

# Pacific Journal of Mathematics

## **BOUNDS FOR DISTORTION IN PSEUDOCONFORMAL MAPPINGS**

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1. **Introduction.** When considering a conformal mapping of a domain, say<sup>1</sup>  $B^2$ , of the  $z$ -plane, it is useful to introduce a metric which is invariant with respect to conformal transformations. The line element of this metric is given by

$$(1.1) \quad ds_B^2(z) = K_B(z, \bar{z}) |dz|^2, \quad B \equiv B^2,$$

where  $K_B(z, \bar{z})$  is the kernel function of  $B^2$ . (In the case of  $[|z| < 1]$  the metric (1.1) is identical with the hyperbolic metric introduced by Poincaré.) In addition to the invariant metric one can also introduce scalar invariants, for instance,

$$(1.2) \quad J_B(z) = -\frac{1}{C_B(z)}, \quad C_B(z) = -\frac{2}{K^3} \begin{vmatrix} K & K_{0\bar{1}} \\ K_{10} & K_{\bar{1}\bar{1}} \end{vmatrix}, \quad K_{10} = \frac{\partial K}{\partial z},$$

$$K_{0\bar{1}} = \frac{\partial K}{\partial \bar{z}}.$$

( $C_B(z)$  is the curvature of the metric (1) at the point  $z$ .)

Using the kernel function  $K_{\mathfrak{B}}(z, \bar{z})$ ,  $z = (z_1, \dots, z_n)$ , one can generalize this approach to the theory of PCT's (pseudoconformal transformations), i.e., to the mappings of  $2n$  dimensional domains by  $n$  analytic functions of  $n$  complex variables (with a nonvanishing Jacobian). It is of interest to obtain bounds for the invariant  $J_{\mathfrak{B}}(z)$ , see (3.1), depending on quantities which are in a simple way connected with the domain, for instance, the maximum and minimum (euclidean) distances between the point  $z$  and the boundary of the domain.

In the present paper we shall determine such bounds in the case of pseudoconformal mapping of the domain  $\mathfrak{B} = \mathfrak{B}^4$  of the  $z_1, z_2$ -space by pairs

$$(1.3) \quad w_k = f_k(z_1, z_2), \quad k = 1, 2,$$

of analytic functions of two complex variables (with nonvanishing Jacobian). The generalization of our procedure to the case of pseudoconformal mappings of domains  $\mathfrak{B}^{2n}$  by  $n$  functions of  $n$  complex variables,  $3 \leq n < \infty$ , is immediate and will not be discussed in the following.

2. **The minima  $\lambda_{\mathfrak{B}}(z)$ .** To obtain the desired bound we use

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<sup>1</sup> The upper index at a set indicates its dimension.

the minimum values  $\lambda_{\mathfrak{B}}^{\dots}(z)$  of the integral

$$(2.1) \quad \int_{\mathfrak{B}} |f(\zeta)|^2 d\omega, \quad \zeta = (\zeta_1, \zeta_2),$$

( $d\omega$  = the volume element), under some additional conditions for  $f$  at the point  $z = (z_1, z_2)$ .

As indicated in [1, pp. 183 and 198 ff.], many invariant quantities arising in the theory of PCT's can be expressed in terms of the minima  $\lambda_{\mathfrak{B}}^{\dots}(z)$ . For instance,

$$(2.2) \quad K_{\mathfrak{B}}(z, \bar{z}) = \frac{1}{\lambda_{\mathfrak{B}}^1(z)}, \quad J_{\mathfrak{B}}(z) = \frac{\lambda_{\mathfrak{B}}^{01}(z)\lambda_{\mathfrak{B}}^{001}(z)}{[\lambda_{\mathfrak{B}}^1(z)]^3}.$$

Here  $\lambda_{\mathfrak{B}}^{X_{00}}(z)$  is the minimum of (2.1) under the condition  $f(z) = X_{00}$ ,  $z \in \mathfrak{B}$ ,  $\lambda_{\mathfrak{B}}^{X_{00}X_{10}}$  is the minimum under the condition  $f(z) = X_{00}$ ,  $(\partial f(z)/\partial z_1) = X_{10}$  and  $\lambda_{\mathfrak{B}}^{X_{00}X_{10}X_{01}}(z)$  is the minimum under the condition  $f(z) = X_{00}$ ,  $(\partial f(z)/\partial z_1) = X_{10}$ ,  $(\partial f(z)/\partial z_2) = X_{01}$ . ( $K$  is a *relative* invariant, see (25), p. 180, of [1].)

