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SHOWERING SPACES

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For each triple α, β, ω where $\alpha > 0$ and $\beta \geq \aleph_0$ are cardinal numbers and $\omega > 0$ is an ordinal number, $S_{\alpha, \beta}^\omega$ is defined. It is proved that $S_{\alpha, \beta}^\omega$ is Hausdorff, paracompact and zero-dimensional. Various topological properties of $S_{\alpha, \beta}^\omega$ are discussed and are used to give examples.

In this paper the showering space $S_{\alpha, \beta}^\omega$ is defined where α and β are cardinal numbers and ω is an ordinal number. An important characteristic of the showering spaces is that various topological properties are determined by set theoretic properties of the indices α, β , and ω . Thus, for example, whether or not $S_{\alpha, \beta}^\omega$ is Baire is essentially a question of whether or not ω is a sequential ordinal. Theorems of this nature are given in §1-§5. Specific examples using these results, such as an almost P -space which contains a closed copy of the space of rationals, are given in §6.

It has come to my attention that the special cases $S_{\aleph_0, \aleph_0}^{\omega_0 + \omega_0}$ and $S_{\aleph_0, \aleph_0}^{\omega_1}$ are defined in [3] and [1].

1. Definition and basic properties of showering spaces. Given spaces will be assumed to be completely regular (Hausdorff). Let $\alpha > 0$ be a fixed cardinal number and let $\omega > 0$ be a fixed ordinal number. As usual we identify a cardinal with the initial ordinal of that cardinal. If γ and δ are ordinals, $[\gamma, \delta)$ will denote the half-open interval $\{\rho \mid \gamma \leq \rho < \delta\}$ and $[\gamma, \delta]$ will denote the closed interval $\{\rho \mid \gamma \leq \rho \leq \delta\}$. If A is a set, $|A|$ will denote the cardinal of A . Many of the proofs of this section, particularly Theorem 1.6, were given for special cases in [1] and [3].

For each $\gamma < \omega$, let $R_\gamma = [0, \alpha)^{[0, \gamma]}$. In particular, $R_0 = \{\emptyset\}$. When we are discussing \emptyset as the element of R_0 , we will usually denote it p_0 , so $R_0 = \{p_0\}$. For $0 < \gamma < \omega$, the elements of R_γ are nets valued in $[0, \alpha)$ and indexed by the initial segment $[0, \gamma)$ of ordinals. Let $S_\alpha^\omega = \bigcup_{\gamma < \omega} R_\gamma$. Define a relation $<$ on S_α^ω by $(x_\lambda)_{\lambda < \gamma_1} < (y_\lambda)_{\lambda < \gamma_2}$ if $\gamma_1 \leq \gamma_2$ and $x_\lambda = y_\lambda$ for all $\lambda < \gamma_1$. In particular, $p_0 < x$ for all $x \in S_\alpha^\omega$. It is easy to verify that $<$ is a partial ordering on S_α^ω .

PROPOSITION 1.1. (1) $(S_\alpha^\omega, <)$ is a tree.

(2) If $\gamma \leq \delta < \omega$ and $y \in R_\delta$, then there is exactly one $x \in R_\gamma$ such that $x < y$.

(3) If $\gamma + 1 < \omega$, $p \in R_\gamma$, and $A_p = \{x \in R_{\gamma+1} \mid p < x\}$, then $|A_p| = \alpha$.

(4) If $x \in R_{r_1}$, $y \in R_{r_2}$, $x < y$, then $\gamma_1 \leq \gamma_2$ and $\gamma_1 = \gamma_2$ if and only if $x = y$.

(5) If $\gamma \leq \gamma_0 < \omega$, $x \in R_\gamma$, then there is a $y \in R_{\gamma_0}$ such that $x < y$.

Proof. (1) Suppose $x \in S_\alpha^\omega$. We must show that $\{y \mid y < x\}$ is well-ordered. Suppose $a < x$, $b < x$, $a = (a_\lambda)_{\lambda < r_1}$, $b = (b_\lambda)_{\lambda < r_2}$, $x = (x_\lambda)_{\lambda < r_3}$, with $\gamma_1 \leq \gamma_2$. Then $\gamma_1 \leq \gamma_2 \leq \gamma_3$ and if $\gamma < \gamma_1$, $a_\lambda = x_\lambda = b_\lambda$, so $a < b$. Hence $\{y \mid y < x\}$ is a chain. If $\emptyset \neq A \subseteq \{y \mid y < x\}$, we must show that there is a $q \in A$ such that $q < y$ for all $y \in A$. Let γ_0 be the smallest ordinal such that there is an element of A of the form $(y_\lambda)_{\lambda < r_0}$. If $q = (q_\lambda)_{\lambda < r_0} \in A$ and $(z_\lambda)_{\lambda < r_1} \in A$, and if $x = (x_\lambda)_{\lambda < r_2}$, then $\gamma_0 \leq \gamma_1 \leq \gamma_2$ and if $\lambda < \gamma_0$, $q_\lambda = x_\lambda = z_\lambda$, so $q < (z_\lambda)_{\lambda < r_1}$. Hence, A has a smallest element.

