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CONNECTOR THEORY

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Connector theory is a generalization of topology and uniformity. Each reflexive binary relation U of a space S induces a mapping from S to 2^S wherein $x \in S \rightarrow xU = \{y: (x, y) \in U\} \in 2^S$. This mapping is called a connector. A uniformity on S is a set of connectors which meets certain conditions. The results in this paper include a necessary-sufficient condition for a connector-set to induce a unique topology, generalizations of continuous mappings and uniformly continuous mappings and characterizations of the connector-sets which correspond to a specific type of topology, for instance, a compact topology, a pseudo-compact topology.

If \mathfrak{A} is a connector-set on S , let $\mathfrak{A}^\#$ denote the connector-set, $\{U: \text{for each } x \in S, \text{ there is } V(x) \in \mathfrak{A} \text{ such that } xU = xV(x)\}$. The following types of connector-sets are defined.

Cone: $U \leq V$ and $U \in \mathfrak{A} \Rightarrow V \in \mathfrak{A}$.

Net: $U, V \in \mathfrak{A} \Rightarrow W \leq U \cap V$ for some $W \in \mathfrak{A}$.

Filter: Cone and net.

Sharp: For each $U \in \mathfrak{A}^\#$, there is $V \in \mathfrak{A}$ such that $V \leq U$.

Prenet: $\mathfrak{A}^\#$ is a net.

Topological: For each $U \in \mathfrak{A}$ and for each $x \in S$, there is $V \in \mathfrak{A}$ such that $xV^2 \subseteq xU$.

Topology: Topological sharp filter.

Pretopology: $\mathfrak{A}^\#$ is a topological net.

Uniform: For each $U \in \mathfrak{A}$, there is $V \in \mathfrak{A}$ such that $VV^{-1} \leq U$.

Uniformity: Uniform filter.

Totally bounded: For each $U \in \mathfrak{A}$, there are $x_i \in S$, $i = 1, 2, 3, \dots, m$ such that $S \subseteq \cup x_i U$.

Bounded: For each $U \in \mathfrak{A}$, there are $x_i \in S$, $i = 1, 2, 3, \dots, m$ and a positive integer n such that $S \subseteq \bigcup x_i U^n$.

Absolutely bounded: For each $U \in \mathfrak{A}$ and for each $X \subseteq S$, there are $x_i \in X$ ($i = 1, 2, 3, \dots, m$), and a positive integer n such that $X \subseteq \bigcup x_i U^n$.

A prenet \mathfrak{A} induces a topology $\mathcal{T}(\mathfrak{A})$ (in the usual sense) which will be called an *open-topology*. The interior theorem states that \mathfrak{A} is a pretopology if and only if x is an interior point of xU for every $U \in \mathfrak{A}$. A topology corresponds uniquely to an open-topology and vice versa (The topology theorem). A topology \mathcal{T} induces a compact open-topology if and only if \mathcal{T} is totally bounded. The compact topology theorem states that compactness of $\mathcal{T}(\mathfrak{U})$ for a uniformity \mathfrak{U} implies that \mathfrak{U} is the strongest uniformity included in \mathfrak{U}^* .

Suppose \mathfrak{A} is a connector-set on S and \mathfrak{D} is a connector set on R . Let $xM \in R$ denote the image of $x \in S$ by a mapping M . Each $U \in \mathfrak{D}$ induces the connector MUM^{-1} on S by $x \rightarrow \{z: (xM, zM) \in U\}$. M is \mathfrak{A} -continuous if every MUM^{-1} , $U \in \mathfrak{D}$ belongs to the sharp filter hull of \mathfrak{A} . M is uniformly \mathfrak{A} -continuous if they belong to the filter hull of \mathfrak{A} . The continuity theorem states that the definition is compatible with the continuity (in the usual sense) of M on the topological space $(S, \mathcal{T}(\mathfrak{A}))$ if \mathfrak{A} is a prenet. The kernel theorem is: \mathfrak{U} is the strongest uniformity included in a topology \mathcal{T} and \mathfrak{D} is a uniform net, then \mathcal{T} -continuity implies uniform \mathfrak{U} -continuity. The strongest uniformity included in a topology \mathcal{T} is bounded if and only if every \mathcal{T} -continuous real-valued function is bounded (The pseudo-compact theorem).

1. Connector systems. Let S be a space on which we develop generalized structures of topology and uniformity. A mapping U from S to 2^S is called a *connector* on S if each x of S belongs to its image xU . The *inverse* U^{-1} of U is a mapping: $x \in S \rightarrow \{y: x \in yU\}$. Let U and V be connectors. We write $U \leq V$ if $xU \subseteq xV$ for every x of S . The connector UV (the product of U and V) is defined by $x \in S \rightarrow \bigcup \{yV: y \in xU\}$. UV is denoted by U^2 if $U = V$. An intersection of connectors $\{U_\lambda: \lambda \in \Lambda\}$ is the connector defined by $x \in S \rightarrow \bigcap \{xU_\lambda: \lambda \in \Lambda\} \in 2^S$ and it is denoted by $\bigcap U_\lambda$. A non-empty set of connectors is called a *connector system*.

A connector system \mathfrak{A} on S is called a *cone* if $V \in \mathfrak{A}$ whenever $U \leq V$ and $U \in \mathfrak{A}$. Let $\mathfrak{A}^< = \{U: V \leq U \text{ for some } V \in \mathfrak{A}\}$. $\mathfrak{A}^<$ is a cone, and $\mathfrak{A}^< \subseteq \mathfrak{B}$ if \mathfrak{B} is a cone and $\mathfrak{A} \subseteq \mathfrak{B}$.

(1.1) \mathfrak{A} is a cone if and only if $\mathfrak{A}^< = \mathfrak{A}$.

(1.2) $\mathfrak{A} \subseteq \mathfrak{A}^<$.

$$(1.3) \quad \mathfrak{A}^{\leq} = \mathfrak{A}^< \quad (\mathfrak{A}^{\leq} = (\mathfrak{A}^<)^<).$$

$$(1.4) \quad \mathfrak{A} \subseteq \mathfrak{B} \text{ implies } \mathfrak{A}^< \subseteq \mathfrak{B}^<.$$

$$(1.5) \quad (\cup \mathfrak{A}_\lambda)^< = \cup \mathfrak{A}_\lambda^<.$$

A connector system \mathfrak{A} is called a *net* if for $U, V \in \mathfrak{A}$, there is $W \in \mathfrak{A}$ such that $W \leq U \cap V$. Let \mathfrak{A}^\times denote the set of all finite intersections of connectors of \mathfrak{A} . Then \mathfrak{A}^\times is a net.

$$(1.6) \quad \mathfrak{A} \subseteq \mathfrak{A}^\times.$$

$$(1.7) \quad \mathfrak{A}^{\times \times} = \mathfrak{A}^\times.$$

$$(1.8) \quad \mathfrak{A} \subseteq \mathfrak{B} \text{ implies } \mathfrak{A}^\times \subseteq \mathfrak{B}^\times.$$

$$(1.9) \quad (\cup \mathfrak{A}_\lambda)^\times = (\cup \mathfrak{A}_\lambda^\times)^\times.$$

$$(1.10) \quad \mathfrak{A} \text{ is a net if and only if } \mathfrak{A}^\times \subseteq \mathfrak{A}^<.$$

$$(1.11) \quad \mathfrak{A}^{\times <} = \mathfrak{A}^{< \times}.$$

Proof. It is obvious that $\mathfrak{A}^{\times \times} \subseteq \mathfrak{A}^{\times <}$. Thus, by (1.6), (1.4) and (1.8), we obtain $\mathfrak{A}^{< \times} \subseteq (\mathfrak{A}^\times)^{< \times} \subseteq \mathfrak{A}^{\times <}$. On the other hand, for each $U \in \mathfrak{A}^{\times <}$, there are $U_i \in \mathfrak{A}$, $i = 1, 2, 3, \dots, n$ such that $\cap \{U_i : i = 1, 2, 3, \dots, n\} \leq U$. Let V_i denote a connector: $x \in S \rightarrow (xU_i) \cup (xU)$. Then $U_i \leq V_i$, $i = 1, 2, 3, \dots, n$ and $U = \cap \{V_i : i = 1, 2, 3, \dots, n\}$.

$$(1.12) \quad \mathfrak{A}^{\times < \times} = \mathfrak{A}^{\times <}.$$

Proof. Refer to (1.3), (1.7) and (1.11).

