Pacific Journal of Mathematics

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Vol. 56, No. 1 November 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 56, No. 1, 1975

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The concept of Model-Completeness is defined in a first order language with a generalized quantifier. A necessary and sufficient condition is given for that Model-Completeness and its relation to categoricity is discussed.

Some results of this paper were obtained in the author's thesis [12] and were announced in [11]. They, together with other results of [12] were improved independently by the author and by S. Shelah. A suggestion of S. Shelah made some proofs simpler and due to it, better results were obtained in Theorem 1.5. The author wishes to thank S. Shelah for his remarks.

Let L be a first order language with equality and let $L(Q)$ be the language obtained from L by adding a new quantifier Q. Let α, β denote infinite cardinals. We define α-satisfaction for $L(Q)$ by interpreting Q as "there exist at least α elements". If a sentence ϕ of $L(Q)$ is α-satisfied in a model $\mathfrak{A}$ for L we write $\mathfrak{A} \models_\alpha \phi$ and we say that $\mathfrak{A}$ is an α-model for ϕ. Let $\mathfrak{A}, \mathfrak{B}$ be two models for L, $|\mathfrak{A}| \geqq \alpha$ and $\mathfrak{A} \subseteq \mathfrak{B}$. Write $\mathfrak{A} <_\alpha \mathfrak{B}$ if for every n, every formula $\phi(x_1, \cdots, x_n)$ in $L(Q)$ and every $a_1, \cdots, a_n$ in $\mathfrak{A}$: $\mathfrak{A} \models_\alpha \phi[a_1, \cdots, a_n]$ iff $\mathfrak{B} \models_\alpha \phi[a_1, \cdots, a_n]$. Let T be an ordinary first order theory (namely a theory in L) that has infinite models. Define $T(Q) = T \cup \{Qx[x = x]\}$. Call T α-model-complete if for every $\mathfrak{A}, \mathfrak{B}$ which are α-models for $T(Q)$ and $\mathfrak{A} \subseteq \mathfrak{B}$ also $\mathfrak{A} <_\alpha \mathfrak{B}$. A necessary and sufficient condition for T to be α-model-complete for $\alpha > \aleph_0$ is given in section 1.

Let T be as before. Define $T(\alpha) = \{\phi: \phi$ is a sentence in $L(Q)$ and for every $\mathfrak{A}$, if $\mathfrak{A} \models_\alpha T(Q)$ then $\mathfrak{A} \models_\alpha \phi\}$. Call T α-complete if for every sentence ϕ in $L(Q)$ either $\phi \in T(\alpha)$ or $\neg\phi \in T(\alpha)$. In §2, it is shown that if T is categorical in one uncountable power, it is α-complete and for every $\alpha \geqq \aleph_0$: $T(\alpha) = T(\aleph_0)$. If T is also model-complete (in the usual sense) then it is α-model-complete for every $\alpha \geqq \aleph_0$ and $T(\alpha)$ is decidable provided T is axiomatic.

1. α-Model-Completeness.

DEFINITION 1.1. Let $\phi(x, x_1, \cdots, x_m)$ be a formula in L such that $x, x_1, \cdots, x_m$ are exactly all its free variables. Let $\mathfrak{A}$ be a model for L and let $a_1, \cdots, a_m$ be elements in $\mathfrak{A}$. Define:

$$\phi(\mathfrak{A}, a_1, \cdots, a_m) = \{a : \mathfrak{A} \vDash \phi[a, a_1, \cdots, a_m]\}.$$

Let T be a theory in L and observe the following condition in which T is involved.

Condition 1.1. For every $m \geqq 0$ and every $\phi(x, x_1, \cdots, x_m)$ in L there exists an integer n_ϕ such that for every α-model $\mathfrak{A}$ of $T(Q)$ and every $a_1, \cdots, a_m$ in $\mathfrak{A}$: if $|\phi(\mathfrak{A}, a_1, \cdots a_m)| > n_\phi$ then $|\phi(\mathfrak{A}, a_1, \cdots, a_m)| \geqq \alpha$.

LEMMA 1.1. *Let T be a theory in L, $\alpha \geqq \aleph_0$. If T fulfills Condition* 1.1 *then for every formula ψ in $L(Q)$ there exists a formula ϕ in L such that $T(Q) \vDash_\alpha \psi \leftrightarrow \phi$.* (The meaning of the notation "$\vDash_\alpha$" is "semantically valid in α".)

