Pacific Journal of Mathematics

A THREE-TERM RELATION FOR SOME SUMS RELATED TO DEDEKIND SUMS

L. CARLITZ

Vol. 57, No. 2 February 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 2, 1975

A THREE-TERM RELATION FOR SOME SUMS RELATED TO DEDEKIND SUMS

L. CARLITZ

The object of the paper is to evaluate

$$\sum_{r=0}^{a-1}\left[\frac{b(r+x)}{a}-y\right]\left[\frac{c(r+x)}{a}-z\right]+\sum_{s=0}^{b-1}\left[\frac{c(s+y)}{b}-z\right]\left[\frac{a(s+y)}{b}-x\right]+\sum_{t=0}^{c-1}\left[\frac{a(t+z)}{c}-x\right]\left[\frac{b(t+z)}{c}-y\right].$$

where $[u]$ is the greatest integer function,

$$(b, c) = (c, a) = (a, b) = 1 .$$

and

$$0 \leqq x < 1, \quad 0 \leqq y < 1, \quad 0 \leqq z < 1 .$$

1. Introduction.

Put

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & (x \neq \text{integer}) \\ 0 & (x = \text{integer}) . \end{cases} \tag{1.1}$$

The Dedekind sum $s(h, k)$ is defined by

$$s(h, k) = \sum_{r \pmod k} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{h}\right)\right) . \tag{1.2}$$

It is well known that $s(h, k)$ satisfies the reciprocity relation [5]

$$s(h, k) + s(k, h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right), \tag{1.3}$$

for $(h, k) = 1$.

Rademacher [5] has proved the three-term relation

$$s(bc', a) + s(ca', b) = s(ab', c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right), \tag{1.4}$$

where

$$(b, c) = (c, a) = (a, b) = 1 \tag{1.5}$$

and

$$aa' \equiv 1 \pmod{bc}, \quad bb' \equiv 1 \pmod{ca}, \quad cc' \equiv 1 \pmod{ab} . \tag{1.6}$$

Rademacher [4] has introduced the sum

$$s(h, k \mid x, y) = \sum_{r \pmod k} \left(\left(h\frac{r+y}{k} + x\right)\right)\left(\left(\frac{r+y}{k}\right)\right) \tag{1.7}$$

and proved the reciprocity formula

$$\begin{aligned} &s(h, k \mid x, y) + s(k, h \mid y, x) \\ &= ((x))((y)) + \frac{1}{2}\left\{\frac{h}{k}\bar{B}_2(y) + \frac{1}{hk}\bar{B}_2(hy + kx) + \frac{k}{h}\bar{B}_2(x)\right\}, \end{aligned} \tag{1.8}$$

where $(h, k) = 1$, x and y are not both integers and $\bar{B}_2(x) = B_2(x - [x])$, where $B_2(x) = x^2 - x + 1/6$, the Bernoulli polynomial of degree 2.

In order to get a three-term relation for the generalized sum $s(h, k \mid x, y)$, the writer [2] defined

$$s(a, b, c; x, y, z) = \sum_{r \pmod c} \Phi\left(a\frac{t+z}{c} - x\right)\Phi\left(y - b\frac{t+z}{c}\right), \tag{1.9}$$

where

$$\Phi(x) = x - [x] - \frac{1}{2},$$

and proved that

$$\begin{aligned} &s(a, b, c; x, y, z) + s(b, c, a; y, z, x) + s(c, a, b; z, x, y) \\ &= \delta - \frac{a}{2bc}\bar{B}_2(cy - bz) - \frac{b}{2ca}\bar{B}_2(az - cx) - \frac{c}{2ab}\bar{B}_2(bx - ay), \end{aligned} \tag{1.10}$$

where $(b, c) = (c, a) = (a, b) = 1$ and $\delta = 1$ if integers r, s, t exist such that

$$\frac{r+x}{a} = \frac{s+y}{b} = \frac{t+z}{c}; \quad 0 \leqq r < a, \quad 0 \leqq s < b, \quad 0 \leqq t < c;$$

$\delta = 0$ otherwise. For $c = 1$, $z = 0$, it is easily verified that (1.10) reduces to (1.8).