A connector system is called a *filter* if it is a cone and a net. Every finite intersection of connectors of a filter belongs to the filter. Therefore, a filter \mathfrak{A} is a connector system which satisfies the following conditions.

$$(1) \quad \text{If } U \in \mathfrak{A} \text{ and } U \leq V \text{ then } V \in \mathfrak{A}.$$

$$(2) \quad \text{If } U, V \in \mathfrak{A} \text{ then } U \cap V \in \mathfrak{A}.$$

$$(1.13) \quad \mathfrak{A} \text{ is a filter if and only if } \mathfrak{A} = \mathfrak{A}^\times = \mathfrak{A}^<.$$

$$(1.14) \quad \mathfrak{A} \text{ is a filter if and only if } \mathfrak{A} = \mathfrak{A}^{\times <}.$$

$$(1.15) \quad \mathfrak{A}^< \text{ is a filter if and only if } \mathfrak{A} \text{ is a net.}$$

$\mathfrak{A}^{\times <}$ is a filter and $\mathfrak{B} \supseteq \mathfrak{A}^{\times <}$ if \mathfrak{B} is a filter and $\mathfrak{B} \supseteq \mathfrak{A}$. Therefore, $\mathfrak{A}^{\times <}$ is called the *filter hull* of \mathfrak{A} .

$$(1.16) \quad \mathfrak{A}^{\times <} = \mathfrak{A}^< \text{ if and only if } \mathfrak{A} \text{ is a net.}$$

A connector system \mathfrak{A} is called *sharp* if for each connector system, $\{U(x) \in \mathfrak{A} : x \in S\}$, there is $V \in \mathfrak{A}$ such that $xV \subseteq xU(x)$ for every $x \in S$. Let $\mathfrak{A}^\# = \{U : \text{for each } x \in S, \text{ there is } V(x) \in \mathfrak{A} \text{ such that } xU = xV(x)\}$. The connector system $\mathfrak{A}^\#$ is sharp.

$$(1.17) \quad \mathfrak{A} \text{ is sharp if and only if } \mathfrak{A}^\# \subseteq \mathfrak{A}^<.$$

$$(1.18) \quad \mathfrak{A} \text{ is sharp if and only if } \mathfrak{A}^{\# <} = \mathfrak{A}^<.$$

Proof. (1.3) and (1.4) $\Rightarrow \mathfrak{A}^\# \subseteq \mathfrak{A}^<$ if and only if $\mathfrak{A}^{\#<} = \mathfrak{A}^<$.

$$(1.19) \quad \mathfrak{A} \subseteq \mathfrak{A}^\#.$$

$$(1.20) \quad \mathfrak{A}^{\#\#} = \mathfrak{A}^\#.$$

$$(1.21) \quad \mathfrak{A} \subseteq \mathfrak{B} \text{ implies } \mathfrak{A}^\# \subseteq \mathfrak{B}^\#.$$

$$(1.22) \quad (\cup \mathfrak{A}_\lambda)^\# = (\cup \mathfrak{A}_\lambda^\#)^\#.$$

A connector system is called a *prenet* if for $U, V \in \mathfrak{A}$ and for each $x \in S$, there is $W \in \mathfrak{A}$ such that $xW \subseteq xU \cap xV$. Every net is a prenet.

$$(1.23) \quad \mathfrak{A}^\# \text{ is a net if and only if } \mathfrak{A} \text{ is a prenet.}$$

$$(1.24) \quad \text{Every sharp prenet is a net.}$$

Proof. (1.10) and (1.23) $\Rightarrow \mathfrak{A}^\times \subseteq \mathfrak{A}^{\#\times} \subseteq \mathfrak{A}^{\#<} \subseteq \mathfrak{A}^<$ if \mathfrak{A} is a prenet. (1.4) and (1.10) $\Rightarrow \mathfrak{A}^{\#<} \subseteq \mathfrak{A}^{\leq} = \mathfrak{A}^<$ if \mathfrak{A} is sharp. Hence, \mathfrak{A} is a net if \mathfrak{A} is a sharp prenet.

$$(1.25) \quad \mathfrak{A}^{\#<} = \mathfrak{A}^{<\#}.$$

Proof. Let $\{V(x): x \in S\}$ be a connector system of $\mathfrak{A}^<$. There is $W(x) \in \mathfrak{A}$ for each $V(x)$ such that $W(x) \leq V(x)$. If W and U are defined respectively, by $W: x \in S \rightarrow xW(x)$ and $U: x \in S \rightarrow xV(x)$, then, $W \leq U$ and $W \in \mathfrak{A}^\#$. Hence, U belongs to $\mathfrak{A}^{\#<}$ if $U \in \mathfrak{A}^{<\#}$. Conversely let U be a connector of $\mathfrak{A}^{\#<}$. The connector $U(x)$ is defined, for each $x \in S$, by $U(x): y \in S \rightarrow yU(x) = xU$ if $y = x$ and $yU(x) = S$ otherwise. Then, $xU = xU(x)$ for every $x \in S$, and every $U(x)$, ($x \in S$), belongs to $\mathfrak{A}^<$.

$$(1.26) \quad \mathfrak{A}^{\#\times} \subseteq \mathfrak{A}^{\times\#}.$$

Proof. Each $U \in \mathfrak{A}^{\#\times}$ is a finite intersection of connectors $\{W_i: i = 1, 2, 3, \dots, n\}$ of $\mathfrak{A}^\#$. There are $U_i \in \mathfrak{A}$ for each $x \in S$, such that $xW_i = xU_i$, $i = 1, 2, \dots, n$. $xU = \cap \{xW_i: i = 1, 2, 3, \dots, n\} = \cap \{xU_i: i = 1, 2, 3, \dots, n\} = x(\cap U_i)$, hence U belongs to $\mathfrak{A}^{\times\#}$.

$$(1.27) \quad \mathfrak{A}^{\#\times\#} = \mathfrak{A}^{\times\#}.$$

Proof. Refer to (1.19), (1.8), (1.21), (1.26) and (1.20) in order.

$$(1.28) \quad \mathfrak{A}^{\times\#\times} = \mathfrak{A}^{\times\#}.$$

Proof. Refer to (1.6), (1.26) and (1.7).

$$(1.29) \quad \mathfrak{A}^{\times\#\times\#} = \mathfrak{A}^{\#\times\#\times} = \mathfrak{A}^{\times\#}.$$

Proof. Refer to (1.27) and (1.28).

$$(1.30) \quad \mathfrak{A}^{\times\#<} = \mathfrak{A}^{\times\#<\times\#<}.$$

Proof. $\mathfrak{A}^{\times\#<\times\#<} = \mathfrak{A}^{\times\#\times\#<\times\#<}$ by (1.11), and $\mathfrak{A}^{\times\#\times\#<\times\#<} = \mathfrak{A}^{\times\#\times\#<\times\#<\times\#<}$ by

(1.25). (1.29) and (1.13) $\Rightarrow \mathfrak{A}^{\times\#\times\#\#} = \mathfrak{A}^{\times\#\#}$. Hence, $\mathfrak{A}^{\times\#\#<\times\#\#} = \mathfrak{A}^{\times\#\#}$.
 (1.31) \mathfrak{A} is a sharp filter if and only if $\mathfrak{A} = \mathfrak{A}^{\times\#\#}$.

Proof. An implication of (1.13) and (1.18) is the following. \mathfrak{A} is a sharp filter if and only if $\mathfrak{A} = \mathfrak{A}^{\times} = \mathfrak{A}^{<} = \mathfrak{A}^{\#}$. Therefore, $\mathfrak{A} = \mathfrak{A}^{\times\#\#}$ if \mathfrak{A} is a sharp filter. Conversely, $\mathfrak{A} = \mathfrak{A}^{\times\#\#}$ implies $\mathfrak{A}^{\times} \subseteq \mathfrak{A}^{\times\#\#} = \mathfrak{A} \subseteq \mathfrak{A}^{\times}$, thus $\mathfrak{A} = \mathfrak{A}^{\times}$. $\mathfrak{A} = \mathfrak{A}^{\#}$ and $\mathfrak{A} = \mathfrak{A}^{<}$ can be proved similarly.

$\mathfrak{A}^{\times\#\#}$ is a sharp filter and $\mathfrak{A}^{\times\#\#} \subseteq \mathfrak{B}$ if \mathfrak{B} is a sharp filter and $\mathfrak{A} \subseteq \mathfrak{B}$. Therefore, $\mathfrak{A}^{\times\#\#}$ is called the *sharp filter hull* of \mathfrak{A} .

(1.32) \mathfrak{A} is a prenet if and only if $\mathfrak{A}^{\times\#\#} = \mathfrak{A}^{\#<}$.

Proof. \mathfrak{A} is a prenet if and only if $\mathfrak{A}^{\#}$ is a net. $\mathfrak{A}^{\#}$ is a net if and only if $\mathfrak{A}^{\#\times} = \mathfrak{A}^{\#<}$. The last relation implies $\mathfrak{A}^{\#\times<\#} = \mathfrak{A}^{\#<\#} = \mathfrak{A}^{\#\#\#} = \mathfrak{A}^{\#<}$. On the other hand, $\mathfrak{A}^{\#\times<\#} = \mathfrak{A}^{\#<}$ implies $\mathfrak{A}^{\#<} \subseteq \mathfrak{A}^{\#\times} \subseteq \mathfrak{A}^{\#\times<\#} = \mathfrak{A}^{\#<}$, thus $\mathfrak{A}^{\#\times} = \mathfrak{A}^{\#<}$. $\mathfrak{A}^{\#\times<\#} = \mathfrak{A}^{\#\#\#}$ by (1.25) and (1.27).

(1.33) \mathfrak{A} is a sharp prenet if and only if $\mathfrak{A}^{\times\#\#} = \mathfrak{A}^{<}$.

Proof. If \mathfrak{A} is a sharp prenet then, by (1.32) and (1.18), $\mathfrak{A}^{\times\#\#} = \mathfrak{A}^{\#<} = \mathfrak{A}^{<}$. $\mathfrak{A}^{<} \subseteq \mathfrak{A}^{\#\times} \subseteq \mathfrak{A}^{\times\#\#}$, thus, $\mathfrak{A}^{\times\#\#} = \mathfrak{A}^{<}$ implies $\mathfrak{A}^{<} = \mathfrak{A}^{\#<}$.

(1.34) $\mathfrak{A}^{\times\#\#} = \mathfrak{A}^{\#}$ if \mathfrak{A} is a filter.

Proof. Refer to (1.14), (1.25) and (1.31).

2. Base of filters. A connector system \mathfrak{A} is called *stronger* than a connector system \mathfrak{B} (or \mathfrak{B} is *weaker* than \mathfrak{A}) if $\mathfrak{B} \subseteq \mathfrak{A}^{<}$.

(2.1) \mathfrak{A} is a net. \mathfrak{A} is stronger than \mathfrak{B} if and only if $\mathfrak{B} \subseteq \mathfrak{A}^{<}$.