(2) If $y = (y_\lambda)_{\lambda < \delta}$, and $x = (y_\lambda)_{\lambda < r}$, $x \in R_r$, $x < y$, and x is the only element of R_r such that $x < y$.

(3) If $p = (p_\lambda)_{\lambda < r}$, $A_p = \{(x_\lambda)_{\lambda \leq r} \mid x_\lambda = p_\lambda \text{ for all } \lambda < r\}$. Therefore, $|A_p| = |[0, \alpha]| = \alpha$.

(4) The only statement that needs proof is that if $\gamma_1 = \gamma_2$, $x = y$. But if $x = (x_\lambda)_{\lambda < r_1}$, $y = (y_\lambda)_{\lambda < r_1}$, then $x_\lambda = y_\lambda$ for all $\lambda < \gamma_1$, so $x = y$.

(5) If $x = (x_\lambda)_{\lambda < r}$, let $y = (y_\lambda)_{\lambda < r_0}$ be defined by $y_\lambda = x_\lambda$ if $\lambda < \gamma$ and $y_\lambda = 0$ if $\gamma \leq \lambda < \gamma_0$. Then $y \in R_{r_0}$ and $x < y$.

DEFINITION. $(S_\alpha^\omega, <)$ is called the *showering tree of type α, ω* . If $\gamma + 1 < \omega$ and $p \in R_\gamma$, A_p will denote the set $\{x \in R_{\gamma+1} \mid p < x\}$. If $\gamma + 1 = \omega$ and $p \in R_\gamma$, let $A_p = \emptyset$.

REMARK. Any tree can be order-embedded in a showering tree. Specifically, if T is a tree, $\omega = \sup\{\lambda \mid T \text{ has a chain of order type } \lambda\}$, and $\alpha = \sup\{\gamma \mid \text{some element of } T \text{ has } \gamma \text{ immediate successors}\}$, then T can be order-embedded in S_α^ω .

We will now introduce a topology on S_α^ω . For $p \in S_\alpha^\omega$ and $A \subseteq A_p$, let $\tilde{U}_p(A) = \{x \in S_\alpha^\omega \mid p < x \text{ but it is not the case that there is an } a \in A \text{ such that } a < x\}$. If $p \in R_\gamma$ and $A \subseteq R_{\gamma+1}$, $\tilde{U}_p(A \cap A_p)$ will be denoted $U_p(A)$. Note that $U_p(\emptyset)$ is just the set of successors of p . Now let $\beta \geq \aleph_0$ be a fixed cardinal number. For $p \in S_\alpha^\omega$, let $\mathcal{U}_p = \{U_p(A) \mid |A| < \beta\}$.

LEMMA 1.2. If $p \in R_\gamma$, $A \subseteq R_{\gamma+1}$, and $q \in A_p \setminus A$, then $U_q(\emptyset) \subseteq U_p(A)$.

Proof. Suppose $x \in U_q(\emptyset)$. Then $q < x$, so q is the unique element of $R_{\gamma+1}$ such that $q < x$. Therefore, there is no $a \in A$ such that $a < x$. Hence, $x \in U_p(A)$.

THEOREM 1.3. $\{\mathcal{U}_p | p \in S_\alpha^\omega\}$ is a system of neighborhood bases for a topology on S_α^ω .

Proof. Clearly, $p \in U$ for each $U \in \mathcal{U}_p$. If $U_p(A), U_p(\tilde{A}) \in \mathcal{U}_p$, $U_p(A) \cap U_p(\tilde{A}) = U_p(A \cup \tilde{A})$, and since $|A \cup \tilde{A}| \leq |A| + |\tilde{A}| < \beta + \beta = \beta$, $U_p(A) \cap U_p(\tilde{A}) \in \mathcal{U}_p$. If $U_p(A) \in \mathcal{U}_p$ and $s \in U_p(A) \setminus \{p\}$, choose $q \in A_p$ such that $q < s$. Such a q exists by Proposition 1.1, $q \notin A$. Hence $U_q(\phi) \subseteq U_p(A)$ by Lemma 1.2. But since $q < s$, $U_s(\emptyset) \subseteq U_q(\emptyset)$. Therefore $U_s(\emptyset) \subseteq U_p(A)$. Since $U_s(\emptyset) \in \mathcal{U}_s$, this completes the proof.

DEFINITION. Let $S_{\alpha,\beta}^\omega$ denote the topological space obtained from S_α^ω by taking as a neighborhood base at $p \in S_\alpha^\omega$ the collection \mathcal{U}_p . $S_{\alpha,\beta}^\omega$ is called the *showering space of type α, β, ω* .