The reciprocity Theorem (1.3) is equivalent to [5, p. 9]

$$h\sum_{r=1}^{k-1} r\left[\frac{hr}{k}\right] + k\sum_{s=1}^{h-1} s\left[\frac{ks}{h}\right] = \frac{1}{12}(h-1)(k-1)(8hk - h - k - 1), \tag{1.11}$$

where $(h, k) = 1$. It is shown in [1] that (1.4) implies

$$A(b, c; a) + A(c, a; b) + A(a, b; c) = (a-1)(b-1)(c-1), \tag{1.12}$$

where $(b, c) = (c, a) = (a, b) = 1$ and

$$A(b, c; a) = \sum_{r=1}^{a-1}\left[\frac{br}{a}\right]\left[\frac{cr}{a}\right]. \tag{1.13}$$

It is not difficult to give a direct proof of (1.12) and indeed considerably more. It is also proved in [1] that

$$(1.14)\qquad 6k\sum_{r=0}^{k-1}\left[\frac{hr+z}{k}\right]^2+6h\sum_{s=0}^{h-1}\left[\frac{ks+z}{h}\right]^2$$
$$=(h-1)(2h-1)(k-1)(2k-1)+6[z]([z]+2hk-h-k+1)\,,$$

where $(h, k)=1$ and $0\leqq z<h+k$.

Generalizing (1.13), we define

$$(1.15)\qquad A(b, c; a\mid y, z; x)=\sum_{r=0}^{a-1}\left[\frac{b(r+x)}{a}-y\right]\left[\frac{c(r+x)}{a}-z\right]^2.$$

In Theorem 2 below we evaluate

$$R\equiv A(a, b; c\mid x, y; z)+A(b, c; a\mid y, z; x)+A(c, a; b\mid z, x; y)\,,$$

where $(b, c)=(c, a)=(a, b)=1$ and

$$0\leqq x<1\,,\quad 0\leqq y<1\,,\quad 0\leqq z<1\,.$$

It is however convenient to first consider the sum

$$(1.16)\qquad B(b, c; a\mid y, z; x)=\sum_{r=0}^{a-1}\left[y-\frac{b(r+x)}{a}\right]\left[z-\frac{c(r+x)}{a}\right],$$

In Theorem 1 below we evaluate

$$S\equiv B(a, b; c\mid x, y; z)+B(b, c; a\mid y, z; x)+A(c, a; b\mid z, x; y)\,,$$

where the parameters satisfy the same conditions as above.

The results are particularly simple if no two of the fractions

$$\frac{r+x}{a}\,,\quad\frac{s+y}{b}\,,\quad\frac{t+z}{c}\qquad(0\leqq r<a,\ 0\leqq s<b,\ 0\leqq s<c)$$

are equal. In this case we show that

$$(1.17)\qquad S=abc\,,\quad R=(a-1)(b-1)(c-1)+1\,.$$

Also, if $x=y=z$, we have

$$(1.18)\qquad S=abc-1\,,\quad R=(a-1)(b-1)(c-1)\,.$$

The last form evidently includes (1.12).

It should be noted that (1.14) is not contained in the results of the present paper.

2. Preliminaries.

LEMMA 1. *For* $a\geqq 1$,

$$[ax] = \sum_{r=0}^{a-1} \left[x + \frac{r}{a} \right]. \tag{2.1}$$

LEMMA 2. *For $x \neq$ integer,*

$$[-x] = -[x] - 1. \tag{2.2}$$

LEMMA 3. *Put $\Phi(x) = x - [x] - 1/2$. Then $\Phi(x+1) = \Phi(x)$ and*

$$\Phi(ax) = \sum_{r \pmod a} \Phi\left(x + \frac{r}{a} \right). \tag{2.3}$$

LEMMA 4. *For $a \geqq 1$, $(b, a) = 1$,*

$$\sum_{r=0}^{a-1} \left[x + \frac{br}{a} \right] = [ax] + \frac{1}{2}(a-1)(b-1).$$

Lemmas 1, 2, 3 are well-known. To prove Lemma 4, take

$$\begin{aligned} \sum_{r=0}^{a-1} \left[x + \frac{br}{a} \right] &= \sum_{r=0}^{a-1} \left\{ x + \frac{br}{a} - \Phi\left(x + \frac{br}{a} \right) - \frac{1}{2} \right\} \\ &= ax + \frac{1}{2} b(a-1) - \Phi(ax) - \frac{1}{2} a \\ &= [ax] + \frac{1}{2} ab - \frac{1}{2} a - \frac{1}{2} b + \frac{1}{2} \\ &= [ax] + \frac{1}{2}(a-1)(b-1). \end{aligned}$$

