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## **ALMOST PERIODIC COMPACTIFICATIONS OF TRANSFORMATION SEMIGROUPS**

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**In this paper we generalize the notion of (weakly) almost periodic compactification of a semitopological semigroup to the corresponding notion for transformation semigroups. The properties of these compactifications are studied and applications are made to semidirect products.**

**0. Introduction.** Let  $X$  be a Hausdorff topological space and  $S$  a semitopological semigroup [2]. The pair  $(S, X)$  is a *semitopological transformation semigroup* (abbr. s.t.t.s.) if there exists a separately continuous mapping  $(s, x) \rightarrow sx$  of  $S \times X$  into  $X$  (called an *action*) such that  $s(tx) = (st)x$  ( $s, t \in S; x \in X$ ). If  $S$  has an identity  $1$  and  $1x = x$  ( $x \in X$ ) we say that  $(S, X)$  *has an identity*. If  $S$  is a topological semigroup and the action is jointly continuous, then  $(S, X)$  is called a *topological transformation semigroup* (abbr. t.t.s.).  $(S, X)$  is *compact* if both  $S$  and  $X$  are compact. Any semitopological semigroup may be considered to be a s.t.t.s., where the action is left multiplication.

Let  $(S, X)$  be a s.t.t.s. and denote by  $C(X)$  the Banach space of all bounded continuous complex-valued functions on  $X$ . A function  $f$  in  $C(X)$  is (weakly) *almost periodic* if  $O(f) = \{ {}_s f : s \in S \}$  is relatively (weakly) compact in  $C(X)$ , where  ${}_s f(x) = f(sx)$  ( $x \in X$ ). The set of all almost periodic (resp. weakly almost periodic) functions in  $C(X)$  is denoted by  $A(X)$  (resp.  $W(X)$ ). Both  $A(X)$  and  $W(X)$  are  $C^*$ -subalgebras of  $C(X)$  which are invariant under the action  $s \rightarrow {}_s f$  [2].

A *homomorphism* of a s.t.t.s.  $(S, X)$  into a s.t.t.s.  $(T, Y)$  is a pair  $(\sigma, \xi)$ , where  $\sigma: S \rightarrow T$  is a continuous homomorphism and  $\xi: X \rightarrow Y$  a continuous map such that  $\xi(sx) = \sigma(s)\xi(x)$  ( $x \in X, s \in S$ ). The *dual* of  $\xi$  is the map  $\tilde{\xi}: C(Y) \rightarrow C(X)$  defined by  $\tilde{\xi}(f) = f \circ \xi$ . Clearly  $\tilde{\xi}$  is a bounded linear operator.

Recall that a weakly almost periodic (resp. almost periodic) compactification of a semitopological semigroup  $S$  may be defined as a pair  $(\bar{S}, \rho)$ , where  $\bar{S}$  is a compact semitopological (resp. compact topological) semigroup and  $\rho: S \rightarrow \bar{S}$  is a continuous homomorphism such that  $\rho(S)$  is dense in  $\bar{S}$  and  $\tilde{\rho}(C(\bar{S})) = W(S)$  (resp.  $\tilde{\rho}(C(\bar{S})) = A(S)$ ) (see [6], [8]). Motivated by this we define a *weakly almost periodic* (resp. *almost periodic*) *compactification* of a s.t.t.s.  $(S, X)$  as a compact s.t.t.s. (resp. compact t.t.s.)  $(\bar{S}, \bar{X})$  and a homomorphism  $(\rho, \eta)$  of  $(S, X)$  into  $(\bar{S}, \bar{X})$  such that  $(\bar{S}, \rho)$  is a weakly almost periodic

(resp. almost periodic) compactification of  $S$ ,  $\eta(X)$  is dense in  $\bar{X}$ , and  $\tilde{\eta}(C(\bar{X})) = W(X)$  (resp.  $\tilde{\eta}(C(\bar{X})) = A(X)$ ). Clearly  $\tilde{\eta}$  (and  $\tilde{\rho}$ ) is an isometric isomorphism. We shall write  $\hat{f}$  for  $\tilde{\eta}^{-1}(f)$ . We shall also occasionally use the notation  $(S^w, X^w)$  and  $(S^a, X^a)$  for weakly almost periodic and almost periodic compactifications respectively.

In §1 we shall show that if  $S$  has a (weakly) almost periodic compactification then  $(S, X)$  has a (weakly) almost periodic compactification. Furthermore, a compactification is unique up to isomorphism and satisfies a universal factorization property analogous to that satisfied by a (weakly) almost periodic compactification of  $S$ . These results are extensions of results in [11].

In §2 some specialized lemmas are proved, and in §3 we use these to characterize almost periodic compactifications of semidirect products. It is shown that if  $S$  is a topological group, then for a large class of semitopological semigroups  $X$  (including abelian semigroups, compact topological semigroups, and topological groups), the almost periodic compactification of a semidirect product  $S \oplus X$  of  $S$  and  $X$  is a semidirect product of  $\bar{S}$  and  $\bar{X}$ , where  $\bar{S}$  is the almost periodic compactification of  $S$ , and  $\bar{X}$  is a certain compactification of  $X$ . These results generalize Theorem 4 of [11]; for other results along this line see [5]. We also show that the kernel of an almost periodic compactification of  $S \oplus X$  may be expressed as a semidirect product of the kernels of  $\bar{S}$  and  $\bar{X}$ . A similar result is obtained for the weakly almost periodic case.