LEMMA 1.4. If $p, q \in R_\gamma$, $p \neq q$, then $U_p(\emptyset) \cap U_q(\emptyset) = \emptyset$. If $s \in R_\beta$, $\gamma < \delta$, and t is the unique element of $R_{\gamma+1}$ such that $t < s$, then $U_p(\{t\}) \cap U_s(\emptyset) = \emptyset$.

Proof. The first statement is immediate from (2) of Proposition 1.1. For the second statement, we observe that $U_p(\{t\}) \cap U_t(\emptyset) = \emptyset$, and since $U_s(\emptyset) \subseteq U_t(\emptyset)$, $U_p(\{t\}) \cap U_s(\emptyset) = \emptyset$.

LEMMA 1.5. (1) Every set of the form $U_p(A)$ is closed in $S_{\alpha,\beta}^\omega$ (irrespective of the cardinal of A).

(2) Every element of \mathcal{U}_p is open in $S_{\alpha,\beta}^\omega$ for all p .

Proof. (1) Suppose $p \in R_\gamma$ and $q \notin U_p(A)$. Suppose $q \in R_\beta$. If $\delta < \gamma$, choose $s \in R_{\delta+1}$ such that $s < p$; then by Lemma 1.4, $U_q(\{s\}) \cap U_p(\emptyset) = \emptyset$, so $U_q(\{s\}) \cap U_p(A) = \emptyset$. Therefore, we may assume $\gamma \leq \delta$. If $\gamma = \delta$, $q \neq p$ so $U_q(\emptyset) \cap U_p(\emptyset) = \emptyset$ by Lemma 1.4; hence $U_q(\emptyset) \cap U_p(A) = \emptyset$. If $\gamma < \delta$, choose $t \in R_{\gamma+1}$ such that $t < q$. Then $t \in A$ or $t \notin A_p$. If $t \in A$, $U_t(\emptyset) \cap U_p(A) = \emptyset$, so $U_q(\emptyset) \cap U_p(A) = \emptyset$. If $t \notin A_p$, $t \in A_s$ for some $s \neq p$. $U_s(\emptyset) \cap U_p(\emptyset) = \emptyset$, so $U_q(\emptyset) \cap U_p(A) = \emptyset$. Thus q has a neighborhood which misses $U_p(A)$. Therefore, $U_p(A)$ is closed.

(2) This is immediate from Lemma 1.2 and the fact that if $q < t$, $U_t(\emptyset) \subseteq U_q(\emptyset)$.

THEOREM 1.6. $S_{\alpha,\beta}^\omega$ is Hausdorff, zero-dimensional (that is, $\text{ind } S_{\alpha,\beta}^\omega = 0$), and paracompact. In fact, every open cover of $S_{\alpha,\beta}^\omega$ has a discrete open refinement which covers $S_{\alpha,\beta}^\omega$.

Proof. That $S_{\alpha,\beta}^\omega$ is Hausdorff is immediate from Lemma 1.4. $S_{\alpha,\beta}^\omega$ is zero-dimensional by Lemma 1.5. We now prove that $S_{\alpha,\beta}^\omega$ is

paracompact. Suppose \mathcal{V} is an open cover of $S_{\alpha,\beta}^\omega$. Choose $A \subseteq A_{p_0}$ such that $|A| < \beta$ and $U_{p_0}(A) \subseteq V$ for some $V \in \mathcal{V}$. Let $\mathcal{V}_0 = \{U_{p_0}(A)\}$. Suppose inductively that \mathcal{V}_γ is defined for $\gamma < \gamma_0 < \omega$ such that $\bigcup_{r < \gamma_0} \mathcal{V}_r$ is an open refinement of \mathcal{V} which covers $\bigcup_{r < \gamma_0} R_r$ and whose elements are pairwise disjoint. Let $\tilde{\mathcal{V}}_{\gamma_0} = \bigcup_{r < \gamma_0} \mathcal{V}_r$. For each $p \in R_{\gamma_0} \setminus \bigcup \{V \mid V \in \tilde{\mathcal{V}}_{\gamma_0}\}$, choose $B_p \subseteq A_p$ such that $|B_p| < \beta$ and such that $U_p(B_p) \subseteq V$ for some $V \in \mathcal{V}$. Let $\mathcal{V}_{\gamma_0} = \{U_p(B_p) \mid p \in R_{\gamma_0} \setminus \bigcup \{V \mid V \in \tilde{\mathcal{V}}_{\gamma_0}\}\}$. Then $\bigcup_{r < \gamma_0+1} \mathcal{V}_r$ is an open refinement of \mathcal{V} which covers $\bigcup_{r < \gamma_0+1} R_r$ and whose elements are pairwise disjoint. Let $\tilde{\mathcal{V}} = \bigcup_{r < \omega} \mathcal{V}_r$. Then $\tilde{\mathcal{V}}$ is an open refinement of \mathcal{V} by pairwise disjoint sets which covers $S_{\alpha,\beta}^\omega$. Hence, $S_{\alpha,\beta}^\omega$ is paracompact.