(2.2) \mathfrak{A} is stronger and also weaker than \mathfrak{B} if and only if $\mathfrak{A}^{<} = \mathfrak{B}^{<}$.

A connector system \mathfrak{A} is called *finer* than a connector system \mathfrak{B} if $\mathfrak{B} \subseteq \mathfrak{A}^{\times\#\#}$. (1.28), (1.19), (1.32) and (1.33) imply (2.3), (2.4), (2.5) and (2.6) respectively.

(2.3) \mathfrak{A} is finer than \mathfrak{B} if and only if $\mathfrak{A}^{\times\#}$ is stronger than \mathfrak{B} .

(2.4) \mathfrak{A} is finer than \mathfrak{B} if \mathfrak{A} is stronger than \mathfrak{B} .

(2.5) \mathfrak{A} is a prenet. \mathfrak{A} is finer than \mathfrak{B} if and only if $\mathfrak{B} \subseteq \mathfrak{A}^{\#<}$.

(2.6) If \mathfrak{A} is a sharp prenet and finer than \mathfrak{B} then \mathfrak{A} is stronger than \mathfrak{B} .

Let \mathfrak{F} be a filter on S . A connector system $\mathfrak{A} \subseteq \mathfrak{F}$ is called a *basis* of \mathfrak{F} if for each $U \in \mathfrak{F}$, there is $V \in \mathfrak{A}$ such that $V \leq U$.

(2.7) \mathfrak{A} is a basis of \mathfrak{F} if and only if $\mathfrak{F} = \mathfrak{A}^{<}$.

(2.8) The following two statements are equivalent.

(1) \mathfrak{A} is a basis of \mathfrak{F} .

(2) \mathfrak{A} is a net, and \mathfrak{A} is stronger and weaker than \mathfrak{F} .

Proof. The statement (2) is, by (1.16) and (2.27), equivalent to $\mathfrak{A}^< = \mathfrak{A}^{*<} = \mathfrak{F}^{*<}$. These relations follow if $\mathfrak{F} = \mathfrak{A}^<$ and $\mathfrak{F} = \mathfrak{F}^{*<}$, hence, the statement (1) implies the statement (2). The converse is obvious.

(2.9) \mathfrak{A} is a basis of $\mathfrak{A}^<$ if \mathfrak{A} is a net.

Proof. Refer to (1.15) and (2.7).

(2.10) Every basis of a sharp filter is sharp.

Proof. If \mathfrak{A} is a basis of a sharp filter \mathfrak{F} then, since (1.31) and (2.7) imply $\mathfrak{F} = \mathfrak{F}^{\#}$ and $\mathfrak{A}^< = \mathfrak{F}$ respectively, $\mathfrak{A}^{*<} = \mathfrak{A}^{<\#} = \mathfrak{F}^{\#} = \mathfrak{F} = \mathfrak{A}^<$. Hence, by (1.18), \mathfrak{A} is sharp.

A connector system \mathfrak{A} is called a *prebasis* of a filter \mathfrak{F} if $\mathfrak{A}^{\#}$ is a basis of \mathfrak{F} .

(2.11) \mathfrak{A} is a prebasis of \mathfrak{F} if and only if $\mathfrak{A}^{*<} = \mathfrak{F}$.

(2.12) A filter \mathfrak{F} is sharp if there is a prebasis.

(2.13) Every basis of \mathfrak{F} is a prebasis if \mathfrak{F} is a sharp filter.

(2.14) Every prebasis is a prenet.

Proof. Refer to (1.23) and (2.8).

(2.15) \mathfrak{B} is a prebasis of $\mathfrak{A}^{*<}$ if and only if \mathfrak{B} is a prenet and $\mathfrak{A}^{*<} = \mathfrak{B}^{*<}$.

Proof. An implication of (2.11) is the following. \mathfrak{B} is a prebasis of $\mathfrak{A}^{*<}$ if and only if $\mathfrak{A}^{*<} = \mathfrak{B}^{*<}$. (1.32) states that \mathfrak{B} is a prenet if and only if $\mathfrak{B}^{*<} = \mathfrak{B}^{<\#}$. $\mathfrak{A}^{*<} = \mathfrak{B}^{*<} \Rightarrow \mathfrak{A}^{*<} = \mathfrak{B}^{<\#} \subseteq \mathfrak{B}^{*<} \subseteq \mathfrak{B}^{< \times \times \#} = \mathfrak{A}^{< \times \times \#} = \mathfrak{A}^{*<}$. Therefore, $\mathfrak{A}^{*<} = \mathfrak{B}^{*<}$ if and only if $\mathfrak{B}^{*<} = \mathfrak{B}^{<\#}$ and $\mathfrak{A}^{*<} = \mathfrak{B}^{*<}$.

(2.16) \mathfrak{F} is a sharp filter. A connector system $\mathfrak{A} \subseteq \mathfrak{F}$ is a prebasis of \mathfrak{F} if and only if \mathfrak{A} is a prenet and finer than \mathfrak{F} .

The following two statements are equivalent if \mathfrak{F} and \mathfrak{G} are filters.

(1) \mathfrak{F} is stronger than \mathfrak{G} .

(2) $\mathfrak{F} \supseteq \mathfrak{G}$.

An intersection of filters is a filter. Therefore, for each set of filters $\{\mathfrak{F}_\lambda : \lambda \in \Lambda\}$, there exists the strongest filter among the filters weaker than all \mathfrak{F}_λ , $\lambda \in \Lambda$. Likewise, there exists the weakest filter among the filters stronger than all \mathfrak{F}_λ , $\lambda \in \Lambda$. The two filters are respectively denoted by $\wedge \mathfrak{F}_\lambda$ and $\vee \mathfrak{F}_\lambda$.

(2.19) If \mathfrak{A}_λ is a basis of \mathfrak{F}_λ for each $\lambda \in \Lambda$ then $(\cup \mathfrak{A}_\lambda)^*$ is a basis of $\vee \mathfrak{F}_\lambda$.

Proof. The following relations are implied respectively by (1.3), (1.11), (1.5), (2.7) and (2.17). $(\cup \mathfrak{A}_\lambda)^{*<} = (\cup \mathfrak{A}_\lambda)^{*\#} = (\cup \mathfrak{A}_\lambda)^{< \times \times} = (\cup \mathfrak{A}_\lambda^{\#})^{<} = (\cup \mathfrak{F}_\lambda)^{<} = \vee \mathfrak{F}_\lambda$. Hence, by (2.7), $(\cup \mathfrak{A}_\lambda)^*$ is a basis of $\vee \mathfrak{F}_\lambda$.

(2.20) $(\vee \mathfrak{F}_\lambda)^{\#} = (\cup \mathfrak{F}_\lambda)^{*<}$.

(2.21) if \mathfrak{A}_λ is a prebasis of \mathfrak{F}_λ for each $\lambda \in \Lambda$, then $(\cup \mathfrak{A}_\lambda)^\times$ is a prebasis of $(\vee \mathfrak{F}_\lambda)^\#$.

Proof. $\mathfrak{F}_\lambda = \mathfrak{A}_\lambda^{\#<}$, $\lambda \in \Lambda$ and $(\vee \mathfrak{F}_\lambda)^\# = (\cup \mathfrak{F}_\lambda)^{\times\#<}$ by (2.11) and (2.20). (1.22) and (1.27) $\Rightarrow (\cup \mathfrak{A}_\lambda^{\#})^{\times\#} = (\cup \mathfrak{A}_\lambda)^{\times\#}$. Therefore,

$$(\vee \mathfrak{F}_\lambda)^\# = (\cup \mathfrak{A}_\lambda^{\#<})^{\times\#<} = (\cup \mathfrak{A}_\lambda^{\#})^{<\times\#<} = (\cup \mathfrak{A}_\lambda^{\#})^{\times\#<} = (\cup \mathfrak{A}_\lambda^{\#})^{\times\#<} = (\cup \mathfrak{A}_\lambda)^{\times\#<}.$$

$$(2.22) \quad (\vee \mathfrak{F}_\lambda)^\# = (\vee \mathfrak{F}_\lambda^{\#})^\#.$$

(2.22) is a generalization of Theorem 5 in §28 of [1].

3. Topologies. A connector system \mathfrak{A} is called *topological* if for each $U \in \mathfrak{A}$ and for each $x \in S$, there is $V \in \mathfrak{A}$ such that $xV^2 \subseteq xU$.

(3.1) If \mathfrak{A} is topological then $\mathfrak{A}^<$, \mathfrak{A}^\times and $\mathfrak{A}^\#$ are topological.

(3.2) If \mathfrak{A}_λ , $\lambda \in \Lambda$ are all topological then $\cup \mathfrak{A}_\lambda$ is topological.

A topological sharp filter is called a *topology*; i.e., a topology \mathfrak{T} on S is a connector system which satisfies the following conditions:

(1) If $U, V \in \mathfrak{T}$ then $U \cap V \in \mathfrak{T}$.

(2) If $U \leq V$ and $U \in \mathfrak{T}$ then $V \in \mathfrak{T}$.

(3) For each system $\{U(x) \in \mathfrak{T} : x \in S\}$, there exists $U \in \mathfrak{T}$ such that $xU = xU(x)$ for every $x \in S$.

(4) For each $U \in \mathfrak{T}$ and for each $x \in S$, there exists $V \in \mathfrak{T}$ such that $xV^2 \subseteq xU$.

The use of the word “topology” is not conventional. Compatibility of the terminology is cleared in the following section. Topologies are filters, hence, a topology is stronger than another if and only if the former includes the latter. A *topology hull* of a connector system \mathfrak{A} is the weakest topology among the topologies stronger than \mathfrak{A} . A topology hull is unique if it exists.