Proof. Use induction on the structure of ψ. The lemma is true for formulae in L and it is clear that if it is true for ψ, ψ_1 in $L(Q)$ it is also true for $\neg\psi$, $\psi \wedge \psi_1$, $\exists v\psi$ (for every individual variable v). We now prove the lemma for $Qv\psi$ assuming it is true for ψ. Suppose ψ is $\psi(x, x_1, \cdots, x_m)$ and v is x. Let $\phi(x, x_1, \cdots, x_m)$ be a formula in L such that $T(Q) \vDash_\alpha \psi \leftrightarrow \phi$. Let n be an integer the existence of which is assumed in Condition 1.1. Let $\exists^{\geqq n+1} x\phi(x, x_1, \cdots, x_n)$ be a formula of L "saying" that there are at least $n+1$ different elements x such that $\phi(x, x_1, \cdots, x_m)$ (here we use the assumption that L contains the equality sign). It is easy to see that for every model $\mathfrak{A}$, if $\mathfrak{A}$ is a model of $T(Q)$ and $a_1, \cdots, a_m \in \mathfrak{A}$ then

$$\mathfrak{A} \vDash_\alpha Qx\psi(x, \alpha_1, \cdots, a_m) \leftrightarrow \exists^{\geqq n+1} x\phi(x, x_1, \cdots, x_m).$$

Hence

$$T(Q) \vDash_\alpha Qx\psi(x, x_1, \cdots, x_m) \leftrightarrow \exists^{\geqq n+1} x\phi(x, x_1, \cdots, x_m).$$

Therefore $\exists^{\geqq n+1} x\phi$ is the required formula for $Qx\psi$.

Note that Lemma 1.1 is true also when L is uncountable.

An Example. Let T be the first order theory of a dense linear ordering having neither first nor last element. Using the well known elimination of quantifiers (e.g. Kreisel and Krivine [6]) it is easy to see that T fulfills Condition 1.1 for $\alpha = \aleph_0$ but not for $\alpha > \aleph_0$.

Now again let T be a theory in L but suppose $\alpha > \aleph_0$ and observe the following condition involving T.

Condition 1.2. For every $m \geqq 0$, every formula $\phi(x, x_1, \cdots, x_m)$ in L, every α-model $\mathfrak{A}$ of $T(Q)$ and every $a_1, \cdots, a_m \in \mathfrak{A}$ either $|\phi(\mathfrak{A}, a_1, \cdots, a_m)| < \aleph_0$ or $|\phi(\mathfrak{A}, a_1, \cdots, a_m)| \geqq \alpha$.

The following lemma settles the relation between Condition 1.1 and Condition 2.2.

LEMMA 1.2. *Let T be a theory in a language L (possibly uncountable). Then for every $\alpha > |L|$ Condition* 1.1 *is equivalent to Condition* 1.2.

Proof. It is clear that if Condition 1.1 holds, then also Condition 1.2 holds. Choose any cardinal μ such that $2^{\mu} > |L|$. By Keisler **[4]** (Theorem 3.3 (iii), p. 121) if D is a regular ultra filter on μ there exist natural numbers n_{ν}, $\nu < \mu$, such that D-Prod $\lambda_{\nu} n_{\nu} = 2^{\mu}$. Suppose that Condition 1.2 holds but Condition 1.1 does not hold. Hence there exists a formula $\phi(x, x_1, \cdots, x_m)$ in L such that for every n_{ν}, $\nu < \mu$, it is possible to find an α-model $\mathfrak{A}_{\nu}$ of $T(Q)$ and elements $a_{\nu_1}, \cdots, a_{\nu_m}$ in $\mathfrak{A}_{\nu}$ such that $n_{\nu} < |\phi(\mathfrak{A}_{\nu}, a_{\nu_1}, \cdots, a_{\nu_m})| < \alpha$. Since Condition 1.2 holds we obtain: $n_{\nu} < |\phi(\mathfrak{A}_{\nu}, a_{\nu_1}, \cdots, a_{\nu_m})| < \aleph_0$. By Skolem-Lowenheim Theorem we are allowed to suppose that $|\mathfrak{A}_{\nu}| = 2^{2^{\mu}, \aleph_0}$ (where $2^{K,0} = K$, $2^{K,n+1} = 2^{2^{K,n}}$ and $2^{K,\aleph_0} = \sup\{2^{K,n} : n < \aleph_0\}$ for every infinite cardinal K). Observe now the structures $(\mathfrak{A}_{\nu}, \phi(\mathfrak{A}_{\nu}, a_{\nu_1}, \cdots, a_{\nu_m}))$ and take the ultra product D-Prod $\lambda_{\nu}(\mathfrak{A}_{\nu}, \phi(\mathfrak{A}_{\nu}, a_{\nu_1}, \cdots, a_{\nu_m}))$. Denote it by $(\mathfrak{B}, \phi(\mathfrak{B}, b_1, \cdots, b_m))$. Then:

$$|\mathfrak{B}| \geqq 2^{2^{\mu}, \mathrm{T}_0} > |\phi(\mathfrak{B}, b_1, \cdots, b_m)| = 2^{\mu} > |L|.$$

Therefore we can use Vaught **[10]** (the generalization of Corollary 4.2, p. 401). Hence, there exists an α-model $\mathfrak{C}$ of $T(Q)$ and elements $c_1, \cdots, c_m$ in $\mathfrak{C}$ such that $|\phi(\mathfrak{C}, c_1, \cdots, c_m)| = \aleph_0$, a contradiction to the assumption that Condition 1.2 holds.

In some applications it is simpler to deal with Condition 1.2 than with Condition 1.1, so there is also a practical purpose in Lemma 1.2.

LEMMA 1.3. *Let T be any first order theory, $\alpha \geqq \aleph_0$. If T is model-complete* (in the usual sense) *and T fulfills Condition* 1.1 *then T is α-model-complete.*

Proof. Use Lemma 1.1.

LEMMA 1.4. *Let T be a theory in L and suppose $\alpha > |L|$. If T is α-model-complete then T fulfills Condition* 1.1.

Proof. Suppose that T does not fulfill Condition 1.1. Then by Lemma 1.2 it also does not fulfill Condition 1.2. Therefore there exists an α-model $\mathfrak{A}$ of $T(Q)$, a formula $\phi(x, x_1, \cdots, x_m)$ in L and elements $a_1, \cdots, a_m$ in $\mathfrak{A}$ such that $\aleph_0 \leqq |\phi(\mathfrak{A}, a_1, \cdots, a_m)| < \alpha$. Let C be any set of power α such that C and the domain of $\mathfrak{A}$ are disjoint. Denote by $D(\mathfrak{A})$ the diagram of $\mathfrak{A}$ and let T' be the following set of sentences: $T \cup D(\mathfrak{A}) \cup \{\phi(c, a_1, \cdots, a_m): c \in C\} \cup \{c_1 \neq c_2$: for every two different elements c_1, c_2, in $C\}$.

T' is a first order theory and every finite subset of T' has a model. Hence, by the Compactness Theorem, T' has a model $\mathfrak{A}'$. Since $\mathfrak{A} \subseteq \mathfrak{A}'$ and T is α-model-complete then $\mathfrak{A} <_\alpha \mathfrak{A}'$. But $\mathfrak{A} \vDash_\alpha \neg Qx\phi(x, a_1, \cdots, a_m)$ while $\mathfrak{A}' \vDash_\alpha Qx\phi(x, a_1, \cdots, a_m)$, a contradiction.

For a theory T in L such that $\alpha > |L|$ Lemmas 1.3 and 1.4 yield the following:

THEOREM 1.5. *Let T be a theory in L. Suppose T is model-complete* (in the usual sense) *and $\alpha > |L|$. Then a sufficient and necessary condition for T to be α-model-complete is Condition* 1.1.

It is possible to look at Theorem 1.5 also from the aspect of definability. Let $\mathfrak{A}$ be a model for L, $|\mathfrak{A}| \geqq \alpha$. Suppose $A_1 \subseteq \mathfrak{A}$. Call A_1 α-parametrically definable in $\mathfrak{A}$ if there exist a formula $\phi(x, x_1, \cdots, x_m)$ in $L(Q)$ and elements $a_1, \cdots, a_m$ in $\mathfrak{A}$ such that for every a in $\mathfrak{A}$, $a \in A_1$ iff $\mathfrak{A} \vDash_\alpha \phi(a, a_1, \cdots, a_m)$. By Lemmas 1.1–1.4 we obtain at once:

THEOREM 1.5*. *Let T be a theory in L which is also model-complete. Suppose $\alpha > |L|$. Then T is α-model complete iff for every α-model $\mathfrak{A}$ of $T(Q)$ and for every set $A_1 \subseteq \mathfrak{A}$, if A_1 is α-parametrically definable in $\mathfrak{A}$ then $|A_1| < \aleph_0$ or $|A_1| \geqq \alpha$.*

We proceed with this section by relating to some known model-complete theories. The theory of totally discrete linear ordering having neither first nor last element is model-complete (in the usual sense) but for every $\alpha \geqq \aleph_0$ it is not α-model-complete.