REMARK 1.7. A slight modification of the proof of Theorem 1.6 shows that for any α , $S_{\alpha,\aleph_0}^\omega$ and $S_{\alpha,\aleph_1}^\omega$ are Lindelöf.

2. Isolated points and the Baire property. In this section we characterize the Baire showering spaces.

PROPOSITION 2.1. (1) If $\beta > \alpha$, $S_{\alpha,\beta}^\omega$ is discrete (and hence Baire).

(2) If $\omega = \gamma + 1$, every point of R_γ is isolated.

(3) If $\alpha \geq \beta$, $\gamma + 1 < \omega$, and $p \in R_\gamma$, then p is not isolated. In particular, if $\alpha \geq \beta$ and ω is a limit ordinal, $S_{\alpha,\beta}^\omega$ is dense-in-itself.

Proof. (1) $\{p\} = U_p(A_p)$, and if $\alpha < \beta$, $|A_p| = \alpha < \beta$, so $U_p(A_p)$ is open.

(2) If $A_p = \emptyset$, $U_p(\emptyset) = \{p\}$, so p is isolated.

(3) If $p \in R_\gamma$, $U_p(A) \in \mathcal{U}_p$, then $|A| < \beta \leq \alpha$, so there is an $x \in A_p \setminus A$. Then $x \in U_p(A)$. Hence, p is not isolated.

THEOREM 2.2. Suppose $\alpha \geq \beta$.

(1) If $\omega = \gamma + 1$, then the set R_γ of isolated points is dense in $S_{\alpha,\beta}^\omega$. Hence, $S_{\alpha,\beta}^\omega$ is Baire.

(2) Suppose ω is a limit ordinal. Then the following are equivalent:

(i) ω is not a sequential ordinal, that is, if $\omega_i < \omega$ for $i = 1, 2, \dots$, then $\sup \omega_i < \omega$.

(ii) $S_{\alpha,\beta}^\omega$ is Baire.

(iii) $S_{\alpha,\beta}^\omega$ is second category in itself.

Proof. (1) Suppose $p \in S_{\alpha,\beta}^\omega \setminus R_\gamma$ and suppose $U_p(A) \in \mathcal{U}_p$. Since $\alpha \geq \beta$, there is an $x \in A_p \setminus A$. By Proposition 1.1, there is a $y \in R_\gamma$ such that $x < y$. Then $y \in U_p(A)$.

(2) (i) implies (ii). Suppose ω is not a sequential ordinal. Let

$\{D_i\}_{i=1}^\infty$ be a decreasing sequence of open dense subsets of $S_{\alpha,\beta}^\omega$. Suppose $p \in S_{\alpha,\beta}^\omega$ and $U_p(A) \in \mathcal{U}_p$. We must show $U_p(A) \cap \bigcap_{i=1}^\infty D_i \neq \emptyset$. $U_p(A) \cap D_1 \neq \emptyset$, so there is a set $U_{x_1}(B_1)$ such that $|B_1| < \beta$ and $U_{x_1}(B_1) \subseteq U_p(A) \cap D_1$. Choose $y^1 \in A_p \setminus B_1$. Then $U_{y^1}(\emptyset) \subseteq U_p(A) \cap D_1$. Inductively, suppose y^1, y^2, \dots, y^{n-1} are chosen so that $U_{y^k}(\emptyset) \subseteq U_p(A) \cap D_k$, and $y^1 < y^2 < \dots < y^{n-1}$. There is a set $U_{x_n}(B_n) \in \mathcal{U}_{y^{n-1}}$ such that $U_{x_n}(B_n) \in \mathcal{U}_{y^{n-1}}(\emptyset) \cap D_n$. Choose $y^n \in A_{x_n} \setminus B_n$. Then $U_{y^n}(\emptyset) \subseteq U_p(A) \cap D_n$, and $y_{n-1} < y_n$. Hence, we have inductively defined a sequence $\{y^n\}_{n=1}^\infty$ such that $y^1 < y^2 < \dots$ and such that $y^n \in U_p(A) \cap D_n$. Suppose $y^n \in R_{\gamma_n}$ and let $\delta = \sup \{\gamma_n \mid n = 1, 2, \dots\}$. $\delta < \omega$ since ω is not sequential. Let $y = (\tilde{y}_\lambda)_{\lambda < \delta} \in R_\delta$ be defined by $\tilde{y}_\lambda = y_\lambda^n$ where $\lambda < \gamma_n$. y is well-defined since $y_k < y_{k+1}$ for all k . $y^n < y$ for all n . Hence $y \in \bigcap_{n=1}^\infty U_{y^n}(\emptyset) \subseteq U_p(A) \cap \bigcap_{n=1}^\infty D_n$. Thus $S_{\alpha,\beta}^\omega$ is Baire.

(ii) implies (iii) is trivial.