## 1. Existence and uniqueness of compactifications.

LEMMA 1.1. *Let  $(S, X)$  be a s.t.t.s.*

(a) *If  $(S, X)$  is compact then  $W(X) = C(X)$ .*

(b) *If  $(S, X)$  is a compact t.t.s. then  $A(X) = C(X)$ .*

(c) *If  $f \in W(X)$  then the map  $s \rightarrow {}_s f, S \rightarrow W(X)$  is continuous in the weak topology.*

(d) *If  $f \in A(X)$  then the map  $s \rightarrow {}_s f, S \rightarrow A(X)$  is continuous in the norm topology.*

*Proof.* We prove only (a) and (c), the proofs of (b) and (d) being similar. Let  $(S, X)$  be compact and  $f \in C(X)$ . Since  $s \rightarrow {}_s f$  is pointwise continuous,  $O(f)$  is compact in the pointwise topology of  $C(X)$ . Since this topology agrees with the weak topology on norm-bounded pointwise compact subsets of  $C(X)$  [10],  $O(f)$  is weakly compact, proving (a). For (c) simply observe that if  $(s_\alpha)$  is a net in  $S$  and  $s_\alpha \rightarrow s$ , then  $({}_s f)$  has a unique weak limit point in  $C(X)$ .

LEMMA 1.2. *Let  $(\sigma, \xi)$  be a homomorphism of  $(S, X)$  into  $(T, Y)$ . Then*

$$(1) \quad \xi(W(Y)) \subset W(X) \cap \xi(C(Y)),$$

*and equality holds if  $\xi(X)$  is dense in  $Y$  and  $\sigma(S)$  is dense in  $T$ . The analogous statement holds for the almost periodic case.*

*Proof.* If  $f \in C(Y)$  and  $g = \xi(f)$ , then

$$(2) \quad {}_s g = \xi({}_{\sigma(s)} f) \quad (s \in S).$$

If  $f \in W(Y)$  then  $\xi(O(f))$  is relatively weakly compact, and (2) shows that  $g \in W(X)$ , verifying (1). If  $\xi(X)$  and  $\sigma(S)$  are dense in  $Y$  and  $T$  respectively, then  $\xi$  is an isometry and (2) implies that  $\xi^{-1}(\bar{O}(g)) = \bar{O}(f)$  (bars denote weak closures). Hence if  $g \in W(X)$ , then  $f \in W(Y)$ , verifying equality in (1).

THEOREM 1.3. *If  $(S, X)$  is a s.t.t.s. and if  $S$  has a weakly almost periodic (resp. almost periodic) compactification, then  $(S, X)$  has a weakly almost periodic (resp. almost periodic) compactification. Moreover, any weakly almost periodic (resp. almost periodic) compactification  $(\bar{S}, \bar{X}, \rho, \eta)$  satisfies the following universal property: Given any homomorphism  $(\sigma, \xi)$  of  $(S, X)$  into a compact s.t.t.s. (resp. compact t.t.s.)  $(T, Y)$ , there exists a homomorphism  $(\bar{\sigma}, \bar{\xi})$  of  $(\bar{S}, \bar{X})$  into  $(T, Y)$  such that  $\bar{\sigma} \circ \rho = \sigma$  and  $\bar{\xi} \circ \eta = \xi$ .*

The following corollaries are immediate.

COROLLARY 1.4. *Let  $(S_i, X_i)$  be a s.t.t.s. with weakly almost periodic compactification  $(\bar{S}_i, \bar{X}_i, \rho_i, \eta_i)$  ( $i = 1, 2$ ). If  $(\sigma, \xi)$  is a homomorphism of  $(S_1, X_1)$  into  $(S_2, X_2)$  then there exists a homomorphism  $(\bar{\sigma}, \bar{\xi})$  of  $(\bar{S}_1, \bar{X}_1)$  into  $(\bar{S}_2, \bar{X}_2)$  such that  $\bar{\sigma} \circ \rho_1 = \rho_2 \circ \sigma$  and  $\bar{\xi} \circ \eta_1 = \eta_2 \circ \xi$ . A similar statement holds for the almost periodic case.*

COROLLARY 1.5. *Weakly almost periodic compactifications and almost periodic compactifications are unique (up to isomorphism).*

*Proof of Theorem 1.3.* Let  $(\bar{S}, \rho)$  be a weakly almost periodic compactification of  $S$  and let  $\bar{X}$  be the maximal ideal space of  $W(X)$ . Define  $\eta: X \rightarrow \bar{X}$  by  $\eta(x)(f) = f(x)$  ( $x \in X, f \in W(X)$ ). Then  $X$  is compact,  $\eta$  is continuous and  $\eta(X)$  is dense in  $\bar{X}$ . Furthermore, if  $\hat{f} \in C(\bar{X})$  denotes the Gelfand transform of  $f \in W(X)$ , then  $\tilde{\eta}(\hat{f}) = f$ .