The theory of dense linear ordering having neither first nor last element is $\aleph_0$-model complete but for every $\alpha \geqq \aleph_1$ it is not α-model complete. For the theory of algebraically closed fields and the theory of real closed fields we have the following theorem:

THEOREM 1.6. *Let T be the theory of algebraically closed fields or the theory of real closed fields. Let $\phi(x_1, \cdots, x_n)$ be any formula in $L(Q)$ (where L is the language of T). Then there exists a quantifier free formula $\psi(x_1, \cdots, x_n)$ such that $T(Q) \vDash_\alpha \phi(x_1, \cdots, x_n) \leftrightarrow \psi(x_1, \cdots, x_n)$ for every $\alpha \geqq \aleph_0$.*

Proof. The proof is similar to the usual elimination of quantifiers for these theories (e.g. Kreisel and Krivine [6]).

COROLLARY 1.7. *The theory of algebraically closed fields and the theory of real closed fields are α-model-complete for every $\alpha \geqq \aleph_0$.*

The last theorem of this section gives a partial answer to a natural question, that is, what conclusions about β-model-completeness can be made assuming α-model-completeness? Using Fuhrken [2] and Keisler [3], one can prove in a straightforward manner that:

THEOREM 1.8. *Let T be a countable first order theory.*

(1) *If T is α-model-complete, $\alpha > \aleph_0$, then it is also $\aleph_0$-model-complete.*

(2) *If T is $\aleph_1$-model-complete, then T is α-model-complete for every regular α.*

(3) (G.C.H) *If T is α-model-complete where α is a successor of a regular cardinal, then T is β-model-complete for every regular β.*

(4) *If T is α-model-complete where α is a singular cardinal, then T is β-model-complete for every strong limit cardinal β.*

(5) (G.C.H) *If T is α-model-complete where α is a singular cardinal then T is β-model-complete for every singular β.*

Proof. All the parts of the theorem are proved in the same way so it will be enough if we prove for example part (1).

Assume that T is α-model-complete but not $\aleph_0$-model complete. Then there exist two $\aleph_0$-models $\mathfrak{A}$, $\mathfrak{B}$ for $T(Q)$, $\mathfrak{A} \subseteq \mathfrak{B}$, and there exist a formula $\phi(x_1, \cdots, x_m)$ in $L(Q)$ and elements $a_1, \cdots, a_m$ in $\mathfrak{A}$ such that $\mathfrak{A} \vdash_{\mathrm{I}_0} \phi[a_1, \cdots, a_m]$ while $\mathfrak{B} \vdash_{\mathrm{I}_0} \neg \phi[a_1, \cdots, a_m]$. By Fuhrken [2] we may assume that $|\mathfrak{A}| = |\mathfrak{B}| = \aleph_0$. We also may assume for the sake of simplicity that none of the elements $a_1, \cdots, a_m$ is an interpretation of an individual constant in the language of T and also that this language does not contain functions symbols. Let $c_1, \cdots, c_m, P(x)$ be m new individual constants and a new unary predicate, respectively. Let ψ be any formula in $L(Q)$. Write ψ^P for the formula obtained from ψ by relativizing all the quantifiers of ψ to P (the relativisation of Q is exactly as the relativisation of the existential quantifier). Denote: $T^P = \{\psi^P : \psi \in T\}$. Let S be the following set of sentences:

$$T \cup T^P \cup \{QxP(x), \bigwedge_{1 \leqq i \leqq m} P(c_i), \phi^P(c_1, \cdots, c_m), \neg\phi(c_1, \cdots, c_m)\}.$$

It is easy to see that a suitable expansion of $\mathfrak{B}$ is an $\aleph_0$-model of S. By Fuhrken [2] it follows that there exists an α-model $\mathfrak{D}'$ of S. Define $C = \{d : \mathfrak{D}' \vdash P[d]\}$. Let $\mathfrak{D}$ be the model obtained from $\mathfrak{D}'$ by reducing