(3.3) $\mathfrak{A}^{\times\#<}$ is the topology hull of \mathfrak{A} if \mathfrak{A} is topological.

Proof. Refer to (1.31) and (3.1).

(3.4) If one of \mathfrak{A} , \mathfrak{A}^\times , $\mathfrak{A}^<$ and $\mathfrak{A}^\#$ has a topology hull, then all of them have the same topology hull.

Proof. If \mathfrak{T} is a topology and \mathfrak{T} includes \mathfrak{A} then \mathfrak{T} includes all $\mathfrak{A}^<$, \mathfrak{A}^\times and $\mathfrak{A}^\#$ because $\mathfrak{T} = \mathfrak{T}^{\times\#<}$. If \mathfrak{T} includes one of $\mathfrak{A}^<$, \mathfrak{A}^\times and $\mathfrak{A}^\#$ then \mathfrak{T} includes \mathfrak{A} , thus \mathfrak{T} includes the other two.

(3.5) $\mathfrak{A}^\#$ is the topology hull of \mathfrak{A} if \mathfrak{A} is a topological filter.

Proof. Refer to (1.34) and (3.3).

(3.6) $\mathfrak{A}^<$ is the topology hull of \mathfrak{A} if and only if \mathfrak{A} is topological sharp prenet.

Proof. Refer to (3.3) and (1.33).

(3.7) If \mathfrak{A} is a topological prenet, then $\mathfrak{A}^{*\prec}$ is the topology hull of \mathfrak{A} .

Proof. Refer to (3.3) and (1.32).

(3.8) Every basis of a topology \mathfrak{T} is a topological sharp prenet, and \mathfrak{T} is its topology hull.

Proof. $\mathfrak{T} = \mathfrak{A}^{\prec} \Rightarrow \mathfrak{A}^{\times\#\prec} = \mathfrak{A}^{\prec\times\#\prec} = \mathfrak{T}^{\times\#\prec} = \mathfrak{T} = \mathfrak{A}^{\prec}$. A basis of a topology is topological, hence, \mathfrak{T} is the topology hull of \mathfrak{A} .

(3.9) A topology \mathfrak{T} is the topology hull of every prebasis of \mathfrak{T} .

(3.10) If \mathfrak{T}_{λ} is the topology hull of \mathfrak{A}_{λ} for each $\lambda \in \Lambda$ then $(\vee \mathfrak{T}_{\lambda})^{\#}$ is the topology hull of $\cup \mathfrak{A}_{\lambda}$.

Proof. $(\vee \mathfrak{T}_{\lambda})^{\#}$ is the topology hull of $\cup \mathfrak{T}_{\lambda}$ by (3.2), (3.3) and (2.20). If \mathfrak{T} is a topology which includes all \mathfrak{A}_{λ} , $\lambda \in \Lambda$ then \mathfrak{T} includes all \mathfrak{T}_{λ} , $\lambda \in \Lambda$. Hence the both unions have the same topology hull.

A connector system \mathfrak{A} is called a *pretopology* if $\mathfrak{A}^{\#}$ is a topological net. Every topology is, by (3.1), a pretopology.

(3.11) Every pretopology is a prenet.

(3.12) Every topological prenet is a pretopology.

(3.13) If \mathfrak{A} is a pretopology then $\mathfrak{A}^{*\prec}$ is the topology hull of \mathfrak{A} and \mathfrak{A} is a prebasis of $\mathfrak{A}^{*\prec}$.

Proof. (1.27) and (3.1) $\Rightarrow \mathfrak{A}^{*\prec}$ is topological if \mathfrak{A} is a pretopology. Then, $\mathfrak{A}^{*\prec}$ is the topology hull of \mathfrak{A} . $\mathfrak{A}^{*\prec} = \mathfrak{A}^{*\prec}$ by (3.11) and (1.32), hence, $\mathfrak{A}^{*\prec}$ is the topology hull of \mathfrak{A} .

(3.14) If every \mathfrak{A}_{λ} , $\lambda \in \Lambda$ is a pretopology then $(\cup \mathfrak{A}_{\lambda})^{\times}$ is a pretopology.

Proof. (1.27) and (1.22) $\Rightarrow (\cup \mathfrak{A}_{\lambda})^{\times\#} = (\cup \mathfrak{A}_{\lambda}^{\#})^{\times\#}$, and (3.1) and (3.2) $\Rightarrow (\cup \mathfrak{A}_{\lambda}^{\#})^{\times\#}$ is topological if $\mathfrak{A}_{\lambda}^{\#}$, $\lambda \in \Lambda$ are topological. $(\cup \mathfrak{A}_{\lambda})^{\times\#}$ is a net because $(\cup \mathfrak{A}_{\lambda})^{\times\#} = (\cup \mathfrak{A}_{\lambda})^{\times\#}$. Hence, $(\cup \mathfrak{A}_{\lambda})^{\times}$ is a pretopology.

4. Open sets. The word ‘topology’ has been already used for a connector system. To avoid confusion, a topology in usual sense is called an *open-topology*. An open-topology \mathcal{T} is a family of subsets of S and

- (1) $X_{\lambda} \in \mathcal{T}$, $\lambda \in \Lambda \Rightarrow \cup X_{\lambda} \in \mathcal{T}$,
- (2) $X, Y \in \mathcal{T} \Rightarrow X \cap Y \in \mathcal{T}$,
- (3) $\emptyset, S \in \mathcal{T}$.

Each prenet \mathfrak{A} on S induces an open-topology in the following way. A set X of S is an open set if for each $x \in X$, there is $U \in \mathfrak{A}$ such that $xU \subseteq X$. This open-topology is denoted by $\mathcal{T}(\mathfrak{A})$.

(4.1) \mathfrak{A} and \mathfrak{B} are prenets. Then $\mathcal{T}(\mathfrak{B}) \subseteq \mathcal{T}(\mathfrak{A})$ if \mathfrak{A} is finer than \mathfrak{B} , and $\mathcal{T}(\mathfrak{A}) = \mathcal{T}(\mathfrak{B})$ if $\mathfrak{A}^{* <} = \mathfrak{B}^{* <}$.

Proof. (2.5) states that $\mathfrak{B} \subseteq \mathfrak{A}^{* <}$ if a prenet \mathfrak{A} is finer than \mathfrak{B} . Thus, for each $U \in \mathfrak{B}$ and for each $x \in S$, there is $V \in \mathfrak{A}$ such that $xV \subseteq xU$. Hence, $\mathcal{T}(\mathfrak{B}) \subseteq \mathcal{T}(\mathfrak{A})$. If $\mathfrak{A}^{* <} = \mathfrak{B}^{* <}$ then $\mathfrak{A} \subseteq \mathfrak{B}^{* <}$ and $\mathfrak{B} \subseteq \mathfrak{A}^{* <}$, thus, $\mathcal{T}(\mathfrak{A}) = \mathcal{T}(\mathfrak{B})$.

(4.2) $\mathcal{T}(\mathfrak{A}) = \mathcal{T}(\mathfrak{A}^<) = \mathcal{T}(\mathfrak{A}^*) = \mathcal{T}(\mathfrak{A}^*)$.

A connector U is called \mathfrak{A} -open if $xU \in \mathcal{T}(\mathfrak{A})$ for every $x \in S$.

(4.3) If a connector is \mathfrak{A} -open then the connector belongs to $\mathfrak{A}^{* <}$.

\mathfrak{A} -Int X denotes the interior of a set $X \subseteq S$, relative to the open-topology $\mathcal{T}(\mathfrak{A})$, i.e., \mathfrak{A} -Int $X = \bigcup \{Y \in \mathcal{T}(\mathfrak{A}): Y \subseteq X\}$.

(4.4) \mathfrak{A} -Int $X = \mathfrak{B}$ -Int X for every $X \subseteq S$ if and only if $\mathcal{T}(\mathfrak{A}) = \mathcal{T}(\mathfrak{B})$.

INTERIOR THEOREM. The following statements are equivalent.

(1) \mathfrak{A} is a pretopology.

(2) \mathfrak{A} -Int $X = \{x; xU \subseteq X \text{ for some } U \in \mathfrak{A}\}$ for every subset $X \subseteq S$.

(3) If $U \in \mathfrak{A}$ then x belongs to the \mathfrak{A} -interior of xU for every $x \in S$.

Proof. (1) \Rightarrow (2). Let $X \subseteq S$ and let $Y = \{x \in S: xV \subseteq X \text{ for some } V \in \mathfrak{A}^*\}$. $xV \subseteq X$ for some $V \in \mathfrak{A}^*$ if and only if $xU \subseteq X$ for some $U \in \mathfrak{A}$. Therefore, $Y = \{x \in S: xU \subseteq X \text{ for some } U \in \mathfrak{A}\}$. We show Y is the \mathfrak{A}^* -interior of X . If $xV \subseteq X$ for some $V \in \mathfrak{A}^*$, then, since \mathfrak{A}^* is topological, there is $W \in \mathfrak{A}^*$ such that $yW \subseteq xV \subseteq X$ for every $y \in xW$. Therefore $xW \subseteq Y$ for some $W \in \mathfrak{A}^*$ if $x \in Y$. This implies $Y \in \mathcal{T}(\mathfrak{A}^*)$. If $Z \subseteq X$ and $Z \in \mathcal{T}(\mathfrak{A}^*)$ then $Z \subseteq Y$, hence, Y is the \mathfrak{A}^* -interior of X . \mathfrak{A} -Int $X = \mathfrak{A}^*$ -Int X by (4.2) and (4.4), thus \mathfrak{A} -Int $X = \{x \in S: xU \subseteq X \text{ for some } U \in \mathfrak{A}\}$. (2) obviously implies (3). (3) \Rightarrow (1): If \mathfrak{A} is a prenet then \mathfrak{A}^* is a net. Therefore, it is sufficient to show that \mathfrak{A}^* is topological. If $x \in S$ and $U \in \mathfrak{A}^*$ then, since $x \in \mathfrak{A}$ -Int (xU) , there exists $Y \in \mathcal{T}(\mathfrak{A})$ such that $x \in Y \subseteq xU$. For each $y \in Y$, there is $U(y) \in \mathfrak{A}$ such that $yU(y) \subseteq Y$. Define V by $yV = yU(y)$ if $y \in Y$ and $yV = yU$ otherwise. Then $V \in \mathfrak{A}^*$ and $xV^2 \subseteq xU$. Hence \mathfrak{A}^* is topological.