(iii) implies (i). If ω is sequential, there are ordinals $\gamma_1, \gamma_2, \dots$, such that $\gamma_k < \omega$ for all k and such that $\omega = \sup_k \gamma_k$. Let $C_k = \bigcup_{\lambda < \gamma_k} R_\lambda$. Then each C_k is closed, and by Proposition 1.1(5), each C_k has empty interior. But $S_{\alpha,\beta}^\omega = \bigcup_{k=1}^\infty C_k$. Hence, $S_{\alpha,\beta}^\omega$ is not second category in itself.

3. P -spaces and almost P -spaces. If X is a space, a point $x \in X$ is called a P -point if every G_α containing x is a neighborhood of x . A space X is called a P -space if every point of X is a P -point. X is called an almost P -space if every nonempty G_α of X has non-empty interior. Every P -space is clearly an almost P -space. In Proposition 2.1 it was proved that if $\alpha < \beta$, $S_{\alpha,\beta}^\omega$ is discrete and hence a P -space. In this section we characterize the P -spaces and the almost P -spaces among the nondiscrete showering spaces. We note that since $S_{\alpha,\beta}^1$ consists of a single point, the case $\omega = 1$ will not be of interest to us.

PROPOSITION 3.1. *Suppose $\alpha \geq \beta$ and $\omega > 1$. Then the following are equivalent:*

- (i) $S_{\alpha,\beta}^\omega$ is a P -space.
- (ii) $S_{\alpha,\beta}^\omega$ has a non-isolated P -point.
- (iii) β is not a sequential cardinal.

Proof. (i) implies (ii) is trivial since by Proposition 2.1, p_0 is not isolated.

(ii) implies (iii). Let p be a non-isolated P -point. Suppose $\beta_i < \beta$ for $i = 1, 2, \dots$. Let B_1, B_2, \dots be pairwise disjoint subsets of A_p such that $|B_k| = \beta_k$. Then $\bigcap_{k=1}^\infty U_p(B_k) = U_p(\bigcup_{k=1}^\infty B_k)$ is a neighborhood of p . Therefore, $\sum_{k=1}^\infty \beta_k = \sum_{k=1}^\infty |B_k| = |\bigcup_{k=1}^\infty B_k| < \beta$. Hence β is not sequential.

(iii) implies (i). Suppose $p \in S_{\alpha, \beta}^\omega$ and suppose $p \in \bigcap_{i=1}^\infty U_p(B_i)$ where each $U_p(B_i) \in \mathcal{Z}_p$, and $B_i \subseteq A_p$ for each i . Then

$$\bigcap_{i=1}^\infty U_p(B_i) = U_p\left(\bigcup_{i=1}^\infty B_i\right),$$

and $|\bigcup_{i=1}^\infty B_i| \leq \sum_{i=1}^\infty \beta_i < \beta$ since $\beta_i < \beta$ for each i and β is not sequential. Hence $\bigcap_{i=1}^\infty U_p(B_i)$ is a neighborhood of p . Thus, $S_{\alpha, \beta}^\omega$ is a P -space.

PROPOSITION 3.2. *Suppose $\alpha \geq \beta$ and $\omega > 1$. Then $S_{\alpha, \beta}^\omega$ is an almost P -space if and only if $\sum_{i=1}^\infty \beta_i < \alpha$ whenever $\beta_i < \beta$ for each i .*

Proof. Suppose there were β_1, β_2, \dots such that $\beta_i < \beta$ for each i but $\sum_{i=1}^\infty \beta_i = \alpha$. Choose pairwise disjoint subsets B_1, B_2, \dots of A_{p_0} such that $\bigcup_{i=1}^\infty B_i = A_{p_0}$ and $|B_i| = \beta_i$. Then $\bigcap_{i=1}^\infty U_{p_0}(B_i) = U_{p_0}(\bigcup_{i=1}^\infty B_i) = U_{p_0}(A_{p_0}) = \{p_0\}$ which has empty interior by Proposition 2.1(3). Thus $S_{\alpha, \beta}^\omega$ is not almost P -space. For the converse, suppose $\sum_{i=1}^\infty \beta_i < \alpha$ whenever each $\beta_i < \beta$. Then for each $p \in S_{\alpha, \beta}^\omega$, if $U_p(B_i) \in \mathcal{Z}_p$ where $B_i \subseteq A_p$ for each i , $\bigcap_{i=1}^\infty U_p(B_i) = U_p(\bigcup_{i=1}^\infty B_i)$. $|\bigcup_{i=1}^\infty B_i| \leq \sum_{i=1}^\infty |B_i| \leq \sum_{i=1}^\infty \beta_i < \alpha$. Hence there is a $q \in A_p \setminus \bigcup_{i=1}^\infty B_i$; $U_q(\emptyset) \subseteq \bigcap_{i=1}^\infty U_p(B_i)$, so $\text{int} \bigcap_{i=1}^\infty U_p(B_i) \neq \emptyset$.