Using (23b), p. 179 of [1], one obtains the representations for the  $\lambda_{\mathfrak{B}}^{\dots}(z)$  in terms of the kernel function  $K \equiv K_{\mathfrak{B}}$  and their partial derivatives  $K_{10\bar{0}} = (\partial K/\partial z_1)$ ,  $K_{01\bar{0}} = (\partial K/\partial z_2)$ ,  $K_{00\bar{1}} = (\partial K/\partial \bar{z}_1)$ ,  $K_{00\bar{0}\bar{1}} = \partial K/\partial \bar{z}_2$ . Obviously it holds

LEMMA 2.1. *Suppose that  $z \in \mathfrak{B} \subset \mathfrak{G}$ , then*

$$(2.3) \quad \lambda_{\mathfrak{B}}^{\dots}(z) \leq \lambda_{\mathfrak{G}}^{\dots}(z).$$

*Here it is assumed that the minima  $\lambda^{\dots}(z)$  on both sides of (2.3) are taken under the same conditions.*

Choosing for  $\mathfrak{G}$  a domain for which the kernel function  $K_{\mathfrak{G}}$  is a simple expression of the equation of its boundary (e.g., choosing for  $\mathfrak{G}$  a sphere or certain Reinhardt circular domains, see [2, p. 21]), we obtain the desired inequality.

Using the above method, we shall derive in the next section an inequality for the invariant  $J_{\mathfrak{B}}(z)$ .

3. **Derivation of bounds for  $J_{\mathfrak{B}}(z)$ .** Let  $\mathfrak{B}$  be a connected domain of the (four-dimensional)  $z_1, z_2$ -space,  $z_k = x_k + iy_k$ ,  $k = 1, 2$ . Let

$$(3.1) \quad J_{\mathfrak{B}}(z, \bar{z}) \equiv J_{\mathfrak{B}} = \frac{K}{T_{1\bar{1}}T_{2\bar{2}} - |T_{1\bar{2}}|^2}, \quad T_{m\bar{n}} = \frac{\partial^2 \log K}{\partial z_m \partial \bar{z}_n},$$

denote the invariant respect to PCT's, see (37a), p. 183 of [1]. Here with  $K$  is the kernel function of  $\mathfrak{B}$  and  $T_{m\bar{n}}$  are the coefficients of the line element

$$(3.2) \quad ds_{\mathfrak{B}}^2 = \sum_{m=1}^z \sum_{n=1}^2 T_{mn} dz_m d\bar{z}_n$$

of the metric which is invariant with respect to PCT's, see [1, p. 182 ff.].

**THEOREM I.** *Suppose that  $r$  is the maximum distance of the point  $z$ ,  $z \in \mathfrak{B}$ , to the boundary  $\partial\mathfrak{B}$ , and  $\rho$  is the corresponding minimum distance. Then*

$$(3.3) \quad H(\rho, r) \leq J_{\mathfrak{B}}(z) \leq H(r, \rho),$$

$$H(\rho, r) = \frac{2r^6 [P(\rho)]^9}{9\rho^6 [P(r)]^9 \pi^2}, \quad P(\rho) = \rho^2 - z_1 \bar{z}_1 - z_2 \bar{z}_2.$$

*Proof.* By (97), p. 198 of [1],

$$(3.4) \quad J_{\mathfrak{B}}(z) = \frac{\lambda_{\mathfrak{B}}^{01}(z) \lambda_{\mathfrak{B}}^{001}(z)}{[\lambda_{\mathfrak{B}}^1(z)]^3}$$

and in accordance with (2.3) for  $\mathfrak{J} \subset \mathfrak{B} \subset \mathfrak{A}$  the inequality

$$(3.5) \quad \frac{\lambda_{\mathfrak{J}}^{01}(z) \lambda_{\mathfrak{J}}^{001}(z)}{[\lambda_{\mathfrak{J}}^1(z)]^3} \leq J_{\mathfrak{B}}(z) \leq \frac{\lambda_{\mathfrak{A}}^{01}(z) \lambda_{\mathfrak{A}}^{001}(z)}{[\lambda_{\mathfrak{A}}^1(z)]^3}$$

holds. If  $r$  is the maximum distance of the point  $z$  from the boundary  $\partial\mathfrak{B}$ , and  $\rho$  is the minimum distance of  $z$  from  $\partial\mathfrak{B}$ , then one can use for  $\mathfrak{A}$  the hypersphere  $|z_1|^2 + |z_2|^2 < r^2$  and for  $\mathfrak{J}$  the hypersphere  $|z_1|^2 + |z_2|^2 < \rho^2$ . By (23b)<sup>2</sup>, p.179 of [1] and by (5a), p. 22 of [2] it holds for the hypersphere  $|z_1|^2 + |z_2|^2 < r^2$ ,

$$(3.6) \quad \lambda_{\mathfrak{A}}^{01}(z) \lambda_{\mathfrak{A}}^{001}(z) = \frac{\pi^4 [P(r)]^8}{36r^6},$$

$$(3.7) \quad \lambda_{\mathfrak{A}}^1(z) = \frac{1}{K_{\mathfrak{A}}(z, \bar{z})} = \frac{\pi^2 [P(r)]^3}{2r^2}.$$

Analogous formulas hold for  $\lambda_{\mathfrak{J}}^{01}(z) \lambda_{\mathfrak{J}}^{001}(z)$  and  $\lambda_{\mathfrak{J}}^1(z)$ . Consequently (3.3) holds.