$\mathfrak{D}'$ to the language of T. Let $\mathfrak{C}$ be the submodel of $\mathfrak{D}$ built on C ($\mathfrak{C}$ is a submodel since we assumed that our language does not contain functions symbols). It follows immediately that $\mathfrak{C}$, $\mathfrak{D} \models_\alpha T(Q)$. Denote by $d_1, \cdots, d_m$ the elements of $\mathfrak{D}$ which correspond to the individual constants $c_1, \cdots, c_m$. Then $\mathfrak{C} \models_\alpha \phi[d_1, \cdots, d_m]$, $\mathfrak{D} \models_\alpha \neg\phi[d_1, \cdots, d_m]$, a contradiction to the assumption that T is α-model complete.

A similar theorem, concerning the connections between α-completeness and β-completeness, can be formulated.

It is unknown whether this result is the best result one can obtain.

2. α-Completeness, categoricity and α-model-completeness. Recall now the notions $T(\alpha)$ and α-completeness in the beginning. As an analogue to Vaught's Theorem about the connection between categoricity and completeness we have here:

THEOREM 2.1. *Let T be a countable first order theory categorical in an uncountable power. Then, for every $\alpha \geqq \aleph_0$, T is α-complete and $T(\alpha) = T(\aleph_0)$.*

Proof. If T is not $\aleph_0$-complete then there exist two $\aleph_0$-models $\mathfrak{A}$, $\mathfrak{B}$ of $T(Q)$ and there exists a sentence ϕ in $L(Q)$ such that $\mathfrak{A} \models_{\aleph_0} \phi$ and $\mathfrak{B} \models_{\aleph_0} \neg\phi$. By Fuhrken [2] there exist two $\aleph_1$-models $\mathfrak{A}_1$, $\mathfrak{B}_1$ such that $\mathfrak{A}_1 \models_{\aleph_1} \phi$, $\mathfrak{B}_1 \models_{\aleph_1} \neg\phi$ and $|\mathfrak{A}_1| = |\mathfrak{B}_1| = \aleph_1$. If T is not α-complete for $\alpha > \aleph_0$, then there exist two α-models $\mathfrak{A}_1$, $\mathfrak{B}_1$ of $T(Q)$ and a sentence ϕ in $L(Q)$ such that $\mathfrak{A}_1 \models_\alpha \phi$, $\mathfrak{B}_1 \models_\alpha \neg\phi$ and $|\mathfrak{A}_1| = |\mathfrak{B}_1| = \alpha$. So whether $\alpha = \aleph_0$ or $\alpha > \aleph_0$ the assumption that T is not α-complete leads us to two uncountable models of T that have the same power and are not isomorphic, a contradiction to Morley [7]. Suppose now that there exists α such that $T(\alpha) \neq T(\aleph_0)$. Since T is $\aleph_0$-complete and also α-complete there exists ϕ in $L(Q)$ such that $\phi \in T(\aleph_0)$ and $\neg\phi \in T(\alpha)$. By Fuhrken [2] there exists an α-model $\mathfrak{A}$ for $T(Q)$ such that $\mathfrak{A} \models_\alpha \phi$, a contradiction to the assumption that $\neg\phi \in T(\alpha)$.

REMARK. If T is categorical in $\aleph_0$ then T is also $\aleph_0$-complete but it **is not necessarily α-complete for $\alpha > \aleph_0$.** One can easily see that by taking T as the theory of dense linear ordering (having neither first nor last element). Again as in the previous section arises the question about the connection between α-completeness and β-completeness and the answer here is the same as there. Another question about α-completeness is to find a sufficient and necessary condition on formulae in L so that T will be α-complete; but what we know about α-completeness are Theorems 2.2 and 2.3.

Let $\phi(x)$ be a formula in $L(Q)$ having x as its only free variable. Let $\mathfrak{A}$ be a model for L. Denote $\phi(\mathfrak{A}, \alpha) = \{a : \mathfrak{A} \models_\alpha \phi[a]\}$.