COROLLARY 3.3. *Suppose $\omega > 1$. Then the following are equivalent:*

- (i) $S_{\alpha, \alpha}^\omega$ is a P -space.
- (ii) $S_{\alpha, \alpha}^\omega$ is an almost P -space.
- (iii) α is not a sequential cardinal.

4. Subspaces and autohomeomorphisms. In this section we state results about the embedding of showering spaces in other showering spaces and about autohomeomorphisms of showering spaces, but proofs will be omitted or only sketched since they are straightforward.

PROPOSITION 4.1. *Suppose $\omega \leq \tilde{\omega}$ and $\alpha \leq \tilde{\alpha}$. Then $S_{\alpha, \beta}^\omega$ is a closed subspace of $S_{\tilde{\alpha}, \tilde{\beta}}^{\tilde{\omega}}$.*

LEMMA 4.2. *Suppose $[0, \omega)$ is order-isomorphic to $[\gamma, \omega)$ where $\gamma < \omega$, and suppose $p \in R_\gamma$. Then there is an order-preserving homeomorphism $f: S_{\alpha, \beta}^\omega \rightarrow U_p(\emptyset)$.*

Proof. Suppose $p = (p_\lambda)_{\lambda < \gamma}$. Define f by $f((x_\lambda)_{\lambda < \delta}) = (y_\lambda)_{\lambda < \gamma + \delta}$ where $y_\lambda = \begin{cases} p_\lambda & \text{if } \lambda < \gamma \\ x_\lambda & \text{if } \lambda = \gamma + \tilde{\lambda} \end{cases}$. Then f is one-to-one and onto, and order-preserving. Thus, since the topology on $S_{\alpha, \beta}^\omega$ and the relative

topology on $U_p(\emptyset)$ are each determined by the partial order \leq and β , f is a homeomorphism.

LEMMA 4.3. *If $[0, \omega)$, $[\gamma_1, \omega)$, and $[\gamma_2, \omega)$ are mutually order-isomorphic, and if $p \in R_{\gamma_1}$, $q \in R_{\gamma_2}$, then there is a homeomorphism F of $S_{\alpha, \beta}^\omega$ such that F^2 is the identity and $F(p) = q$ (and hence $F(q) = p$).*

Proof. By Lemma 4.2, there are order-preserving homeomorphisms $f_1: S_{\alpha, \beta}^\omega \rightarrow U_q(\emptyset)$ and $f_2: S_{\alpha, \beta}^\omega \rightarrow U_q(\emptyset)$. Let $f = f_2 \circ f_1^{-1}$. Then

$$f: U_p(\emptyset) \longrightarrow U_q(\emptyset)$$

is an order-preserving homeomorphism. There are essentially two cases: Case (i). $U_p(\emptyset) \cap U_q(\emptyset) = \emptyset$. Define F by $F|_{S_{\alpha, \beta}^\omega \setminus (U_p(\emptyset) \cup U_q(\emptyset))}$ is the identity map, $F|_{U_p(\emptyset)} = f$, $F|_{U_q(\emptyset)} = f^{-1}$. Case (ii). $U_p(\emptyset) \cong U_q(\emptyset)$. Define F by $F|_{(S_{\alpha, \beta}^\omega \setminus U_p(\emptyset)) \cup U_{f(q)}(\emptyset)}$ is the identity map,

$$\begin{aligned} F|_{U_p(\emptyset) \setminus U_q(\emptyset)} &= f|_{U_p(\emptyset) \setminus U_q(\emptyset)}, \quad F|_{U_q(\emptyset) \setminus U_{f(q)}(\emptyset)} \\ &= f^{-1}|_{U_q(\emptyset) \setminus U_{f(q)}(\emptyset)}. \end{aligned}$$

DEFINITION. A space X is *bihomogeneous* if for each $p, q \in X$ there is a homeomorphism $f: X \rightarrow X$ such that $f(p) = q$ and $f(q) = p$.

PROPOSITION 4.4. *If $[0, \omega)$ is order-isomorphic to $[\gamma, \omega)$ for each $\gamma < \omega$, then $S_{\alpha, \beta}^\omega$ is bihomogeneous.*

5. First countability and developability. In this section we give a necessary and sufficient condition for a showering space to be first countable. We also give a necessary condition for a nondiscrete showering space to be developable. It will be shown in §6 that this condition is also sufficient for $S_{\alpha, \beta}^\omega$ to be developable and is in fact equivalent to the metrizability of $S_{\alpha, \beta}^\omega$. We recall that if $\alpha < \beta$ or if $\omega = 1$, $S_{\alpha, \beta}^\omega$ is discrete and hence metrizable. For this reason, these cases will not be of interest to us here.

PROPOSITION 5.1. *Suppose $\omega > 1$ and $\alpha \geq \beta$. Then the following are equivalent:*

- (i) $\alpha = \beta = \aleph_0$.
- (ii) $S_{\alpha, \beta}^\omega$ is first countable.
- (iii) $S_{\alpha, \beta}^\omega$ has a nonisolated point of first countability.