3. **Three term relation for $B(a, b; c \mid x, y; z)$.** We assume in the remainder of the paper that

$$(b, c) = (c, a) = (a, b) = 1 \tag{3.1}$$

and

$$0 \leqq x < 1, \quad 0 \leqq y < 1, \quad 0 \leqq z < 1. \tag{3.2}$$

Also we write

$$\sum_{r,s,t} \equiv \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{t=0}^{c-1}.$$

It is convenient to define

$$B(a, b; c \mid x, y; z) = \sum_{t=0}^{c-1} \left[x - \frac{a(t+z)}{c} \right] \left[y - \frac{b(t+z)}{c} \right]. \tag{3.3}$$

By Lemma 1, this gives

$$(3.4)\qquad B(a, b; c \mid x, y, z) = \sum_{r,s,t}\left[\frac{r+x}{a} - \frac{t+z}{c}\right]\left[\frac{s+y}{b} - \frac{t+z}{c}\right].$$

Put

$$\xi = \frac{r+x}{a}, \quad \eta = \frac{s+y}{b}, \quad \zeta = \frac{t+z}{c}.$$

Thus (3.4) becomes

$$(3.5)\qquad B(a, b; c \mid x, y; z) = \sum_{r,s,t}[\xi - \zeta][\eta - \zeta].$$

To begin with, we assume that no two of the fractions

$$(3.6)\qquad \frac{r+x}{a}, \quad \frac{s+y}{b}, \quad \frac{t+z}{c} \qquad (0 \leqq r < a, 0 \leqq s < b, 0 \leqq t < c)$$

differ by an integer. It is easily seen that this is equivalent to requiring that no two of the fractions are equal. We shall refer to this as Case 1. Case 2 is that in which exactly two of the fractions are equal, Case 3 that in which all three are equal. If a pair r, s exists such that

$$\frac{r+x}{a} = \frac{s+y}{b},$$

it is easily shown to be unique.

Thus in Case 1 we have, by Lemma 2,

$$(3.7)\qquad [\xi - \zeta] = -[\zeta - \xi] - 1,$$

so that (3.5) becomes

$$B(a, b; c \mid x, y; z) = -\sum_{r,s,t}[\eta - \zeta][\zeta - \xi] - \sum_{r,s,t}[\eta - \zeta].$$

Hence

$$\begin{aligned}(3.8)\qquad & B(a, b; c \mid x, y; z) + B(b, c; a \mid y, z; x) + B(c, a; b \mid z, x; y)\\ &= -\sum_{r,s,t}\{[\eta - \zeta][\zeta - \xi] + [\zeta - \xi][\xi - \eta] + [\xi - \eta][\eta - \zeta]\\ &\quad + [\eta - \zeta] + [\zeta - \xi] + [\xi - \eta]\}\\ &= -\sum_{r,s,t}\Big\{\Big([\eta - \zeta] + \frac{1}{2}\Big)\Big([\zeta - \xi] + \frac{1}{2}\Big)\\ &\quad + \Big([\zeta - \xi] + \frac{1}{2}\Big)\Big([\xi - \eta] + \frac{1}{2}\Big)\\ &\quad + \Big([\xi - \eta] + \frac{1}{2}\Big)\Big([\eta - \zeta] + \frac{1}{2}\Big)\Big\} + \frac{3}{4}abc.\end{aligned}$$

Hence if we put

$$T = \sum_{r,s,t} \left\{ [\eta - \zeta] + [\zeta - \xi] + [\xi - \eta] + \frac{3}{2} \right\}^2 , \tag{3.9}$$

it is clear that

$$T = \sum_{r,s,t} \left\{ \left([\eta - \zeta] + \frac{1}{2} \right)^2 + \left([\zeta - \xi] + \frac{1}{2} \right)^2 + \left([\xi - \eta] + \frac{1}{2} \right)^2 \right\} + \frac{3}{2} abc - 2S , \tag{3.10}$$

where

$$S = B(a, b; c \mid x, y; z) + B(b, c; a \mid y, z; x) + B(c, a; b \mid z, x; y) . \tag{3.11}$$

Since

$$[\eta - \zeta] + [\zeta - \xi] + [\xi - \eta] = -1 \text{ or } -2 \text{ (Case 1 or 2)} ,$$

it follows that

$$T = \frac{1}{4} abc \qquad \text{(Case 1 or 2)} . \tag{3.12}$$

Also since each of $[\eta - \zeta]$, $[\zeta - \xi]$, $[\xi - \eta] = 0$ or -1, we get

$$\sum_{r,s,t} \left\{ \left([\eta - \zeta] + \frac{1}{2} \right)^2 + \left([\zeta - \xi] + \frac{1}{2} \right)^2 + \left([\xi - \eta] + \frac{1}{2} \right)^2 \right\} = \frac{3}{4} abc .$$