THEOREM 2.2. *Let T be a countable first order theory. Assume T is α-complete, $\alpha > \aleph_0$. Then for every formula $\phi(x)$ in $L(Q)$* (having x as its only free variable) *there exists a cardinal m_ϕ, finite or equal to α, such that for every model $\mathfrak{A}$ for T of power α, $|\phi(\mathfrak{A}, \alpha)| = m_\phi$.*

Proof. Let $\mathfrak{A}$ be any countable model for T (there exists such a model since by the definition of α-completeness T has infinite models. It has also a countable model because it is countable). It is clear that either $\mathfrak{A} \models_{\aleph_0} Qx\phi(x)$ or $\mathfrak{A} \models_{\aleph_0} \neg Qx\phi(x)$. In the first case define $\Sigma_1 = T(Q) \cup \{Qx\phi(x)\}$. By Fuhrken [2] there exists an α-model $\mathfrak{A}_1$ for Σ_1. Since $\mathfrak{A}_1 \models_\alpha Qx\phi(x)$ and T is α-complete then for every α-model $\mathfrak{A}$ for $T(Q)$, $\mathfrak{A} \models_\alpha Qx\phi(x)$. Hence $m_\phi = \alpha$ in this case. In the second case there exists a finite number k such that $\mathfrak{A} \vdash_{\aleph_0} \exists^k ! x\phi(x)$, where $\exists^k ! x\phi(x)$ is a formula in $L(Q)$ "saying" that there are exactly k elements x such that $\phi(x)$. Define $\Sigma_2 = T(Q) \cup \{\exists^k ! x\phi(x)\}$. By the same argument as before there exists an α-model $\mathfrak{A}_2$ for Σ_2. Because of the α-completeness of T we obtain $\mathfrak{A} \models_\alpha \exists^k ! x\phi(x)$ for every α-model $\mathfrak{A}$ of $T(Q)$. Hence, in this case, $m_\phi = k$.

THEOREM 2.3. *Let T be a complete theory in L which is also α-model-complete, $\alpha \geqq \aleph_0$. Then T is also α-complete.*

Proof. Suppose on the contrary that T is not α-complete. Then there exist two α-models $\mathfrak{A}, \mathfrak{B}$ for $T(Q)$ and a sentence ϕ in $L(Q)$ such that $\mathfrak{A} \models_\alpha \phi$ and $\mathfrak{B} \vdash_\alpha \neg\phi$. Since T is complete then $\mathfrak{A}$ is elementary equivalent to $\mathfrak{B}$. By Bell and Slomson [1] (p. 161), there exists a model $\mathfrak{D}$ which is an elementary extension of $\mathfrak{A}$ and $\mathfrak{B}$. Since T is α-model-complete and $\mathfrak{A} \models_\alpha \phi$ it follows that $\mathfrak{D} \models_\alpha \phi$. By the same argument we obtain also $\mathfrak{D} \models_\alpha \neg\phi$, a contradiction.

DEFINITION 2.1. Let $L(Q)$ be recursive and let T be a theory in L. Call T α-decidable if $T(\alpha)$ is recursive (more precisely, the set of Gödel-Numbers of all the sentences in $T(\alpha)$ is recursive).

THEOREM 2.4. *Let T be a theory in L categorical in an uncountable power. Suppose $L(Q)$ and T are recursive. Then T is α-decidable for every $\alpha \geqq \aleph_0$.*

Proof. By Theorem 2.1 we have: $T(\alpha) = T(\aleph_0)$ for every $\alpha \geqq \aleph_0$. So it is sufficient to show that $T(\aleph_1)$ is recursive. By Keisler [5] we know that $T(\aleph_1)$ is recursively enumerable. Since T is $\aleph_1$-complete then for every ϕ in $L(Q)$, $\phi \in T(\aleph_1)$ iff $\neg\phi \notin T(\aleph_1)$. This means that also the complement of $T(\aleph_1)$ is recursively enumerable. Hence $T(\aleph_1)$ is recursive.