Define  $\pi: S \times \bar{X} \rightarrow \bar{X}$  by  $\pi(s, \theta)(f) = \theta({}_s f)$  ( $s \in S, \theta \in \bar{X}, f \in W(X)$ ). Clearly,  $\pi(s, \cdot)$  is continuous, and  $s \rightarrow \pi(s, \cdot)$  is a continuous homo-

morphism from  $S$  into  $\bar{X}^{\bar{X}}$  (Lemma 1.1(c)), where  $\bar{X}^{\bar{X}}$  carries the product topology. Let  $E$  denote the closure in  $\bar{X}^{\bar{X}}$  of  $\{\pi(s, \cdot) : s \in S\}$ .  $E$  will be a compact semitopological semigroup if we show that each  $v \in E$  is continuous. Since  $\bar{X}$  is compact it suffices to show that  $g \circ v$  is continuous for arbitrary  $g \in C(\bar{X})$ . Let  $(s_\alpha)$  be a net in  $S$  such that  $\pi(s_\alpha, \cdot) \rightarrow v$ , and let  $g = \hat{f}$ , where  $f \in W(X)$ . For each  $\theta \in \bar{X}$ ,  $\theta(\cdot_\alpha f) = \hat{f}(\pi(s_\alpha, \theta)) \rightarrow g(v(\theta))$ . Also, there exists some  $h \in W(X)$  and a subnet  $(s_\beta)$  such that  $\cdot_\beta f \rightarrow h$  weakly. Thus  $g \circ v = \hat{h} \in C(\bar{X})$ .

By Theorem 2.2 of [8] there exists a continuous homomorphism  $u \rightarrow \bar{\pi}(u, \cdot)$  of  $\bar{S}$  onto  $E$  such that  $\bar{\pi}(\rho(s), \cdot) = \pi(s, \cdot)$  ( $s \in S$ ). Define an action of  $\bar{S}$  on  $\bar{X}$  by  $u\theta = \bar{\pi}(u, \theta)$  ( $u \in \bar{S}$ ,  $\theta \in \bar{X}$ ). With this action  $(\bar{S}, \bar{X}, \rho, \eta)$  is a weakly almost periodic compactification of  $(S, X)$ .

Now let  $(\bar{S}, \bar{X}, \rho, \eta)$  be any weakly almost periodic compactification of  $(S, X)$ , and let  $(\sigma, \xi)$  be a homomorphism into the compact s.t.t.s.  $(T, Y)$ . Let  $\bar{\sigma} : \bar{S} \rightarrow T$  be a continuous homomorphism such that  $\bar{\sigma} \circ \rho = \sigma$ . Define  $\bar{\xi} : \eta(X) \rightarrow Y$  by  $\bar{\xi}(\eta(x)) = \xi(x)$ . If  $\xi(x_1) \neq \xi(x_2)$  choose  $g \in C(Y)$  such that  $g(\xi(x_1)) \neq g(\xi(x_2))$ . Then  $f = \bar{\xi}(g) \in W(X)$  (Lemmas 1.1, 1.2) and  $\hat{f}(\eta(x_1)) = f(x_1) \neq f(x_2) = \hat{f}(\eta(x_2))$ , so  $\eta(x_1) \neq \eta(x_2)$ . Thus  $\bar{\xi}$  is well defined. Now for any  $g \in C(Y)$ , if  $f = \bar{\xi}(g)$  then  $\hat{f}|_{\eta(X)} = g \circ \bar{\xi}$ , hence  $g \circ \bar{\xi}$  is uniformly continuous. Since  $Y$  is compact, its uniform structure is defined by  $C(Y)$ , hence  $\bar{\xi}$  is uniformly continuous.

We may now extend  $\bar{\xi}$  continuously to  $\bar{X}$ . Since  $\bar{\xi}(\rho(s)\eta(x)) = \bar{\xi}(\eta(sx)) = \xi(sx) = \sigma(s)\xi(x) = \bar{\sigma}(\rho(s))\bar{\xi}(\eta(x))$  ( $s \in S$ ,  $x \in X$ ),  $(\bar{\sigma}, \bar{\xi})$  is a homomorphism.

The almost periodic case is proved similarly except that the set  $E$  must be shown to be a topological semigroup. This follows readily from the equicontinuity of  $\{\pi(s, \cdot) : s \in S\}$ .

**REMARK 1.6.** In the almost periodic case of Corollary 1.4 it may be shown that if  $X_2$  is compact and if the action of  $S_2$  on  $X_2$  is equicontinuous, then  $\eta_2 : X_2 \rightarrow \bar{X}_2$  is a homeomorphism and therefore  $\bar{X}_2$  may be replaced by  $X_2$  in the conclusion of the corollary. This means that  $\bar{S}_2$  acts on  $X_2$  such that  $\rho_2(s)x = sx$  ( $s \in S_2$ ,  $x \in X_2$ ), and that  $\bar{\xi} \circ \eta_1 = \xi$ .