(4.5) If $\mathcal{T}(\mathfrak{B}) \subseteq \mathcal{T}(\mathfrak{A})$ and \mathfrak{B} is a pretopology then \mathfrak{A} is finer than \mathfrak{B} .

Proof. If $U \in \mathfrak{B}$ then, by the interior theorem, the connector $V: x \rightarrow \mathfrak{B}$ -Int (xU) is a \mathfrak{B} -open connector. V is also \mathfrak{A} -open because $\mathcal{T}(\mathfrak{B}) \subseteq \mathcal{T}(\mathfrak{A})$. $V \subseteq U$ and $V \in \mathfrak{A}^{* <}$, hence $U \in \mathfrak{A}^{* <}$.

BASIS THEOREM. \mathcal{T} is a topology. Then, the set of all \mathcal{T} -open connectors is a basis of \mathcal{T} .

Proof. A mapping $V: x \in S \rightarrow \mathcal{T}\text{-Int}(xU)$ is a connector if $U \in \mathcal{T}$. V is \mathcal{T} -open and $V \leq U$. (4.3) implies $V \in \mathcal{T}$. Hence, the set of all \mathcal{T} -open connectors is a basis of \mathcal{T} .

COMPARISON THEOREM. \mathfrak{A} and \mathfrak{B} are topologies. $\mathfrak{B} \subseteq \mathfrak{A}$ if and only if $\mathcal{T}(\mathfrak{B}) \subseteq \mathcal{T}(\mathfrak{A})$.

Proof. \mathfrak{A} is finer than \mathfrak{B} if and only if $\mathfrak{B} \subseteq \mathfrak{A}$. Hence, (4.1) implies the one direction and (4.5) implies the other.

TOPOLOGY THEOREM. \mathcal{T} is an open topology on S . There is a unique topology \mathcal{T} such that $\mathcal{T} = \mathcal{T}(\mathcal{T})$.

Proof. Let $\mathfrak{A} = \{U; xU \in \mathcal{T} \text{ for every } x \in S\}$. U is a connector and $x \in S$. Define V by $yV = xU$ if $y \in xU$ and $yV = S$ otherwise. Then, $xV^2 = xU$ and V belongs to \mathfrak{A} if U does. Therefore, \mathfrak{A} is topological. \mathfrak{A} is a sharp net because $\mathfrak{A} = \mathfrak{A}^\times = \mathfrak{A}^*$. Let $\mathcal{T} = \mathfrak{A}^<$. Then $\mathcal{T}(\mathcal{T}) = \mathcal{T}(\mathfrak{A})$ and \mathcal{T} is, by (3.6), the topology hull of \mathfrak{A} . $\mathcal{T} = \mathcal{T}(\mathfrak{A})$ is clear by the definition of \mathfrak{A} , hence, $\mathcal{T} = \mathcal{T}(\mathcal{T})$. The comparison theorem implies the uniqueness.

5. Uniformities. A connector system \mathfrak{A} is called *uniform* if for each $U \in \mathfrak{A}$, there is $V \in \mathfrak{A}$ such that $VV^{-1} \leq U$. If \mathfrak{A} is uniform, then for each $U \in \mathfrak{A}$, there are $V, W \in \mathfrak{A}$ such that $WW^{-1} \leq V$ and $VV^{-1} \leq U$. $W^{-1} \leq V$ implies $W \leq V^{-1}$, thus $W^2 \leq VV^{-1} \leq U$.

(5.1) Every uniform system is topological.

(5.2) If \mathfrak{A} is uniform, then $\mathfrak{A}^<$ and \mathfrak{A}^\times are uniform.

(5.3) If $\mathfrak{A}_\lambda, \lambda \in \Lambda$ are uniform then $\cup \mathfrak{A}_\lambda$ is uniform.

A *uniformity* is a filter which is uniform. ([1], [2]). If \mathfrak{U} is the weakest among the uniformities stronger than \mathfrak{A} , then \mathfrak{U} is called the *uniformity hull* of \mathfrak{A} .

(5.4) The filter hull $\mathfrak{A}^{\times<}$ is the uniformity hull of \mathfrak{A} if \mathfrak{A} is uniform.

(5.5) \mathfrak{U} is a uniformity. Then, a basis \mathfrak{B} of \mathfrak{U} is a uniform net and \mathfrak{U} is the uniformity hull of \mathfrak{B} .

(5.6) If \mathfrak{A} is a uniform net then $\mathfrak{A}^<$ is the uniformity hull and \mathfrak{A} is a basis of $\mathfrak{A}^<$.

Proof. Refer to (1.16), (5.4) and (2.7).

(5.7) If \mathfrak{U}_λ is the uniformity hull of $\mathfrak{A}_\lambda, \lambda \in \Lambda$, then $\vee \mathfrak{U}_\lambda$ is the uniformity-hull of $\cup \mathfrak{A}_\lambda$.

Proof. (5.3) and (5.4) $\Rightarrow \vee \mathbb{U}_\lambda$ is a uniformity. If \mathbb{U} is a uniformity and \mathbb{U} includes all \mathbb{U}_λ , $\lambda \in \Lambda$ then \mathbb{U} includes all \mathbb{U}_λ , $\lambda \in \Lambda$ and $\mathbb{U} = \mathbb{U}^{<}$. Hence, \mathbb{U} includes $(\cup \mathbb{U}_\lambda)^{<} = \vee \mathbb{U}_\lambda$.

\mathfrak{U} is a connector system. If there is a uniformity which is the strongest among the uniformities weaker than \mathfrak{U} then the uniformity is called the *uniformity kernel* of \mathfrak{U} . A uniformity kernel is unique if it exists.

(5.8) Every filter has a uniformity kernel.

Proof. Every filter includes the uniformity which contains the only connector, $x \in S \rightarrow S$ (for every $x \in S$). Let $\{\mathbb{U}: \mathbb{U} \subseteq \mathfrak{F}\}$ denote the collection of uniformities weaker than a filter \mathfrak{F} . (5.7) $\Rightarrow \vee \{\mathbb{U}: \mathbb{U} \subseteq \mathfrak{F}\}$ is a uniformity. (2.17) states $\vee \{\mathbb{U}: \mathbb{U} \subseteq \mathfrak{F}\} = (\cup \{\mathbb{U}: \mathbb{U} \subseteq \mathfrak{F}\})^{<} \subseteq \mathfrak{F}^{<} = \mathfrak{F}$. Therefore, the uniformity is included in \mathfrak{F} and the strongest among $\{\mathbb{U}: \mathbb{U} \subseteq \mathfrak{F}\}$.

The topology hull of a uniformity \mathbb{U} is \mathbb{U}^* because \mathbb{U} is a topological filter. $\mathcal{T}(\mathbb{U}^*)$ is a uniform topology (called the induced topology in [1] and [2]). Therefore, a topology \mathfrak{T} corresponds to a uniform topology if and only if there is a uniformity \mathbb{U} such that $\mathfrak{T} = \mathbb{U}^*$.

(5.9) \mathfrak{T} is a topology. There is a uniformity \mathbb{U} such that $\mathfrak{T} = \mathbb{U}^*$ if and only if \mathfrak{T} is the topology hull of its uniformity kernel.

Proof. Since a topology is a filter, a topology \mathfrak{T} has the uniform kernel \mathfrak{B} . If $\mathfrak{T} = \mathbb{U}^*$ for some uniformity \mathbb{U} then $\mathbb{U} \subseteq \mathfrak{B} \subseteq \mathfrak{T}$. Hence $\mathfrak{T} = \mathfrak{B}^*$.

(5.10) $\mathfrak{T}_\lambda, \lambda \in \Lambda$ are topologies and $\mathbb{U}_\lambda, \lambda \in \Lambda$ are uniformities. If $\mathfrak{T}_\lambda = \mathbb{U}_\lambda^*$ for each $\lambda \in \Lambda$, then there is a uniformity \mathbb{U} such that $(\vee \mathfrak{T}_\lambda)^* = \mathbb{U}^*$.

Proof. $(\vee \mathbb{U}_\lambda)^* = (\vee \mathbb{U}_\lambda^*)^* = (\vee \mathfrak{T}_\lambda)^*$ by (2.22). $\vee \mathbb{U}_\lambda$ is a uniformity by (5.7).