**4. An application of Theorem I.** A domain which admits the group

$$(4.1) \quad z_k^* = z_k e^{i\varphi_k}, \quad 0 \leq \varphi_k \leq 2\pi, \quad k = 1, 2,$$

---

<sup>2</sup> In the last term of the expression for  $\lambda^{x_{00}x_{10}x_{01}}(t)$  of (23b) are misprints, in the denominator  $\left| \frac{K}{K_{1000}} \frac{K_{000\bar{0}}}{K_{101\bar{0}}} \right|$  should be replaced by  $\left| \frac{K}{K_{100\bar{0}}} \frac{K_{001\bar{0}}}{K_{101\bar{0}}} \right|$ . In the nominator of the last term of (23b) the last term  $K_{010\bar{1}}$  in the third row should be replaced by  $K_{011\bar{0}}$ . In the denominator the first term  $K_{010\bar{1}}$  of the third row should be replaced by  $K_{010\bar{0}}$ .

of PCT's onto itself (automorphisms) is called a Reinhardt circular domain (see [3], pp. 33-34).

A domain, say  $\mathfrak{R}$ , bounded by the hypersurface

$$(4.2) \quad |z_2| = r(|z_1|),$$

where  $y_2 = r(x_1)$  is a convex curve, is a Reinhardt circular domain. Its kernel function is

$$(4.3) \quad K_{\mathfrak{R}}(z, \bar{z}) = B_{00} + B_{10}z_1\bar{z}_1 + B_{01}z_2\bar{z}_2 + B_{02}z_1^2\bar{z}_1^2 + B_{11}z_1\bar{z}_1z_2\bar{z}_2 + \dots,$$

$$(4.4) \quad B_{m\bar{p}}^{-1} = \int_{\mathfrak{R}} |z_1|^{2m} |z_2|^{2p} d\omega,$$

$d\omega$  volume element ( $B_{m\bar{p}}$  are the inverse of moments of  $\mathfrak{R}$ ), see [2], p. 20 ff.

LEMMA. *The kernel function  $K_{\mathfrak{R}}$  and its derivatives at the center 0 of  $\mathfrak{R}$  equal*

$$(4.5) \quad \begin{aligned} K_{\mathfrak{R}} &\equiv K = B_{00}, \\ K_{1000} &\equiv K_{z_1}(0) = 0, \quad K_{1010} \equiv \frac{\partial^2 K}{\partial z_1 \partial \bar{z}_1} = B_{10}, \quad K_{0100} = 0, \\ K_{0101} &= B_{01}, \dots \end{aligned}$$

Therefore

$$(4.6) \quad J_{\mathfrak{R}}(0) = \frac{K}{\begin{vmatrix} K & K_{00\bar{1}0} & K_{000\bar{1}} \\ K_{1000} & K_{1010} & K_{1001} \\ K_{0100} & K_{0110} & K_{0101} \end{vmatrix}} = \frac{B_{00}^4}{\begin{vmatrix} B_{00} & 0 & 0 \\ 0 & B_{10} & 0 \\ 0 & 0 & B_{01} \end{vmatrix}} = \frac{B_{00}^3}{B_{10}B_{01}}$$

(see [1], p. 183, (37a)).

THEOREM II. *Let  $\mathfrak{B} = B(\mathfrak{R})$  be a pseudoconformal image of a Reinhardt circular domain  $\mathfrak{R}$ , and let  $r$  and  $\rho$  be the maximum and minimum distances from the boundary, respectively, of the image  $z^0 = (z_1^0, z_2^0) = B(0)$  of the center 0 of  $\mathfrak{R}$  in  $\mathfrak{B}$ . Then*

$$(4.7) \quad H(\rho, r) \leq \frac{B_{00}^3}{B_{10}B_{01}} \leq H(r, \rho).$$

Here  $B_{m\bar{n}}$  are the inverse moments (introduced in (4.4)) of  $\mathfrak{R}$ .

*Proof.* Since  $J_{\mathfrak{R}}$  is invariant and  $\mathfrak{B}$  is a pseudoconformal image of  $\mathfrak{R}$

$$(4.8) \quad J_{\mathfrak{z}}(0) = J_{\mathfrak{z}}(z^0) = \frac{B_{00}^3}{B_{10}B_{01}}.$$

By Theorem I it follows that for  $J_{\mathfrak{z}}(z^0)$  the inequality (4.7) holds.

Similar results as above can be obtained for other interior distinguished points, for instance, for critical points of  $J_{\mathfrak{z}}(z, \bar{z})$ .

REMARK. One obtains a generalization of Theorem I by assuming that  $\mathfrak{S}$  and  $\mathfrak{X}$  are domains  $|z_1|^{2/m} + |z_2|^2 < \rho^2$  and  $|z_1|^{2/M} + |z_2|^2 < r^2$ , respectively. The kernel function for the above domains is given in (5), p. 21, of [2].

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