ON BOUNDARY FUNCTIONALS AND OPERATORS WITH FINITE-DIMENSIONAL NULL SPACES

FRANKLIN TAKASHI IHA

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Let L be a closed operator on a Hilbert space $\mathcal{H}$ defined on a linear manifold $\mathcal{D}$ of $\mathcal{H}$ with the property that L has a continuous right inverse T and that the dimension of the null space of L is finite. A boundary functional η for L is defined to be a linear functional η on $\mathcal{D}$ such that ηT is continuous. The boundary-value problems for ordinary differential equations are generalized to the operator L with the boundary conditions defined by a set of boundary functionals. It is shown, in particular, that if K is a continuous right inverse of L, then there exist n linearly independent boundary functionals, $\eta_1, \cdots, \eta_n$, where n is the dimension of the null space of L, such that the range of K is precisely the linear manifold $\{u \epsilon \mathcal{D} \mid \eta_j(u) = 0, j = 1, 2, \cdots, n\}$.

We make a further abstraction on the generalized homogeneous boundary-value problems of ordinary differential equations treated in [4]. Moreover, we prove that the condition imposed in [4] on the linear functionals defining the boundary conditions is not only sufficient but necessary in order to obtain a continuous right inverse.

In earlier works concerning generalized boundary-value problems, such as found in the survey article [5] by Whyburn, no consideration is given to the continuity of the right inverse of the differential operator. Yet, if we examine how boundary-value problems arise in application, we see that the right-hand side of the the equation $Ly = f$ is usually a force function which is at best an approximate description of the force acting on the physical system. Thus, without the consideration of the continuity of the right inverse, the solution of a boundary-value problem must be regarded as incomplete. It is interesting to note that in all the boundary-value problems arising in application involving an nth order differential equation, conditions are never imposed on the nth derivative. From equation (8) of [4], we see that the linear functional η defined by $\eta(u) = u^{(n)}(c)$, $u \in C^n([a, b])$, where $c \in [a, b]$, does not have the property that ηT is continuous (with respect to the L^2 norm), where T is a continuous right inverse of the differential operator L. We show in this paper that the linear functional η must have the property that ηT is continuous in order to obtain a continuous right inverse. Thus, even if such a boundary condition did arise in application, the solution of the boundary-value problem would not have been usable.

Boundary-value problems considered here are actually much more general than the usual generalized boundary-value problems for a single nth order equation.   Very little use is made of the properties of differential operators, and consequently, the treatment is set in the language of a Hilbert space.   Similar abstraction of the generalized boundary-value problems for systems of differential equations considered by such authors as Whyburn [5], Cole [3], and Bryan [2] can possibly be made, but it is not undertaken here.

2.   Notations, definitions and a lemma.   Throughout this paper $\mathcal{H}$ denotes a Hilbert space and L a closed operator from $\mathcal{D} \subset \mathcal{H}$ into $\mathcal{H}$, where $\mathcal{D}$ is a linear manifold not necessarily dense in $\mathcal{H}$.   We let

$$(1) \qquad\qquad \mathcal{S} = \{u \in \mathcal{D} \mid Lu = 0\}$$

and we assume dim $\mathcal{S} = n < \infty$.   We also assume that there exists a linear operator T on $\mathcal{H}$ having the following properties:

$$(2) \qquad \begin{cases} LTf = f \text{ for all } f \in \mathcal{H}; \\ \|Tf\| \leqq c \|f\| \text{ for all } f \in \mathcal{H}, \end{cases}$$

where c is a constant.   Following the terminology used in [4], we make the following definitions.

DEFINITION 1.   A linear functional η defined on $\mathcal{D}$ is said to be a boundary functional for L if ηT is continuous.

DEFINITION 2.   A set $\{\eta_1, \cdots, \eta_l\}$ of boundary functionals is said to be linearly independent if the set of their restrictions to $\mathcal{S}$ is linearly independent, that is, if $\Sigma_{j=1}^{l} \alpha_j \eta_j(u) = 0$ for all $u \in \mathcal{S}$ implies $\alpha_j = 0$ for all $j, 1 \leqq j \leqq l$; or equivalently, if for any basis $\{u_1, \cdots, u_n\}$ of $\mathcal{S}$ the rank of the matrix $[\eta_i(u_j)]$ is l.