The following theorem exhibits the connection between our approach to almost periodic compactifications and that of Landstad [11].

**THEOREM 1.7.** *Let  $(\bar{S}, \bar{X}, \rho, \eta)$  be an almost periodic compactification of the s.t.t.s.  $(S, X)$  and let  $\mathcal{U}$  denote the coarsest uniform structure on  $X$  relative to which each  $f \in A(X)$  is uniformly continuous. Then  $(\bar{X}, \eta)$  is a Hausdorff completion of  $(X, \mathcal{U})$ . Furthermore,  $\mathcal{U}$  is the finest uniform structure  $\mathcal{V}$  on  $X$  satisfying the following properties:*

(a)  $\mathcal{V}$  defines a topology  $\mathcal{T}(\mathcal{V})$  on  $X$  coarser than the given

one,

(b)  $\mathcal{V}$  is totally bounded, and

(c) the family of mappings  $x \rightarrow sx$  ( $x \in X, s \in S$ ) is  $\mathcal{V}$  uniformly equicontinuous (equivalently,  $\mathcal{V}$  has a base consisting of those  $V \in \mathcal{V}$  such that  $(x, y) \in V$  implies that  $(sx, sy) \in V$  for all  $s \in S$ ).

*Proof.* Clearly  $\mathcal{U}$  satisfies (a).  $\mathcal{U}$  satisfies (b) because  $\eta(X)$  is totally bounded and  $\mathcal{U}$  is the coarsest uniform structure on  $X$  making  $\eta$  uniformly continuous. These facts also imply that  $(X, \eta)$  is a Hausdorff completion of  $(X, \mathcal{U})$ . (See for example [3].) That  $\mathcal{U}$  satisfies (c) follows from the total boundedness of  $O(f)$  in  $C(X)$  for each  $f \in A(X)$ .

Now let  $\mathcal{V}$  be a uniform structure on  $X$  satisfying (a), (b), and (c), and let  $(Y, \xi)$  be the Hausdorff completion of  $(X, \mathcal{V})$ . By (b),  $Y$  is compact. For each  $s \in S$ ,  $x \rightarrow \xi(sx)$  is  $\mathcal{V}$ -uniformly continuous, hence there exists a uniformly continuous function  $\sigma(s): Y \rightarrow Y$  such that  $\sigma(s)(\xi(x)) = \xi(sx)$  ( $x \in X$ ). From (a) and (c) it follows that  $\sigma: S \rightarrow Y^Y$  is a continuous homomorphism and that  $F = \{\sigma(s): s \in S\}$  is uniformly equicontinuous. Let  $T$  be the closure of  $F$  in the product space  $Y^Y$ . Then  $T$  is a compact topological semigroup and  $vy = v(y)$  ( $v \in T, y \in Y$ ) defines a jointly continuous action of  $T$  on  $Y$ . Furthermore,  $(\sigma, \xi)$  is a homomorphism of  $(S, X)$  into  $(T, Y)$ , so there exists a uniformly continuous map  $\bar{\xi}: \bar{X} \rightarrow Y$  such that  $\bar{\xi} \circ \eta = \xi$  (Theorem 1.3). Thus  $\xi$  is  $\mathcal{U}$ -uniformly continuous, and it follows that  $\mathcal{V}$  is coarser than  $\mathcal{U}$ .

**2. More lemmas.** We shall assume throughout this section that  $(S, X)$  is a s.t.t.s. with identity 1, that  $X$  is a semitopological semigroup with identity 1, and that there exists a continuous homomorphism  $\phi: X \rightarrow S$  such that  $\phi(1) = 1$  and  $\phi(x)y = xy$  ( $x, y \in X$ ). If  $f \in C(X)$  and  $y \in X$ ,  $f_y$  shall denote the function  $x \rightarrow f(xy)$ . A subspace  $L$  of  $C(X)$  is said to be *right translation invariant* if  $f \in L$  implies that  $f_y \in L$  for all  $y \in X$ .

We shall denote by  $K(Q)$  the minimal ideal of the compact semitopological semigroup  $Q$  (see [6]). If  $B$  is a Banach space,  $L(B)$  and  $B^*$  are respectively the space of continuous linear operators and the space of continuous linear functionals on  $B$ .

**LEMMA 2.1.** *Let  $(\bar{S}, \bar{X}, \rho, \eta)$  be an almost periodic (resp. weakly almost periodic) compactification of  $(S, X)$ . Then in order for  $\bar{X}$  to be a topological (resp. semitopological) semigroup and  $\eta$  a homomorphism it is necessary and sufficient that  $A(X)$  (resp.  $W(X)$ ) be right translation invariant.*