6. Mappings. Let R be a space and let U be a connector on R . M is a mapping from a space S to R . xM and $(xM)U$ denote the image of x by M and the image of $xM \in R$ by U respectively. Define a connector on S by corresponding x of S to the set $\{y \in S: yM \in (xM)U\}$. This connector is denoted by MUM^{-1} . $x(MUM^{-1})$ is the image of x by MUM^{-1} . The following formulas are found in [2].

$$(6.1) \quad M(U \cap V)M^{-1} = MUM^{-1} \cap MVM^{-1}.$$

$$(6.2) \quad U \leq V \text{ implies } MUM^{-1} \leq MVM^{-1}.$$

$$(6.3) \quad (MUM^{-1})^{-1} = M(U^{-1})M^{-1}.$$

$$(6.4) \quad (MUM^{-1})(MVM^{-1}) \leq MUV M^{-1}.$$

Each connector system \mathfrak{D} on R can be used to construct a connector system on S by a transfer of each $U \in \mathfrak{D}$ to the connector MUM^{-1} on S . $M\mathfrak{D}M^{-1}$ denotes the connector system $\{MUM^{-1}: U \in \mathfrak{D}\}$. The following propositions are proved by referring to (6.1), (6.2), (6.3) and (6.4).

$$(6.5) \quad M\mathfrak{D}^<M^{-1} \subseteq (M\mathfrak{D}M^{-1})^< \text{ and } (M\mathfrak{D}^<M^{-1})^< = (M\mathfrak{D}M^{-1})^<.$$

$$(6.6) \quad M\mathfrak{D}^*M^{-1} \subseteq (M\mathfrak{D}M^{-1})^* \text{ and } (M\mathfrak{D}^*M^{-1})^* = (M\mathfrak{D}M^{-1})^*.$$

$$(6.7) \quad M\mathfrak{D}^\times M^{-1} = (M\mathfrak{D}M^{-1})^\times.$$

$$(6.8) \quad \text{If } \mathfrak{D} \text{ is a net, then } M\mathfrak{D}M^{-1} \text{ is a net.}$$

$$(6.9) \quad \text{If } \mathfrak{D} \text{ is a prenet, then } M\mathfrak{D}M^{-1} \text{ is a prenet.}$$

Proof. Refer to (6.6), (1.23) and (6.8).

$$(6.10) \quad \text{If } \mathfrak{D} \text{ is topological then } M\mathfrak{D}M^{-1} \text{ is topological.}$$

$$(6.11) \quad \text{If } \mathfrak{D} \text{ is uniform then } M\mathfrak{D}M^{-1} \text{ is uniform.}$$

$$(6.12) \quad \text{If } \mathfrak{D} \text{ is a pretopology then } M\mathfrak{D}M^{-1} \text{ is a pretopology.}$$

Proof. $(M\mathfrak{D}^*M^{-1})^*$ is a topological net by (6.8), (6.10), (1.23) and (3.1), if \mathfrak{D} is a pretopology. Hence, by (6.6), $(M\mathfrak{D}M^{-1})^*$ is a topological net.

(6.13) \mathfrak{D} and \mathfrak{E} are connector systems on R . $M\mathfrak{D}M^{-1}$ is stronger than $M\mathfrak{E}M^{-1}$ if \mathfrak{D} is stronger than \mathfrak{E} .

$$(6.14) \quad M\mathfrak{D}M^{-1} \text{ is finer than } M\mathfrak{E}M^{-1} \text{ if } \mathfrak{D} \text{ is finer than } \mathfrak{E}.$$

YM^{-1} denotes the inverse image of a set $Y \subseteq R$ by a mapping M .

(6.15) \mathfrak{D} is a prenet on R and $\mathcal{T}(\mathfrak{D})$ is the open-topology induced by \mathfrak{D} . $\mathcal{T}(M\mathfrak{D}M^{-1})$ is the open-topology on S induced by $M\mathfrak{D}M^{-1}$. Then, $\{YM^{-1}: Y \in \mathcal{T}(\mathfrak{D})\} \subseteq \mathcal{T}(M\mathfrak{D}M^{-1})$.

$$(6.16) \quad \{YM^{-1}: Y \in \mathcal{T}(\mathfrak{D})\} = \mathcal{T}(M\mathfrak{D}M^{-1}) \text{ if } \mathfrak{D} \text{ is a pretopology.}$$

Proof. If $X \in \mathcal{T}(M\mathfrak{D}M^{-1})$ and $x \in X$, then there is $U(x) \in \mathfrak{D}$ such that $x(MU(x)M^{-1}) \subseteq X$. According to the interior theorem, since \mathfrak{D} is a pretopology, $y = xM$ belongs to $\mathfrak{D}\text{-Int}(yU(x)) \in \mathcal{T}(\mathfrak{D})$, and $\mathfrak{D}\text{-Int}(yU(x)) \subseteq yU(x)$. Let $Y = \cup \{\mathfrak{D}\text{-Int}(yU(x)): y = xM, x \in X\}$. Then $Y \in \mathcal{T}(\mathfrak{D})$ and $X \subseteq YM^{-1} \subseteq \cup \{(yU(x))M^{-1}: y = xM, x \in X\} \subseteq X$. Hence, $X = YM^{-1}$.

\mathfrak{A} is a connector system on S and \mathfrak{D} is a connector system on R . A mapping M from S to R is called \mathfrak{A} -continuous w.r.t. \mathfrak{D} if \mathfrak{A} is finer than $M\mathfrak{D}M^{-1}$. This is a generalization of continuity. Neither connector system needs to be a topology.

(6.17) M is \mathfrak{A} -continuous w.r.t. \mathfrak{D} if and only if for each $U \in \mathfrak{D}$ and for each $x \in S$, there are $V_i \in \mathfrak{A}$, $i = 1, 2, 3, \dots, n$ such that $\cap \{xV_i: i = 1, 2, 3, \dots, n\} \subseteq x(MUM^{-1})$.

(6.18) \mathfrak{A} is a prenet. M is \mathfrak{A} -continuous w.r.t. \mathfrak{D} if and only if for each $U \in \mathfrak{D}$ and for each $x \in S$, there is $V \in \mathfrak{A}$ such that $(xV)M \subseteq (xU)M$.

CONTINUITY THEOREM. \mathfrak{A} and \mathfrak{D} are connector systems on S and R respectively. \mathfrak{A} is a prenet and \mathfrak{D} is a pretopology. M is a mapping from S to R . M is \mathfrak{A} -continuous w.r.t. \mathfrak{D} if and only if $\{YM^{-1}: Y \in \mathcal{T}(\mathfrak{D})\} \subseteq \mathcal{T}(\mathfrak{A})$.

Proof. $M\mathfrak{D}M^{-1}$ is a pretopology if \mathfrak{D} is a pretopology. (4.1) and (4.5) \Rightarrow a prenet \mathfrak{A} is finer than a pretopology $M\mathfrak{D}M^{-1}$ if and only if $\mathcal{T}(M\mathfrak{D}M^{-1}) \subseteq \mathcal{T}(\mathfrak{A})$. Hence, by (6.16), \mathfrak{A} is finer than $M\mathfrak{D}M^{-1}$ if and only if $\{YM^{-1}: Y \in \mathcal{T}(\mathfrak{D})\} \subseteq \mathcal{T}(\mathfrak{A})$.

Comment on the continuity theorem: An implication of the theorem is a compatibility of continuous mappings and continuous functions from a topological space $(S, \mathcal{T}(\mathfrak{A}))$ to a topological space $(R, \mathcal{T}(\mathfrak{D}))$. The theorem states that if \mathfrak{A} is a prenet and \mathfrak{D} is a pretopology, then a continuous mapping is topologically continuous and vice versa.

\mathfrak{A} is a connector system on S and \mathfrak{D} is a connector system on R . A mapping M from S to R is called *uniformly \mathfrak{A} -continuous* w.r.t. \mathfrak{D} if \mathfrak{A} is stronger than $M\mathfrak{D}M^{-1}$. \mathfrak{A} is stronger, then \mathfrak{A} is finer, therefore, M is \mathfrak{A} -continuous if M is uniformly \mathfrak{A} -continuous. (2.6) implies the converse if \mathfrak{A} is a sharp prenet.

(6.19) M is uniformly \mathfrak{A} -continuous w.r.t. \mathfrak{D} if and only if for each $U \in \mathfrak{D}$, there are $V_i \in \mathfrak{A}$, $i = 1, 2, 3, \dots, n$, such that $\cap V_i \leq MUM^{-1}$.

(6.20) \mathfrak{A} is a sharp prenet. M is \mathfrak{A} -continuous w.r.t. \mathfrak{D} if and only if M is uniformly \mathfrak{A} -continuous w.r.t. \mathfrak{D} .

KERNEL THEOREM. \mathfrak{T} is a topology on S , \mathfrak{U} is the uniformity kernel of \mathfrak{T} and \mathfrak{D} is a uniform net on R . If a mapping M from S to R is \mathfrak{T} -continuous w.r.t. \mathfrak{D} , then M is uniformly \mathfrak{U} -continuous w.r.t. \mathfrak{D} .

Proof. The continuity of M implies $M\mathfrak{D}M^{-1} \subseteq \mathfrak{T}$. $M\mathfrak{D}M^{-1}$ is a uniform net by (6.8) and (6.11), if \mathfrak{D} is a uniform net. Then, $(M\mathfrak{D}M^{-1})^<$ is the uniformity hull of $M\mathfrak{D}M^{-1}$, and $M\mathfrak{D}M^{-1} \subseteq (M\mathfrak{D}M^{-1})^< \subseteq \mathfrak{U} \subseteq \mathfrak{T}$. Hence, M is uniformly \mathfrak{U} -continuous w.r.t. \mathfrak{D} .

Comment on the kernel theorem: A uniform net is a pretopology by (3.12) and (5.1), and a topology is a prenet. Therefore, by the continuity theorem, the kernel theorem can be applied to a continuous function from a topological space $(S, \mathcal{T}(\mathfrak{T}))$ to a topological space $(R, \mathcal{T}(\mathfrak{D}))$. (5.6) states that $\mathfrak{D}^<$ is the uniformity hull of a uniform net \mathfrak{D} . Since $\mathcal{T}(\mathfrak{D}) = \mathcal{T}(\mathfrak{D}^<)$, $\mathcal{T}(\mathfrak{D})$ is a topology induced by a uniformity. The kernel theorem presents an answer to the following question in a generalized form. If S is a topological space and R is a uniform-topological space, then what is a uniformity on S , for which every continuous function from S to R is uniformly continuous?