The following lemma shows that the property of a linear functional η being a boundary functional for L depends only on L.   Because of the rather remarkable property, we give the lemma in a slightly more general form than needed in this paper.

LEMMA.   *Let* $L, \mathcal{D},$ *and* $\mathcal{S}$ *be as defined above.   Let* L *have an additional property that* $\dim \mathcal{N}(L - \lambda I) < \infty$ *for all* $\lambda \in \mathbf{C}$, *where* $\mathcal{N}(L - \lambda I)$ *is the null space of* $L - \lambda I$ *and* $\mathbf{C}$ *is the set of all complex numbers.   Let* $\lambda_1, \lambda_2 \in \mathbf{C}$ *and* K_1 *and* K_2 *be continuous right inverses of* $L - \lambda_1 I$ *and* $L - \lambda_2 I$, *respectively.   If* η *is a linear functional defined on* $\mathcal{D}$ *such that* ηK_1 *is continuous, then* ηK_2 *is also continuous.*

Proof. Let $u \in \mathcal{D}$. Then,

$$(L - \lambda_1 I)[K_1(L - \lambda_1 I)u - u] = 0$$

so that

$$K_1(L - \lambda_1 I)u - u \in \mathcal{N}(L - \lambda_1 I).$$

Since $K_2 \mathcal{H} \subset \mathcal{D}$, it follows

$$K_1(L - \lambda_1 I)K_2 f - K_2 f \in \mathcal{N}(L - \lambda_1 I)$$

for all $f \in \mathcal{H}$.

Let $\{z_1, \cdots, z_m\}$ be an orthonormal basis for $\mathcal{N}(L - \lambda_1 I)$. Then,

$$(3) \qquad K_1(L - \lambda_1 I)K_2 f - K_2 f = \sum_{j=1}^{m} \alpha_j z_j,$$

where

$$\alpha_j = (K_1(L - \lambda_1 I)K_2 f - K_2 f, z_j), \qquad j = 1, \cdots, m.$$

Since $(L - \lambda_1 I)K_2 f = f + (\lambda_2 - \lambda_1)K_2 f$, we see that the map $\mathcal{E}_k : f \to \alpha_k$ is continuous. From (3) we obtain

$$\eta(K_2 f) = \eta(K_1(L - \lambda_1 I)K_2 f) - \sum_{j=1}^{m} \mathcal{E}_j(f)\eta(z_j).$$

From this and from the continuities of $K_1, K_2, \mathcal{E}_k$ and $(L - \lambda_1 I)K_2$, the continuity of ηK_2 follows.

We use this lemma essentially for the case $\lambda_1 = \lambda_2 = 0$, and for this case we do not need the assumption that $\dim \mathcal{N}(L - \lambda_1 I) < \infty$ for all $\lambda \in \mathbb{C}$.

3. Generalized boundary-value problems and spectral theorem.

We begin with a theorem which is an immediate generalization of Theorem 1 of [4] and can be proved in the same way.

THEOREM 1. *Let* $L, \mathcal{D}, \mathcal{S}$ *and* T *be as defined in* §2. *Let* $\{\eta_1, \cdots, \eta_n\}$ *be a linearly independent set of boundary functionals for* L, *and let*

$$\mathcal{M} = \{u \in \mathcal{D} \mid \eta_j(u) = 0, \qquad 1 \leqq j \leqq n\}.$$

Then there exists a continuous operator K form $\mathcal{H}$ onto $\mathcal{M}$ such that $LKf = f$ for all $f \in \mathcal{H}$.

We now prove converses of this theorem.

THEOREM 2. *Let* L, $\mathcal{D}$ *and* $\mathcal{S}$ *be as defined in* §2. *Let* $\{\eta_1, \cdots, \eta_n\}$ *be a set of linear functionals on* $\mathcal{D}$ *such that their restrictions to* $\mathcal{S}$ *are linearly independent. Let* $\mathcal{M} = \{u \in \mathcal{D} \mid \eta_j(u) = 0, 1 \leqq j \leqq n\}$. *If the right inverse* K, *which does exist, from* $\mathcal{H}$ *into* $\mathcal{M}$ *is continuous, then each* η_j *must be a boundary functional for* L.