*Proof.* The necessity is clear. For the sufficiency we consider only the weakly almost periodic case. By Corollary 1.5 and the proof of Theorem 1.3 we may suppose that  $\bar{X}$  is the maximal ideal space of  $W(X)$  and that  $\eta$  is the map defined by  $\eta(x)(f) = f(x)$  ( $x \in X, f \in W(X)$ ). Define  $u(s) \in L(W(X))$  ( $s \in S$ ) by  $u(s)(f) = {}_s f$ , and let  $T$  denote the closure of  $\{u(s); s \in \phi(X)\}$  in the weak operator topology of  $L(W(X))$ . Then  $T$  with this topology is a compact semitopological semigroup (under the operation composition) [6; Theorem 3.1]. Given  $v \in T$  define  $\Psi(v) \in W(X)^*$  by  $\Psi(v)(f) = (vf)(1)$  ( $f \in W(X)$ ). Clearly,  $\Psi$  is continuous in the weak\* topology of  $W(X)^*$ , and  $\Psi(u(\phi(x))) = \eta(x)$  ( $x \in X$ ). It follows that  $\Psi(T) = \bar{X}$ . If  $v, w \in T$  and  $\Psi(v) = \Psi(w)$ , then  $(vf)(x) = \Psi(v)(f_x) = \Psi(w)(f_x) = (wf)(x)$  ( $x \in X, f \in W(X)$ ), so  $v = w$ . Therefore  $\Psi$  is a homeomorphism of  $T$  onto  $\bar{X}$  and hence induces a multiplication on  $\bar{X}$  making the latter a semitopological semigroup and  $\Psi$  an anti-isomorphism. Finally,  $\eta(xy) = \Psi(u(\phi(xy))) = \Psi(u(\phi(y))u(\phi(x))) = \eta(x)\eta(y)$ , completing proof.

The proofs of the following two lemmas are straightforward and therefore omitted.

**LEMMA 2.2.** *Let  $(\bar{S}, \bar{X}, \rho, \eta)$  be a weakly almost periodic compactification of  $(S, X)$  and let  $W(X)$  be right translation invariant. If  $\bar{X}$  has a unique minimal left ideal and if  $K(\bar{S})$  is a group with identity  $e$ , then  $e\theta = \theta$  for all  $\theta \in K(\bar{X})$ .*

**LEMMA 2.3.** *Let  $(S^a, X^a, \rho, \eta)$  and  $(S^w, X^w, \rho', \eta')$  denote respectively almost periodic and weakly almost periodic compactifications of  $(S, X)$ . Suppose  $K(S^w)$  is a group with identity  $e$ ,  $W(X)$  and  $A(X)$  are right translation invariant,  $X^w$  has a unique minimal left ideal, and for some idempotent  $d \in K(X^w)$ ,  $u(\theta d) = (u\theta)d$  ( $\theta \in X^w, u \in S^w$ ). Then  $K(X^a)$  and  $K(X^w)$  are canonically isomorphic as semitopological semigroups.*

**3. Applications to semidirect products.** Let  $S$  and  $X$  be semitopological semigroups with identities, and let  $\tau: S \times X \rightarrow X$  satisfy  $\tau(s, xy) = \tau(s, x)\tau(s, y)$ ,  $\tau(st, x) = \tau(s, \tau(t, x))$ , and  $\tau(1, x) = x$ , ( $x, y \in X; s, t \in S$ ). Thus  $s \rightarrow \tau(s, \cdot)$  is a homomorphism from  $S$  into  $\text{Hom}(X)$ , the semigroup of all homomorphisms from  $X$  to  $X$ . We shall assume that  $\tau(s, \cdot)$  is continuous for each  $s \in S$  and that  $(s, x) \rightarrow x\tau(s, y)$  is continuous for each  $y \in X$ . The *semidirect product*  $S \circledast X$  of  $S$  and  $X$  is the topological space  $S \times X$  with multiplication defined by  $(s, x)(t, y) = (st, x\tau(s, y))$  ( $s, t \in S; x, y \in X$ ). The above assumptions on  $\tau$  imply that  $S \circledast X$  is a semitopological semigroup with identity  $(1, 1)$ .

Following Landstad [11] we define an action of  $S \circledast X$  on  $X$  by  $(s, x)y = x\tau(s, y)$  ( $x, y \in X; s \in S$ ). Let  $A(X)$  (resp.  $W(X)$ ) denote the

almost periodic (resp. weakly almost periodic) functions on  $X$  relative to this action, and let  $((S \circledcirc X)^a, X^a, \rho_0, \eta)$  be an almost periodic compactification of  $(S \circledcirc X, X)$  and  $(S^a, \rho)$  an almost periodic compactification of  $S$ . Both of these compactifications exist because  $S \circledcirc X$  and  $S$  have identities. The map  $\phi: X \rightarrow S \circledcirc X$ ,  $\phi(x) = (1, x)$ , is a homomorphism satisfying  $\phi(x)y = xy$ , hence the results of §2 apply. Thus, assuming  $A(X)$  is right translation invariant,  $X^a$  is a topological semigroup and  $\eta$  a continuous homomorphism.