Proof. (i) implies (ii). If $\alpha = \beta = \aleph_0$, and $p \in S_{\alpha, \beta}^\omega$ is not isolated, let $A_p = \{\alpha_1, \alpha_2, \dots\}$. Then $\{U_p(\{\alpha_1, \dots, \alpha_n\})\}_{n=1}^\infty$ is a base at p .

(ii) implies (iii) is trivial.

(iii) implies (i). If $\alpha > \beta$, $S_{\alpha, \beta}^\omega$ is an almost P -space by Proposition 3.2, and hence $S_{\alpha, \beta}^\omega$ is not first countable at any nonisolated point. Therefore, if $S_{\alpha, \beta}^\omega$ is first countable at a nonisolated point p_1 , $\alpha = \beta$. We must show that $\beta = \aleph_0$. Recall that β is always assumed to be infinite. Suppose $\beta > \aleph_0$. Let $\{U_{p_1}(B_i)\}_{i=1}^\infty$ be a countable base at p_1 , with $B_i \subseteq A_{p_1}$, $|B_i| < \beta (= \alpha)$ for each i . Choose $x_i \in A_{p_1} \setminus B_i$ for each i . Let $B = \{x_i \mid i = 1, 2, \dots\}$. Then $B \subseteq A_{p_1}$ and $|B| \leq \aleph_0 < \beta$ so $U_{p_1}(B) \in \mathcal{U}_{p_1}$. But for each i , $x_i \in U_{p_1}(B_i) \setminus U_{p_1}(B)$, so $U_{p_1}(B)$ contains no $U_{p_1}(B_i)$, contradicting the assumption that $\{U_{p_1}(B_i)\}_{i=1}^\infty$ is a base at p_1 .

REMARK 5.2. It is not difficult to prove that if $\omega > 1$ and $\alpha \geq \beta$, then every point of $S_{\alpha, \beta}^\omega$ is a G_δ if and only if α is a sequential cardinal and $\alpha = \beta$. (Compare to Corollary 3.3.) Hence there are showering spaces which fail at each point to be first countable but which have countable pseudo character. $S_{\aleph_0, \aleph_0}^{\omega_0}$ is such a space.

PROPOSITION 5.3. Suppose $\alpha \geq \beta$. If $S_{\alpha, \beta}^\omega$ is developable, $\alpha = \beta = \aleph_0$ and $\omega \leq \omega_0$.

Proof. Any developable space is first countable, so by Proposition 5.1, $\alpha = \beta = \aleph_0$. Suppose $\omega > \omega_0$. By Proposition 4.1, $S_{\alpha, \beta}^{\omega_0+1}$ is a subspace of $S_{\alpha, \beta}^\omega$, so it suffices to prove that $S_{\aleph_0, \aleph_0}^{\omega_0+1}$ is not developable. Suppose $\{\mathcal{D}_n\}_{n=1}^\infty$ is a countable collection of open covers of $S_{\aleph_0, \aleph_0}^{\omega_0+1}$. Choose $D_0 \in \mathcal{D}_1$ such that $p_0 \in D_0$. Choose $B_1 \subseteq A_{p_0}$, $|B_1| < \aleph_0$ such that $U_{p_0}(B_1) \subseteq D_0$. Choose $p_1 \in A_{p_0} \setminus B_1$. Then $U_{p_1}(\emptyset) \subseteq D_0$. Choose $D_1 \in \mathcal{D}_2$ such that $p_1 \in D_1$. Choose $B_2 \subseteq A_{p_1}$, $|B_2| < \aleph_0$ such that $U_{p_1}(B_2) \subseteq D_1$. Choose $p_2 \in A_{p_1} \setminus B_2$. Then $U_{p_2}(\emptyset) \subseteq D_1$. Inductively, suppose p_0, p_1, \dots, p_n , and D_0, \dots, D_{n-1} are chosen so that $D_{k-1} \in \mathcal{D}_k$, $p_k \in R_k$, and $U_{p_k}(\emptyset) \subseteq D_{k-1}$ for $k = 1, 2, \dots, n$. Choose $D_n \in \mathcal{D}_{n+1}$ such that $p_n \in D_n$. Choose $B_{n+1} \subseteq A_{p_n}$, $|B_{n+1}| < \aleph_0$ such that $U_{p_n}(B_{n+1}) \subseteq D_n$. Choose $p_{n+1} \in A_{p_n} \setminus B_{n+1}$. Then $U_{p_{n+1}}(\emptyset) \subseteq D_n$. Now $p_0 < p_1 < p_2 < \dots$ and $p_k \in R_k$, so there is a unique $q \in R_{\omega_0}$ such that $p_k < q$ for each k . By Proposition 2.1(2), q is isolated in $S_{\aleph_0, \aleph_0}^{\omega_0+1}$, but $q \in D_n$ for $n = 0, 1, \dots$, so $\text{St}(q, \mathcal{D}_n) \supseteq D_{n-1} \neq \{q\}$. Hence, $\{\text{St}(q, \mathcal{D}_n)\}_{n=1}^\infty$ is not a base at q . Therefore, $S_{\aleph_0, \aleph_0}^{\omega_0+1}$ is not developable.