Proof. By the definition of K, $Kf \in \mathcal{M}$ for all $f \in \mathcal{H}$, and so $\eta_j(Kf) = 0$ for all $f \in \mathcal{H}$. Hence each $\eta_j K$ is trivially continuous and the assertion follows from the Lemma.

THEOREM 3. *Let* L, $\mathcal{D}$ *and* $\mathcal{S}$ *be as defined in* §2. *If* K *is a continuous operator from* $\mathcal{H}$ *into* $\mathcal{D}$ *such that* $LKf = f$ *for all* $f \in \mathcal{H}$, *then there exist n linearly independent boundary functionals* η_j, $j = 1, 2, \cdots, n$, *such that*

$$\mathcal{R}(K) = \{u \mid \eta_j(u) = 0, \qquad 1 \leqq j \leqq n\},$$

where $\mathcal{R}(K)$ *is the range of* K *and* $n = \dim \mathcal{S}$.

Proof. Let $\{y_1, \cdots, y_n\}$ be an orthonormal basis for $\mathcal{S}$. Define η_j by

$$\eta_j(u) = (u - KLu, y_j), \quad u \in \mathcal{D}, \quad j = 1, 2, \cdots, n.$$

Then, $\eta_j(Kf) = 0$ for all $f \in \mathcal{H}$, and so each $\eta_j K$ is trivially continuous. Thus, from the Lemma, each η_j is a boundary functional for L.

Let $\alpha_j \in \mathbf{C}$ such that $\sum_{j=1}^{n} \alpha_j \eta_j(u) = 0$ for all $u \in \mathcal{S}$. Then, putting y_k for u, we obtain $\alpha_k = 0$, $k = 1, \cdots, n$. Hence, $\{\eta_1, \cdots, \eta_n\}$ is linearly independent in the sense of Definition 2. Let $\mathcal{M} = \{u \in \mathcal{D} \mid \eta_j(u) = 0, 1 \leqq j \leqq n\}$. Let $u \in \mathcal{R}(K)$. Then $u = Kf$ for some $f \in \mathcal{H}$ and so $\eta_j(u) = \eta_j(Kf) = 0$, for all $j, j = 1, \cdots, n$. Hence, $u \in \mathcal{M}$ and $\mathcal{R}(K) \subset \mathcal{M}$.

Let $u \in \mathcal{M}$. Then

$$(u - KLu, y_j) = \eta_j(u) = 0, \qquad 1 \leqq j \leqq n.$$

Hence, $u - KLu \in \mathcal{S}^{\perp}$, the orthogonal complement of $\mathcal{S}$. But $L(u - KLu) = Lu - Lu = 0$, so that $u - KLu \in \mathcal{S}$. Hence $u - KLu = 0$ or $u = K(Lu)$, so that $u \in \mathcal{R}(K)$ and so $\mathcal{M} \subset \mathcal{R}(K)$. Hence $\mathcal{M} = \mathcal{R}(K)$, which completes the proof.

The question arises as to what happens if the set $\mathcal{M}$ is defined by two sets of linear functionals, one of which is a set of boundary functionals. The next result settles this question.

THEOREM 4. *Let $L, \mathcal{D}$ and $\mathcal{S}$ be as defined in §2 and assume L has a continuous right inverse. Let $\{\eta_1, \cdots, \eta_n\}$ be a set of linearly indepen-dent boundary functionals for L and let*

$$\mathcal{M} = \{u \in \mathcal{D} \mid \eta_j(u) = 0, \quad 1 \leq j \leq n\}.$$

If $\{\xi_1, \cdots, \xi_n\}$ is a set of linear functionals on $\mathcal{D}$ such that their restrictions to $\mathcal{S}$ are linearly independent and $\mathcal{M} = \{u \in \mathcal{D} \mid \xi_j(u) = 0, 1 \leq j \leq n\}$, then each ξ_i must be a linear combination of η_j's.