7. Mappings. Let $R_\lambda, \lambda \in \Lambda$ be a system of spaces and let \mathcal{D}_λ be a connector system on R_λ for each $\lambda \in \Lambda$. If a topology hull \mathfrak{T} of $\bigcup \{M_\lambda \mathcal{D}_\lambda M_\lambda^{-1} : \lambda \in \Lambda\}$ exists, then \mathfrak{T} is called the weak topology on S by $\{M_\lambda : \lambda \in \Lambda\}$, w.r.t. $\{\mathcal{D}_\lambda : \lambda \in \Lambda\}$. \mathfrak{T} is the weakest topology among those for which $M_\lambda, \lambda \in \Lambda$ are continuous.

(7.1) \mathfrak{T} is the weak topology by $\{M_\lambda : \lambda \in \Lambda\}$, w.r.t. $\{\mathcal{D}_\lambda : \lambda \in \Lambda\}$. If $\mathcal{D}_\lambda \subseteq \mathcal{E}_\lambda \subseteq \mathcal{D}_\lambda^{**<}$ for every $\lambda \in \Lambda$, then \mathfrak{T} is the weak topology by those mappings w.r.t. $\{\mathcal{E}_\lambda : \lambda \in \Lambda\}$.

Proof. Refer to (1.5), (1.9), (1.22), (6.5), (6.6) and (6.7).

(7.2) If $\mathcal{D}_\lambda, \lambda \in \Lambda$ are all topological or all of them are pre-topologies, then there exists a weak topology by every system of mappings, w.r.t. those connector systems.

Proof. Refer to (6.10), (6.12), (3.3) and (3.13).

WEAK TOPOLOGY THEOREM. \mathfrak{T} is a topology on S . \mathcal{D}_λ is a connector system on R_λ and M_λ is a \mathfrak{T} -continuous mapping, w.r.t. \mathcal{D}_λ from S to R_λ , for each $\lambda \in \Lambda$. \mathfrak{T} is the weak topology on S by those mappings, w.r.t. the given connector systems if for each $U \in \mathfrak{T}$ and for each $x \in S$, there is $\lambda \in \Lambda$ and $V \in \mathcal{D}_\lambda$ such that $x(M_\lambda V M_\lambda^{-1}) \subseteq xU$.

Proof. The hypotheses are: $M_\lambda \mathcal{D}_\lambda M_\lambda^{-1} \subseteq \mathfrak{T}$ for every $\lambda \in \Lambda$ and

$$\mathfrak{T} \subseteq (\bigcup M_\lambda \mathcal{D}_\lambda M_\lambda^{-1})^{**<}.$$

$$(\bigcup M_\lambda \mathcal{D}_\lambda M_\lambda^{-1})^{**<} \subseteq (\bigcup M_\lambda \mathcal{D}_\lambda M_\lambda^{-1})^{x**<} \subseteq \mathfrak{T}^{x**<} = \mathfrak{T}.$$

Hence, \mathfrak{T} is the topology hull of $\bigcup M_\lambda \mathcal{D}_\lambda M_\lambda^{-1}$.

If there exists a uniformity hull of $\bigcup M_\lambda \mathcal{D}_\lambda M_\lambda^{-1}$, then the uniformity hull is called the *weak uniformity* by those mappings, w.r.t. the given connector systems. A weak uniformity is the weakest uniformity among those for which each M_λ is uniformly continuous w.r.t. \mathcal{D}_λ .

(7.3) \mathfrak{U} is a weak uniformity w.r.t. $\{\mathcal{D}_\lambda : \lambda \in \Lambda\}$. If $\mathcal{D}_\lambda \subseteq \mathcal{E}_\lambda \subseteq \mathcal{D}_\lambda^{x*}$ for each $\lambda \in \Lambda$, then \mathfrak{U} is the weak uniformity by the same mappings, w.r.t. $\{\mathcal{E}_\lambda : \lambda \in \Lambda\}$.

WEAK UNIFORMITY THEOREM. \mathcal{D}_λ is a unifrom connector system on R_λ and M_λ is a mapping from S to R_λ for each $\lambda \in \Lambda$. Then there exists a weak uniformity \mathfrak{U} by $\{M_\lambda : \lambda \in \Lambda\}$, w.r.t. $\{\mathcal{D}_\lambda : \lambda \in \Lambda\}$, and the topology hull of \mathfrak{U} is the weak topology by those mappings, w.r.t. the given connector systems.

Proof. Let $\mathfrak{B} = \cup M_\lambda \mathfrak{D}_\lambda M_\lambda^{-1}$ and let $\mathfrak{U} = \mathfrak{B}^{* <}$. \mathfrak{B} is uniform and \mathfrak{U} is the uniformity hull by (6.11), (5.3) and (5.4). $\mathfrak{U}^{* \# <}$ is the topology hull of \mathfrak{B} by (3.3) and (3.4), as well as that of \mathfrak{U} . Hence the topology hull of \mathfrak{U} is the weak topology by $\{M_\lambda : \lambda \in \Lambda\}$.

Let $M_\lambda, \lambda \in \Lambda$ be mapping from S to R and let U be a connector on R . Let $\cap M_\lambda U M_\lambda^{-1}$ denote the connector, $x \in S \rightarrow \cap \{(yU)M_\lambda^{-1} : y = xM_\lambda, \lambda \in \Lambda\}$.

(7.4) U and V are connectors on R and W is the product of $\cap M_\lambda U M_\lambda^{-1}$ and $\cap M_\lambda V M_\lambda^{-1}$. Then $W \leq \cap M_\lambda U V M_\lambda^{-1}$.

(7.5) $\{\cap M_\lambda U M_\lambda^{-1} : U \in \mathfrak{D}^<\} \subseteq \{\cap M_\lambda U M_\lambda^{-1} : U \in \mathfrak{D}\}^<$.

(7.6) $\{\cap M_\lambda U M_\lambda^{-1} : U \in \mathfrak{D}^*\} = \{\cap M_\lambda U M_\lambda^{-1} : U \in \mathfrak{D}\}^*$.

(7.7) If \mathfrak{D} is topological then $\{\cap M_\lambda U M_\lambda^{-1} : U \in \mathfrak{D}\}$ is topological.

(7.8) If \mathfrak{D} is uniform then $\{\cap M_\lambda U M_\lambda^{-1} : U \in \mathfrak{D}\}$ is uniform.

\mathfrak{A} is a connector system on S and \mathfrak{D} is a connector system on R . A system of mappings $M_\lambda, \lambda \in \Lambda$ from S to R is called \mathfrak{A} -*equi-continuous* w.r.t. \mathfrak{D} if \mathfrak{A} is finer than $\{\cap M_\lambda U M_\lambda^{-1} : U \in \mathfrak{D}\}$, i.e., the latter is included in $\mathfrak{A}^{* \# <}$. The mappings are called *uniformly* \mathfrak{A} -*equi-continuous* if \mathfrak{A} is stronger than $\{\cap M_\lambda U M_\lambda^{-1} : U \in \mathfrak{D}\}$, i.e., the latter is included in $\mathfrak{A}^{* <}$. The uniformly equi-continuity implies uniform continuity of each mapping in the system. If there exists a topology hull of $\{\cap M_\lambda U M_\lambda^{-1} : U \in \mathfrak{D}\}$ then the topology hull is called the *equi-topology* by $\{M_\lambda : \lambda \in \Lambda\}$.

(7.9) \mathfrak{D} and \mathfrak{E} are connector systems on R and $\mathfrak{D} \subseteq \mathfrak{E} \subseteq \mathfrak{D}^{* <}$. If \mathfrak{T} is an equi-topology by a system of mappings to R , w.r.t. \mathfrak{D} then \mathfrak{T} is the equi-topology by the same mappings, w.r.t. \mathfrak{E} .

(7.10) $\{\cap M_\lambda U M_\lambda^{-1} : U \in \mathfrak{D}\}^{* \# <}$ is the equi-topology by $\{M_\lambda : \lambda \in \Lambda\}$, w.r.t. \mathfrak{D} if \mathfrak{D} is topological.

Proof. Refer to (7.7) and (3.3).

If there exists a uniformity hull of $\{\cap M_\lambda U M_\lambda^{-1} : U \in \mathfrak{D}\}$ then the uniformity hull is called the *equi-uniformity* by $\{M_\lambda : \lambda \in \Lambda\}$, w.r.t. \mathfrak{D} , i.e., it is the weakest uniformity for which the mappings are uniformly equi-continuous.

(7.11) \mathfrak{D} and \mathfrak{E} are connector systems on R and $\mathfrak{D} \subseteq \mathfrak{E} \subseteq \mathfrak{D}^{* <}$. If \mathfrak{U} is an equi-uniformity by a system of mappings w.r.t. \mathfrak{D} , then \mathfrak{U} is the equi-uniformity by the same mappings w.r.t. \mathfrak{E} .

(7.12) If \mathfrak{D} is uniform then $\{\cap M_\lambda U M_\lambda^{-1} : U \in \mathfrak{D}\}^{* <}$ is the equi-uniformity by $\{M_\lambda : \lambda \in \Lambda\}$, w.r.t. \mathfrak{D} and the equi-topology is the topology hull of the equi-uniformity.

Proof. Refer to (7.8), (5.4), (3.5) and (7.10).