6. EXAMPLES. In this section we apply the results of previous sections to get several examples.

EXAMPLE 6.1. $S_{\aleph_0, \aleph_0}^{\omega_0}$ is homeomorphic to the space Q of rationals.

Proof. $S_{\aleph_0, \aleph_0}^{\omega_0}$ is easily seen to be countable. It is first countable by Proposition 5.1. Hence it is second countable and thus metrizable. $S_{\aleph_0, \aleph_0}^{\omega_0}$ is dense-in-itself by Proposition 2.1. Therefore, by a theorem

of Sierpinski [5], $S_{\aleph_0, \aleph_0}^{\omega_0}$ is homeomorphic to Q .

COROLLARY 6.2. *If $\alpha > \beta$, the following are equivalent:*

- (i) $S_{\alpha, \beta}^{\omega}$ is metrizable.
- (ii) $S_{\alpha, \beta}^{\omega}$ is developable.
- (iii) $\alpha = \beta = \aleph_0$ and $\omega \leq \omega_0$.

Proof. This follows from Proposition 5.3 and Example 6.1.

EXAMPLE 6.3. The space Q of rationals is a closed subspace of a dense-in-itself almost P -space.

Proof. Q is homeomorphic to $S_{\aleph_0, \aleph_0}^{\omega_0}$ by Example 6.2. $S_{\aleph_0, \aleph_0}^{\omega_0}$ is homeomorphic to a closed subspace of $S_{\aleph_1, \aleph_0}^{\omega_0}$ by Proposition 4.1, and by Propositions 2.1 and 3.2, $S_{\aleph_1, \aleph_0}^{\omega_0}$ is a dense-in-itself almost P -space.

The following example is given in [4] where the proof may be found.

EXAMPLE 6.4. If ω is not sequential, the first countable, paracompact space $S_{\aleph_0, \aleph_0}^{\omega}$ contains no dense developable subspace.

EXAMPLE 6.5. For each uncountable cardinal α , there is a P -space of cardinal α which is first category in itself and an almost P -space of cardinal α which has no P -points and which is first category in itself.

Proof. $|S_{\alpha}^{\omega_0}| = \alpha$, so by Proposition 3.1 and Theorem 2.2, $S_{\alpha, \aleph_1}^{\omega_0}$ is the required P -space, and by Propositions 3.1 and 3.2, and Theorem 2.2, $S_{\alpha, \aleph_0}^{\omega_0}$ is the required almost P -space.

LEMMA 6.6. *If X is a Lindelöf P -space and $f: X \rightarrow R$ is continuous, $|f(X)| \leq \aleph_0$.*

Proof. $\{f^{-1}(r) | r \in R\}$ is an open cover for X . Therefore, it has a countable subcover.

EXAMPLE 6.7. If $\alpha > \aleph_0$, every continuous function $f: S_{\alpha, \aleph_1}^{\omega_0} \rightarrow R$ has countable image.

Proof. This is immediate from Example 6.5, Lemma 6.6, and Remark 1.7.

EXAMPLE 6.8. Let $X = S_{2^c, \aleph_1}^2 \times S_{c, \aleph_1}^2 \setminus (p_0, p_0)$, where $c = 2^{\aleph_0}$. Then X is a nonnormal P -space.

Proof. By Proposition 3.1 and the fact that P -spaces are closed under subspaces and finite products, X is a P -space. Let $A = (\{p_0\} \times S_{c, \aleph_1}^2) \cap X$ and $B = (S_{2^c, \aleph_1}^2 \times \{p_0\}) \cap X$. Then A and B are disjoint closed subsets of X . But a proof similar to the usual proof that the Tychanoff plank is not normal shows that A and B are not completely separated. Hence, X is not normal.

EXAMPLE 6.9. Let $Y = S_{2^c, \aleph_1}^{w_0} \times S_{c, \aleph_1}^{w_0}$. Then Y is a dense-in-itself P -space such that for any $p \in Y$, $Y \setminus \{p\}$ is nonnormal.

Proof. By Propositions 2.1, 3.1, and 4.4, Y is a product of dense-in-themselves homogeneous P -spaces, so Y is a dense-in-itself homogeneous P -space. If X is as in Example 6.8, X is a closed subspace of $Y \setminus \{(p_0, p_0)\}$, so $Y \setminus \{(p_0, p_0)\}$ is nonnormal. By the homogeneity of Y , for any $p \in Y$, $Y \setminus \{p\}$ is nonnormal.

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