice University Studies on Complex Analysis, Vol. 59, No. 1 (1972), 147–162.
6. O. Zariski, *Some open questions in the theory of singularities*, Bull. Amer. Math. Soc., **77** (1971), 481–491.

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