Define  $\pi: S \times X^a \rightarrow X^a$  by  $\pi(s, \theta) = \rho_0(s, 1)\theta$ . Then  $s \mapsto \pi(s, \cdot)$  is a continuous homomorphism of  $S$  into  $\text{Hom}(X^a)$ . Furthermore,  $\{\pi(s, \cdot): s \in S\}$  is equicontinuous hence its closure  $E$  in the product space  $X^{a \times X^a}$  is a compact topological semigroup contained in  $\text{Hom}(X^a)$ . Thus there exists a continuous homomorphism  $u \mapsto \bar{\tau}(u, \cdot)$  of  $S^a$  into  $E$  such that  $\bar{\tau}(\rho(s), \cdot) = \pi(s, \cdot)$  ( $s \in S$ ). Note that

$$(1) \quad \bar{\tau}(\rho(s), \eta(x)) = \eta(\tau(s, x)) \quad (s \in S, x \in X).$$

We may now form the semidirect product  $S^a \circledcirc X^a$ , where multiplication is defined by

$$(2) \quad (u, \theta)(v, \psi) = (uv, \theta \bar{\tau}(u, \psi)) \quad (u, v \in S^a; \theta, \psi \in X^a).$$

It follows from (1) that the map  $\beta: S \circledcirc X \rightarrow S^a \circledcirc X^a$  defined by  $\beta(s, x) = (\rho(s), \eta(x))$  is a homomorphism.

**THEOREM 3.1.** *If  $S$  is a topological group and  $A(X)$  is right translation invariant, then  $(S^a \circledcirc X^a, \beta)$  is an almost periodic compactification of  $S \circledcirc X$ , and*

$$(3) \quad K(S^a \circledcirc X^a) = S^a \circledcirc K(X^a).$$

*Proof.* For the first part it suffices to show that given any continuous homomorphism  $\alpha$  from  $S \circledcirc X$  into a compact topological semigroup  $T$ , there exists a continuous homomorphism  $\bar{\alpha}: S^a \circledcirc X^a \rightarrow T$  such that  $\bar{\alpha} \circ \beta = \alpha$ . Following Landstad we define an action of  $S \circledcirc X$  on  $T$  by

$$(s, x)t = \alpha(s, x)t\alpha(s^{-1}, 1) \quad (s \in S, x \in X, t \in T).$$

Define  $\alpha_1: S \rightarrow T$  by  $\alpha_1(s) = \alpha(s, 1)$  and  $\alpha_2: X \rightarrow T$  by  $\alpha_2(x) = \alpha(1, x)$ . Then  $\alpha_1$  is a continuous homomorphism, hence there exists a continuous homomorphism  $\bar{\alpha}_1: S^a \rightarrow T$  such that  $\bar{\alpha}_1 \circ \rho = \alpha_1$ . Also,  $(\iota, \alpha_2)$  is a homomorphism of  $(S \circledcirc X, X)$  into  $(S \circledcirc X, T)$ , where  $\iota: S \circledcirc X \rightarrow S \circledcirc X$  is the identity map. Since the action on  $T$  is equicontinuous, Remark 1.6 implies the existence of a continuous map  $\bar{\alpha}_2: X^a \rightarrow T$  such that  $\bar{\alpha}_2 \circ \eta = \alpha_2$ . The required map  $\bar{\alpha}$  is defined by  $\bar{\alpha}(u, \theta) = \bar{\alpha}_2(\theta)\bar{\alpha}_1(u)$ , ( $u \in S^a, \theta \in X^a$ ).



To verify (3) note first that  $\bar{\tau}(u, K(X^a)) \subset K(X^a)(u \in S^a)$  so that  $S^a \bar{\tau} K(X^a)$  is defined and is an ideal, and  $K(S^a \bar{\tau} X^a) \subset S^a \bar{\tau} K(X^a)$ . Now let  $(u, \theta) \in S^a \times K(X^a)$  and choose  $e \in K(X^a)$  such that  $e^2 = e$  and  $e\theta = \theta$  [6; Theorem 2.3]. Since  $(\rho(1), e)(u, \theta) = (u, \theta)$ , it suffices to show that  $(\rho(1), e) \in K(S^a \bar{\tau} X^a)$ . Now,  $S^a \times eX^a$  is a right ideal in  $S^a \bar{\tau} X^a$  and hence contains an idempotent  $d \in K(S^a \bar{\tau} X^a)$  [6; Lemma 2.2, Theorem 2.3]. It is easily seen that  $d = (\rho(1), e_1)$ , where  $e_1^2 = e_1 \in K(X^a)$ . Then  $S^a \times e_1 X^a = d(S^a \bar{\tau} X^a) \subset S^a \times eX^a$ , and by the minimality of  $eX^a$ ,  $e_1 X^a = eX^a$ . Therefore  $(\rho(1), e) \in d(S^a \bar{\tau} X^a) \subset K(S^a \bar{\tau} X^a)$ .

The next result shows that the conclusions of Theorem 3.1 are valid for a large class of semitopological semigroups  $X$ . Theorems 3.1 and 3.2 generalize Theorem 4 of [11].

**THEOREM 3.2.** *If  $S$  is a topological group then  $A(X)$  is right translation invariant if any one of the following conditions holds:*

- (i)  $X$  is abelian.
- (ii)  $X$  is a compact topological semigroup.
- (iii)  $X$  is a topological group.