Proof. Let $\{y_1, \cdots, y_n\}$ be an orthonormal basis for $\mathcal{S}$. Then the matrices $[\eta_i(y_j)]$ and $[\xi_i(y_j)]$ are invertible. So let $[\alpha_{ij}]$ and $[\beta_{ij}]$ be their inverses, respectively. Let

$$\zeta_i = \sum_{j=1}^{n} \alpha_{ij}\eta_j$$

$$\zeta'_i = \sum_{j=1}^{n} \beta_{ij}\xi_j.$$

Then, we have $\zeta_i(y_j) = \delta_{ij}$ and $\zeta'_i(y_j) = \delta_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Since $\{\eta_1, \cdots, \eta_n\}$ is a linearly independent set of boundary functionals for L, by Theorem 1 there exists a continuous right inverse K of L from $\mathcal{H}$ onto $\mathcal{M}$. Then $\eta_j(Kf) = 0 = \xi_j(Kf)$ for all $f \in \mathcal{H}$, $j = 1, \cdots, n$. Hence, $\zeta_i(Kf) = \zeta'_i(Kf) = 0$ for all $f \in \mathcal{H}$, $1 \leq i \leq n$. Let $u \in \mathcal{D}$. Then $Lu \in \mathcal{H}$, so that $KLu \in \mathcal{M}$. Moreover, $u - KLu \in \mathcal{S}$, and so $u = KLu + \sum_{j=1}^{n} r_j y_j$ for some $r_j \in \mathbf{C}$. Hence it follows that

$$\zeta_i(u) = r_i \quad \text{and}$$

$$\zeta'_i(u) = r_i.$$

Thus, $\zeta_i(u) = \zeta'_i(u)$ for all $u \in \mathcal{D}$, or $\sum_{j=1}^{n} \beta_{ij}\xi_j(u) = \sum_{j=1}^{n} \alpha_{ij}\eta_j(u)$ for all $u \in \mathcal{D}$, $i = 1, \cdots, n$. Since the matrix $[\beta_{ij}]$ is invertible, it follows there exist $r_{ij} \in \mathbf{C}$ such that

$$\xi_i(u) = \sum_{j=1}^{n} r_{ij}\eta_j(u), \quad u \in \mathcal{D}, \quad i = 1, 2, \cdots, m.$$

We want to state the immediate corollary of these theorems with respect to differential operators. Let $P(x, D) = \sum_{k=0}^{n} a_k(x)D^k$ and let I

be an interval, may be finite or infinite.   If a_k's are sufficiently differentiable, i.e. $a_k \in C^k(I)$, then it can readily be shown that the operator $u \to P(x, D)u$ from $C_0^\infty(I) \subset L^2(I)$ into $L^2(I)$ is closable, where $C_0^\infty(I)$ is the set of all infinitely differentiable complex valued functions with compact support in I. Let L be the closure of the operator $u \to P(x, D)u$, $u \in C_0^\infty(I)$ in $L^2(I)$.   Let $\mathscr{D}$ be the domain of definition of $L, \mathscr{S}$ the null space of L and dim $\mathscr{S} = n$.   With these definitions, we have

COROLLARY. *If L has a continuous right inverse T and if $\{\eta_1, \cdots, \eta_n\}$ is a linearly independent set of boundary functionals, then there exists a continuous right inverse K of L such that*

$$\mathscr{R}(K) = \{u \in \mathscr{D} \mid \eta_j(u) = 0, \quad 1 \leq j \leq n\}.$$

Conversely, if K is a continuous right inverse of L, then there exists a linearly independent set of boundary functionals, $\eta_1, \cdots, \eta_n$, such that

$$\mathscr{R}(K) = \{u \in \mathscr{D} \mid \eta_j(u) = 0, \quad 1 \leq j \leq n\}.$$

Moreover, if $\{\xi_1, \cdots, \xi_n\}$ is a set of linear functionals on $\mathscr{D}$ whose restrictions to $\mathscr{S}$ are linearly independent and

$$\mathscr{R}(K) = \{u \in \mathscr{D} \mid \xi_j(u) = 0, \quad 1 \leq j \leq n\},$$

then each ξ_i must be a linear combination of η_j's.
    We now put Theorem 5 of [4] in the general setting, keeping in mind a possible application to singular boundary-value problems which were not treated in [4].   Since a part of the proof is almost exactly the same as that of Theorem 5 of [4], that part of the proof will only be sketched.