LEMMA 2.5. *Let T be any theory in a countable first order language L such that T is categorical in an uncountable power. Let $\mathfrak{A}$ be a model for T and let $a_1, \cdots, a_n$ be elements in $\mathfrak{A}$. Suppose $|\mathfrak{A}| = \alpha > \aleph_0$ and $\phi(x, x_1, \cdots, x_n)$ is a formula in L having exactly $x, x_1, \cdots, x_n$ as free variables. Then $|\phi(\mathfrak{A}, a_1, \cdots, a_n)| = \alpha$ or $|\phi(a, a_1, \cdots, z_n)| < \aleph_0$.*

Proof. Since $\alpha > \aleph_0$ then T is categorical in α, by Morley [7]. Denote by $T((\mathfrak{A}, a_1, \cdots, a_n))$ the (first order) theory of $(\mathfrak{A}, a_1, \cdots, a_n)$. Again by Morley [7] it is easy to see that $T((\mathfrak{A}, a_1, \cdots, a_n))$ is categorical in α so $(\mathfrak{A}, a_1, \cdots, a_n)$ is a saturated model. It is well known (see for instance Morley and Vaught [8], Theorem 3.7) that in a saturated model each infinite set defined by a formula (in the language for the model) has the power of the whole model. Hence $|\phi(\mathfrak{A}, a_1, \cdots, a_n)| = \alpha$ or $|\phi(\mathfrak{A}, a_1, \cdots, a_n)| < \aleph_0$.

COROLLARY 2.6. *Let T be as in Lemma 2.5. Then for every formula $\phi(x_1, \cdots, x_n)$ in $L(Q)$ there exists a formula $\psi(x_1, \cdots, x_n)$ in L such that $T(Q) \vDash_\alpha \phi(x_1, \cdots, x_n) \leftrightarrow \psi(x_1, \cdots, x_n)$ for every $\alpha \geqq \aleph_0$.*

Proof. By Lemmas 2.5, 1.2, 1.1 and Theorem 2.1.

Corollary 2.6 says that the use of the language $L(Q)$ is dispensible for talking about models of T; namely, everything that can be said in $L(Q)$ about elements in a model of T can be said about them in L.

THEOREM 2.7. *Let T be a theory in a countable first order language L such that T is categorical in an uncountable power and also model-complete* (in the usual sense). *Then*

(1) *For every formula $\phi(x_1, \cdots, x_n)$ in $L(Q)$ there exist two formulae $\psi_i(x_1, \cdots, x_n)$, $i = 1, 2$, in L, ψ_1 is existential, ψ_2 is universal and $T(Q) \vDash_\alpha \phi(x_1, \cdots, x_n) \leftrightarrow \psi(x_1, \cdots, x_n)$ for every $\alpha \geqq \aleph_0$.*

(2) *T is α-model-complete for every $\alpha \geqq \aleph_0$.*

(3) *If $L(Q)$ and T are recursive then there exists an effective procedure to find ψ_i, $i = 1, 2$, that were mentioned in* (1).

Proof. (1) Let $\phi(x_1, \cdots, x_n)$ be a formula in $L(Q)$. By Corollary 2.6 there exists a formula $\psi(x_1, \cdots, x_n)$ in L such that $T(Q) \vdash_\alpha \phi(x_1, \cdots, x_n) \leftrightarrow \psi(x_1, \cdots, x_n)$ for every $\alpha \geqq \aleph_0$. By Robinson [9] (Theorem 3.3.11), since T is model-complete, there exist two formulae $\psi_1(x_1, \cdots, x_n)$, $\psi_2(x_1, \cdots, x_n)$ in L, ψ_1 is existential, ψ_2 is universal and $T \vdash \psi(x_1, \cdots, x_n) \leftrightarrow \psi_i(x_1, \cdots, x_n)$, $i = 1, 2$. Therefore

$$T(Q) \vdash_\alpha \phi(x_1, \cdots, x_n) \leftrightarrow \psi_i(x_1, \cdots, x_n),$$

$i = 1, 2$, for every $\alpha \geqq \aleph_0$.

(2) By the assumption on T and by Corollary 2.6 T is α-model-complete for every $\alpha \geqq \aleph_0$.

(3) Since $L(Q)$ is recursive there is an effective procedure to count all existential formulae (in L) that have exactly $x_1, \cdots, x_n$ as free variables. Let ψ' be such a formula. By Theorem 2.4 there is an effective procedure to decide whether $[\phi \leftrightarrow \psi'] \in T(\aleph_1)$ or not. Since there exists an existential formula ψ_1 such that $[\phi \leftrightarrow \psi_1] \in T(\aleph_1)$ we shall find it after finite number of steps. In the same way we shall find a universal formula ψ_2 such that $[\phi \leftrightarrow \psi_2] \in T(\aleph_1)$.

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Received September 24, 1973.

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Pacific Journal of Mathematics
Vol. 56, No. 1 November, 1975