*Proof.* Clearly (i) implies that  $A(X)$  is right translation invariant. Suppose (ii) holds and let  $f \in A(X)$ ,  $z \in X$  and  $(s_\alpha, x_\alpha)$  a net in  $S \times X$ . There exists  $g \in A(X)$ ,  $y \in X$  and a subnet  $(s_\beta, x_\beta)$  such that  $z_\beta = \tau(s_\beta^{-1}, z) \rightarrow y$  and  $f(x_\beta \tau(s_\beta, x)) \rightarrow g(x)$  uniformly in  $x \in X$ . Then  $g(xz_\beta) \rightarrow g(xy)$  uniformly in  $x \in X$ , hence by the triangle inequality  $f_z(x_\beta \tau(s_\beta, x)) = f(x_\beta \tau(s_\beta, xz_\beta)) \rightarrow g(xy)$  uniformly in  $x \in X$ . Therefore  $f_z \in A(X)$ .

Now suppose (iii) holds. Let  $\mathcal{U}$  denote the coarsest uniform structure on  $X$  making each  $f \in A(X)$  uniformly continuous. We shall show that for each  $x_0 \in X$ ,  $x \rightarrow xx_0$  is  $\mathcal{U}$ -uniformly continuous. Let  $\mathcal{B}$  denote the base for  $\mathcal{U}$  consisting of all symmetric  $U \in \mathcal{U}$  such that  $(x, y) \in U$  implies that  $((s, z)x, (s, z)y) \in U$  for all  $(s, z) \in S \bar{\tau} X$  (see Theorem 1.7). If  $U \in \mathcal{B}$  and  $X = \bigcup_{i=1}^n U[x_i]$ , define

$$[U] = [U; x_1, \dots, x_n] = \{(x, y): (xx_i, yx_i) \in U^2 (1 \leq i \leq n)\}.$$

Let  $\mathcal{C}$  be the set of all  $[U]$  ( $U \in \mathcal{B}$ ). Claim that  $\mathcal{C}$  is a base for a uniform structure  $\mathcal{V}$  on  $X$ . For it is clear that  $(x, x) \in [U]$  ( $x \in X$ ) and  $[U]^{-1} = [U]$ , and the remaining axioms are easily verified with the help of the inclusion  $[U; x_1, \dots, x_n]^2 \subset [U^2; x_1, \dots, x_n]$  and the following fact (whose simple proof we omit):

$$(4) \quad U, V \in \mathcal{B} \text{ and } V^2 \subset U \text{ implies } [V; y_1, \dots, y_m] \subset [U; x_1, \dots, x_n].$$

We shall show that  $\mathcal{V}$  is coarser than  $\mathcal{U}$  by verifying that (a), (b), and (c) of Theorem 1.7 hold. Let  $\mathcal{T}$  denote the given topology

of  $X$  and  $\mathcal{T}(\mathcal{U})$  and  $\mathcal{T}(\mathcal{V})$  the topologies induced by the uniform structures  $\mathcal{U}$  and  $\mathcal{V}$  respectively. If  $U \in \mathcal{B}$  then for any  $x \in X$ ,  $[U; x_1, \dots, x_n][x] = \bigcap_{i=1}^n xU^2[x_i]x_i^{-1}$  is a  $\mathcal{T}$ -neighborhood of  $x$  since  $\mathcal{T}(\mathcal{U})$  is coarser than  $\mathcal{T}$  and  $\mathcal{T}$  is a group topology. Thus  $\mathcal{T}(\mathcal{V})$  is coarser than  $\mathcal{T}$ , verifying (a). Call a neighborhood  $N$  of 1 in  $(X, \mathcal{T})$  *left relatively dense* with respect to  $y_1, \dots, y_k \in X$  if  $X = \bigcup_{i=1}^k y_i N$ . Let  $[U] = [U; x_1, \dots, x_n] \in \mathcal{C}$ . For each  $j = 1, \dots, n$ ,  $U[x_j]x_j^{-1} = x_j U[1]x_j^{-1}$  is left relatively dense with respect to  $x_i x_j^{-1}$  ( $1 \leq i \leq n$ ), and since  $(x_j U[1]x_j^{-1})^{-1}(x_j U[1]x_j^{-1}) = x_j U[1]^{-1}U[1]x_j^{-1} \subset x_j U^2[1]x_j^{-1} = U^2[x_j]x_j^{-1}$ , it follows from Proposition 3 of [1] that  $\bigcap_{j=1}^n U^2[x_j]x_j^{-1}$  is left relatively dense with respect to say  $y_1, \dots, y_k$ . Thus  $X = \bigcup_{i=1}^k [U][y_i]$ , verifying (b). To verify (c) let  $[U] \in \mathcal{C}$  and choose  $[V] = [V; x_1, \dots, x_n] \in \mathcal{C}$  such that  $V^4 \subset U$ . Let  $(x, y) \in [V]$ ,  $z \in X$  and  $s \in S$ . For each  $1 \leq i \leq n$  there exists  $j$  such that  $(x_j, \tau(s^{-1}, x_i)) \in V$ . Then  $(xx_j, x\tau(s^{-1}, x_i))$ ,  $(yx_j, y\tau(s^{-1}, x_i)) \in V$  hence  $(x\tau(s^{-1}, x_i), y\tau(s^{-1}, x_i)) \in V^4$  and so  $(z\tau(s, x)x_i, z\tau(s, y)x_i) \in V^4$ . By (4) then,  $((s, z)x, (s, z)y) \in [U]$ . Therefore  $\mathcal{V}$  is coarser than  $\mathcal{U}$ .