THEOREM 5. *Let $L, \mathscr{H}, \mathscr{D}$ and $\mathscr{S}$ be as defined in §2, and assume that L has a continuous right inverse.   Let $\{\eta_1, \cdots, \eta_l\}$ be a set of boundary functionals in which $\{\eta_1, \cdots, \eta_r\}$ is linearly independent in the sense of Definition 2, and let*

$$\mathscr{M}_0 = \{u \in \mathscr{D} \mid \eta_j(u) = 0, \quad 1 \leq j \leq l\}.$$

Let $\mathscr{N} = \mathscr{S} \cap \mathscr{M}_0$, so that $\dim \mathscr{N} = p = n - r$ and let $q = l - r$.   Let $\{\psi_1, \cdots, \psi_p\}$ be an orthonormal basis for $\mathscr{N}$.   Define ζ_j by

$$\zeta_j(f) = (f, \psi_j), \quad f \in \mathscr{H}, \quad j = 1, \cdots, p,$$

and let

$$\mathcal{M}_1 = \{u \in \mathcal{D} \mid \eta_j(u) = \zeta_k(u) = 0, \quad 1 \leqq j \leqq r, \quad 1 \leqq k \leqq p\}.$$

Then there exists a continuous right inverse K of L from $\mathcal{H}$ onto $\mathcal{M}_1$. If moreover, L is self-adjoint on $\mathcal{M}_0$, that is, if $(Lu, v) = (u, Lv)$ for all $u, v \in \mathcal{M}_0$, then there exists a spectral measure E such that

$$Kf = \left(\int \lambda dE \right) f + \sum_{j=1}^{q} (f, \psi_j) Kf, \quad f \in \mathcal{H}.$$

Proof. It can readily be shown that $\{\eta_1, \cdots, \eta_r, \zeta_1, \cdots, \zeta_p\}$ is a linearly independent set of boundary functionals. Hence the existence of the continuous right inverse K of L from $\mathcal{H}$ onto $\mathcal{M}_1$ follows from Theorem 1. To prove the second part, we first show, just as in the proof of Theorem 5 of [4], that every $f \in \mathcal{H}$ can be written as

$$(5) \qquad\qquad f = g + \sum_{j=1}^{q} (f, \psi_j) \psi_j$$

with $g \in L(\mathcal{M}_0 \cap \mathcal{M}_1)$, the image of $\mathcal{M}_0 \cap \mathcal{M}_1$ under L. From this we can show that $L(\mathcal{M}_0 \cap \mathcal{M}_1)$ is invariant under K.

It can readily be shown that $L(\mathcal{M}_0 \cap \mathcal{M}_1)$ is a closed subspace of $\mathcal{H}$. Hence if we denote by K_0 the restriction of K to $L(\mathcal{M}_0 \cap \mathcal{M}_1)$, K_0 is bounded self-adjoint operator on the Hilbert space $L(\mathcal{M}_0 \cap \mathcal{M}_1)$ and so there exists a spectral measure E_0, whose values are projection operators on $L(\mathcal{M}_0 \cap \mathcal{M}_1)$, such that

$$K_0 f = \left(\int \lambda dE_0 \right) f, \qquad f \in L(\mathcal{M}_0 \cap \mathcal{M}_1).$$

Let P be the orthogonal projection of $\mathcal{H}$ onto $L(\mathcal{M}_0 \cap \mathcal{M}_1)$. If δ is a Borel set in $\sigma(K_0)$, the spectrum of K_0, define $E(\delta) = E_0(\delta)P$. Then E becomes a spectral measure in the usual sense (c.f. [1]). Let $f \in \mathcal{H}$ and let

$$g = f - \sum_{j=1}^{q} (f, \psi_j) \psi_j.$$

Then, it follows from (5) that

$$Kg = \left(\int \lambda dE_0 \right) g = \left(\int \lambda dE_0 \right) Pf = \left(\int \lambda dE \right) f,$$

so that

$$Kf = \left(\int \lambda dE \right) f + \sum_{j=1}^{q} (f, \psi_j) K\psi_j.$$

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Pacific Journal of Mathematics
Vol. 56, No. 2   December, 1975