8. Uniformizable topologies. The set of all real numbers is denoted by $(-\infty, \infty)$. For each positive ϵ , let $U(\epsilon)$ denote the connector on $(-\infty, \infty)$ such that $xU(\epsilon) = \{y: |x - y| \leq \epsilon\}$ for every $x \in (-\infty, \infty)$. Then the connector system $\{U(\epsilon): 0 < \epsilon\}$ is a uniform net. This is the only connector system, in this paper, we deal with for the real numbers. $\mathbb{U}[a, b]$ denotes the connector system $\mathbb{U}(-\infty, \infty)$ if each connector is restricted on a closed interval $[a, b]$.

f is a function from a space S to $(-\infty, \infty)$. We write $a \leq f \leq b$ if the image of f is bounded by a and b . A topology \mathfrak{T} on S is called *completely regular* if for each $U \in \mathfrak{T}$ and for each $x \in S$, there is a \mathfrak{T} -continuous function f from S to $[0, 1]$, w.r.t. $\mathbb{U}[0, 1]$ such that $xf = 1$ and $yf = 0$ if y does not belong to xU .

The above definition is obviously compatible with the definition of a completely regular topology on the topological space $(S, \mathcal{T}(\mathfrak{T}))$.

(8.1) If \mathfrak{T} is completely regular then \mathfrak{T} is a weak topology by a system of functions.

Proof. Refer to the weak topology theorem in §7.

(8.2) A weak topology by functions, w.r.t. $\mathbb{U}(-\infty, \infty)$ is completely regular.

UNIFORMIZATION THEOREM. A topology is the topology hull of a uniformity if and only if it is completely regular.

Proof. If \mathfrak{T} is completely regular then \mathfrak{T} is a weak topology and by the weak uniformity theorem in §7, \mathfrak{T} is the topology hull of the weak uniformity by the same functions which induce the weak topology. Conversely, if \mathfrak{T} is the topology hull of a uniformity \mathbb{U} then, since $\mathfrak{T} = \mathbb{U}^*$, for each $U \in \mathfrak{T}$ and for each $x \in S$, there is $V \in \mathbb{U}$ such that $xU = xV$. Theorem 5 in §31 of [1] and Theorem 19.1 of [2] state that there is a uniformly continuous function f such that $0 \leq f \leq 1$, $xf = 1$ and $yf = 0$ if y does not belong to xV .

9. Bounded connectors. A connector U on a space S is called *bounded* if there are $x_i \in S$, $i = 1, 2, 3, \dots, n$ such that $S = \bigcup \{x_i U: i = 1, 2, 3, \dots, n\}$. A connector U is called *absolutely bounded* if for each nonempty set $X \subseteq S$, there are $x_i \in X$, $i = 1, 2, 3, \dots, n$ such that $X \subseteq \bigcup \{x_i U: i = 1, 2, 3, \dots, n\}$. If $U \leq V$ and U is bounded or absolutely bounded, then V is bounded or absolutely bounded respectively.

A connector system \mathfrak{U} on S is called *totally bounded* if every connector in \mathfrak{U} is bounded. This is a generalization of totally bounded uniformities.

(9.1) If a connector system \mathfrak{U} is uniform and totally bounded then every connector in \mathfrak{U} is absolutely bounded.

Proof. For each $U \in \mathfrak{U}$, since \mathfrak{U} is uniform, there is $V \in \mathfrak{U}$ such that $V^{-1}V \subseteq U$. Let X be a nonempty subset of S . There are $x_i \in S$, $i = 1, 2, 3, \dots, n$ such that X is included in the union of $x_i V$, $i = 1, 2, 3, \dots, n$ and $X \cap x_i V$ is nonempty for every $i = 1, 2, 3, \dots, n$. If $y_i \in X \cap x_i V$, $i = 1, 2, 3, \dots, n$, then, $x_i V \subseteq y_i V^{-1}V \subseteq y_i U$ thus X is included in the union of $y_i U$, $i = 1, 2, 3, \dots, n$ and each y_i belongs to X . Hence, \mathfrak{U} is absolutely bounded.

A topology \mathfrak{T} on S is called *compact* if the open-topology $\mathcal{T}(\mathfrak{T})$ is compact (in usual sense), i.e., every open covering of S has a finite subcovering.

(9.2) A topology \mathfrak{T} is compact if and only if \mathfrak{T} is totally bounded.

Proof. The basis theorem in §4 states that there is an open connector $V \in \mathfrak{T}$ for each $U \in \mathfrak{T}$, such that $V \subseteq U$. If \mathfrak{T} is compact then, since all xV are \mathfrak{T} -open, V is bounded. Hence, \mathfrak{T} is totally bounded if \mathfrak{T} is compact. Conversely, suppose $S = \bigcup \{X_\lambda : \lambda \in \Lambda\}$ for some $X_\lambda \in \mathcal{T}(\mathfrak{T})$, $\lambda \in \Lambda$. Correspond each $x \in S$ to one of these X_λ which contains x and define a connector U . Then U is \mathfrak{T} -open and $U \in \mathfrak{T}^{*c} = \mathfrak{T}$. U is bounded if \mathfrak{T} is totally bounded. Therefore, there are $x_i \in S$, $i = 1, 2, 3, \dots, n$ such that S is the union of $x_i U$, $i = 1, 2, 3, \dots, n$, which is a finite union of some X_λ , $\lambda \in \Lambda$.

Comment on (9.2). (9.2) is a generalization of a theorem on uniform spaces. The open-topology $\mathcal{T}(\mathfrak{T})$ in (9.2) is not necessarily a uniform topology. In fact, the theorem is a characterization of the compact topologies because each open-topology is induced by a topology.

COMPACT TOPOLOGY THEOREM. If the topology hull \mathfrak{U}^* of a uniformity \mathfrak{U} is compact then \mathfrak{U} is the uniformity kernel of \mathfrak{U}^* .

Proof. First, we prove that a real valued \mathfrak{U} -continuous function on S is uniformly continuous if \mathfrak{U}^* is compact. By the definition of continuity, for each positive ϵ and $x \in S$, there is $U \in \mathfrak{U}$ such that $|xf - yf| < \epsilon/2$ if $y \in xU$. There is $V \in U$ such that $V^2 \subseteq U$, and $|xf - yf| < \epsilon/2$ if $y \in xV^2$. By corresponding xV to x , we obtain a connector $W \in \mathfrak{U}^*$. Since \mathfrak{U}^* is totally bounded by (9.2), there are $x_i \in S$, $i = 1, 2, 3, \dots, n$ such that S is included in the union of $x_i W$, $i = 1, 2, 3, \dots, n$. By the definition of W , there are corresponding $V_i \in \mathfrak{U}$, $i = 1, 2, 3, \dots, n$ such that $x_i W = x_i V_i$, and $|xf - yf| < \epsilon/2$ if

$y \in x_i V_i^2$. Let $V = \cap V_i$. Then $|xf - yf| < \epsilon$ whenever $y \in xV$. Hence, f is uniformly \mathfrak{U} -continuous. Let \mathfrak{U} be the uniformity kernel of \mathfrak{U}^* . \mathfrak{U}^* is totally bounded and $\mathfrak{U} \subseteq \mathfrak{U}^*$, thus \mathfrak{U} is totally bounded. Every uniformly \mathfrak{U} -continuous function is \mathfrak{U} -continuous, thus it is uniformly \mathfrak{U} -continuous. Therefore, $\mathfrak{U} \subseteq \mathfrak{U}$ by Theorem 6 in §33 of [1]. $\mathfrak{U} \subseteq \mathfrak{U}$ always holds because \mathfrak{U} is the strongest uniformity included in \mathfrak{U}^* . Hence, \mathfrak{U} is the uniformity kernel of \mathfrak{U}^* .

10. Semi-bounded connectors. A connector U on a space S is called *semi-bounded* if there is a positive integer n such that U^n is bounded. U^n is defined by induction, i.e., $U^n = U^{n-1}U$. A connector U is called *absolutely semi-bounded* if for each nonempty set $X \subseteq S$, there are $x_i \in X$, $i = 1, 2, 3, \dots, m$ and a positive integer n such that $X \subseteq \cup \{x_i U^n : i = 1, 2, 3, \dots, m\}$. If $U \leq V$ and U is semi-bounded or absolutely semi-bounded, then V is semi-bounded or absolutely semi-bounded respectively.

A connector system \mathfrak{U} is called bounded if every connector in \mathfrak{U} is semi-bounded.

(10.1) A uniformity \mathfrak{U} is bounded if and only if every uniformly \mathfrak{U} -continuous function is bounded.

Proof. Refer to Theorem 2 in §32 of [1].

A connector system \mathfrak{U} is called *absolutely bounded* if every connector in \mathfrak{U} is absolutely semi-bounded.

(10.2) A uniformity \mathfrak{U} is absolutely bounded if and only if \mathfrak{U} is totally bounded.

Proof. Refer to Theorem 2 in §33 of [1].

A topology \mathfrak{T} on a space S is called *pseudo-compact* if every \mathfrak{T} -continuous function (real-valued) is bounded.

PSEUDO-COMPACT TOPOLOGY THEOREM. A topology \mathfrak{T} is pseudo-compact if and only if the uniformity kernel \mathfrak{U} of \mathfrak{T} is bounded.

Proof. If \mathfrak{U} is bounded then, since every \mathfrak{T} -continuous function is uniformly \mathfrak{U} -continuous by the kernel theorem in §6, \mathfrak{T} is pseudo-compact. On the other hand, every uniformly \mathfrak{U} -continuous function is \mathfrak{T} -continuous because $\mathfrak{U} \subseteq \mathfrak{T}$. Hence, every uniformly \mathfrak{U} -continuous function is bounded if \mathfrak{T} is pseudo-compact. By (10.1), \mathfrak{U} is bounded.

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