Now let  $x_0 \in X$  and  $U \in \mathcal{B}$ . If  $X = \bigcup_{i=1}^n U[x_i]$ , then  $[U] = [U; x_0, x_1, \dots, x_n] \in \mathcal{U}$ , and  $(x, y) \in [U]$  implies that  $(xx_0, yx_0) \in U^2$ . Therefore  $x \mapsto xx_0$  is  $\mathcal{U}$ -uniformly continuous and it follows from Theorem 1.7 that there exists a uniformly continuous function  $F: X^a \rightarrow X^a$  such that  $F(\eta(x)) = \eta(xx_0)$  ( $x \in X$ ). Thus if  $f \in A(X)$  then  $\hat{f} \circ F \in C(X^a)$ , hence  $f_{x_0} = \hat{\eta}(f \circ F) \in A(X)$ .

Now let  $((S \hat{\odot} X)^w, X^w, \rho'_0, \eta')$  and  $(S^w, \rho')$  be weakly almost periodic compactifications of  $(S \hat{\odot} X, X)$  and  $S$  respectively, and assume that  $W(X)$  is right translation invariant. As in the almost periodic case, we may form the semidirect product  $S^w \hat{\odot} X^w$ , and the analogs of (1) and (2) are valid. However,  $S^w \hat{\odot} X^w$  need not be a weakly almost periodic compactification of  $S \hat{\odot} X$ , even in the case of a direct product [7]. We can, however, express  $K((S \hat{\odot} X)^w)$  under certain conditions as a semidirect product.

**THEOREM 3.3.** *Let  $S$  be a topological group,  $A(X)$  and  $W(X)$  right translation invariant, and assume that  $K((S \hat{\odot} X)^w)$  and  $K(X^w)$  are groups. Then  $K((S \hat{\odot} X)^w)$  and  $K(S^w) \hat{\odot} K(X^w)$  are canonically isomorphic.*

*Proof.* Note first that since  $S$  is a topological group,  $W(S)$  has an invariant mean [12], hence  $K(S^w)$  is a group [6]. Let  $\gamma: S^w \rightarrow S^a$  be a continuous homomorphism such that  $\gamma \circ \rho' = \rho_0$ , and let  $(\sigma, \xi)$  be a homomorphism of  $((S \hat{\odot} X)^w, X^w)$  onto  $((S \hat{\odot} X)^a, X^a)$  such that  $\sigma \circ \rho'_0 = \rho_0$  and  $\xi \circ \eta' = \eta$ . If  $d$  denotes the identity of  $K(X^w)$ , then by the analog of (1),  $u(\theta d) = (u\theta)d$  ( $u \in (S \hat{\odot} X)^w$ ,  $\theta \in X^w$ ), hence by

Lemma 2.3,  $\xi_0 = \xi|K(X^w)$  is an isomorphism of  $K(X^w)$  onto  $K(X^a)$ . By the same lemma  $\gamma_0 = \gamma|K(S^w)$  and  $\sigma_0 = \sigma|K((S\overline{\otimes}X)^w)$  are isomorphisms onto  $K(S^a) = S^a$  and  $K((S\overline{\otimes}X)^a)$  respectively. Define  $\delta: K(S^w)\overline{\otimes}K(X^w) \rightarrow S^a\overline{\otimes}K(X^a)$  by  $\delta(u, \theta) = (\gamma_0(u), \xi_0(\theta))$ . Then  $\delta$  is an isomorphism, and since  $K((S\overline{\otimes}X)^a) = S^a\overline{\otimes}K(X^a)$  (Theorem 3.1),  $\mu = \sigma_0^{-1} \circ \delta$  is the required isomorphism of  $K(S^w)\overline{\otimes}K(X^w)$  onto  $K((S\overline{\otimes}X)^w)$ .

REMARK 3.4. Let  $W(S\overline{\otimes}X)_p$  denote the closed subspace of  $W(S\overline{\otimes}X)$  spanned by the coefficients of the finite dimensional unitary representations of  $S\overline{\otimes}X$  (see [6, p. 85]). Under the conditions of the previous theorem,  $W(S\overline{\otimes}X)_p$  is the completed  $\varepsilon$ -tensor product of  $A(S)$  and the space of all  $f \in W(X)$  such that  $f(d\theta) = f(\theta)$  for all  $\theta \in X^w$ , where  $d$  is the identity of  $K(X^w)$ . This follows from Theorem 3.3 together with Theorem 5.7 and Lemma 5.10 of [6] and the identity  $\mu(e_1\rho'(s), d\eta'(x)) = e\rho'_0(s, x)(s \in S, x \in X)$ , where  $e$  is the identity of  $K((S\overline{\otimes}X)^w)$ ,  $e_1$  the identity of  $K(S^w)$ , and  $\mu$  the isomorphism obtained in the proof of Theorem 3.3.

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