Thus (3.11) reduces to simply

$$S = abc \qquad \text{(Case 1)} . \tag{3.13}$$

Turning next to Case 2, let r_0, s_0 satisfy

$$\xi_0 = \frac{r_0 + x}{a} = \frac{s_0 + y}{b} = \eta_0 .$$

For this pair we have

$$[\eta_0 - \xi_0] = -[\xi_0 - \eta_0] = 0$$

rather than (3.7). Thus

$$B(b, c; a \mid y, z; x) = -\sum_{r,s,t} ([\zeta - \xi][\xi - \eta] + [\zeta - \xi]) + \sum_t [\zeta - \xi_0] . \tag{3.14}$$

Since

$$\sum_t [\zeta - \xi_0] = \sum_{t=0}^{c-1} \left[\frac{t + z}{c} - \xi_0 \right] = [z - c\xi_0] ,$$

we get, in place of (3.13),

$$S = abc + [z - c\xi_0] \qquad \text{(Case 2)}. \tag{3.15}$$

For Case 3, let r_0, s_0, t_0 satisfy $\xi_0 = \eta_0 = \zeta_0$, that is

$$\frac{r_0 + x}{a} = \frac{s_0 + y}{b} = \frac{t_0 + z}{c}.$$

Equation (3.14) remains unchanged. Since

$$T = \frac{1}{4}abc + 2 \qquad \text{(Case 3)},$$

we therefore get

$$S = abc - 1 + [x - a\zeta_0] + [y - b\xi_0] + [z - c\eta_0] \quad \text{(Case 3)}. \tag{3.16}$$

Since

$$[x - a\zeta_0] = [x - a\xi_0] = [x - (r_0 + x)] = -r_0, \quad \text{etc.},$$

we may replace (3.16) by

$$S = abc - 1 - r_0 - s_0 - t_0 \qquad \text{(Case 3)}. \tag{3.16$'$}$$

We may now state

THEOREM 1. *Let* $(b, c) = (c, a) = (a, b) = 1$ *and*

$$0 \leqq x < 1, \quad 0 \leqq y < 1, \quad 0 \leqq z < 1.$$

Then

$$S = B(a, b; c \mid x, y; z) + B(b, c; a \mid y, z; x) + B(c, a; b \mid z, x; y)$$

is evaluated by (3.13), (3.15) *and* (3.16).

In particular, for $x = y = z$, we have

COROLLARY 1.

$$\begin{aligned} &\sum_{r=0}^{a-1}\left[x - \frac{b(r+x)}{a}\right]\left[x - \frac{c(r+x)}{a}\right] \\ &\quad + \sum_{s=0}^{b-1}\left[x - \frac{c(s+x)}{b}\right]\left[x - \frac{b(s+x)}{b}\right] \\ &\quad + \sum_{t=0}^{c-1}\left[x - \frac{a(t+x)}{c}\right]\left[x - \frac{b(t+x)}{c}\right] = abc - 1. \end{aligned} \tag{3.17}$$

4. **Three-term relations for $A(a, b; c \mid x, y; z)$.** We again first consider Case 1. Thus in

$$A(a, b; c \mid x, y; z) = \sum_{t=0}^{c-1}\left[\frac{a(t+z)}{c} - x\right]\left[\frac{b(t+z)}{c} - y\right],$$

none of the quantities

$$\frac{a(t+z)}{c} - x\,, \quad \frac{b(t+z)}{c} - y \qquad (0 \leqq t < c)$$

is an integer. It follows that

$$\begin{aligned} A(a, b; c \mid x, y; z) &= \sum_{t=0}^{c-1}\left\{1 + \left[x - \frac{a(t+z)}{c}\right]\right\}\left\{1 + \left[y - \frac{b(t+z)}{c}\right]\right\} \\ &= c + \sum_{t=0}^{c-1}\left[x - \frac{a(t+z)}{c}\right] + \sum_{t=0}^{c-1}\left[y - \frac{b(t+z)}{c}\right] \\ &\quad + B(a, b; c \mid x, y; z)\,. \end{aligned}$$

Thus by Lemma 4, we get

$$\begin{aligned} &A(a, b; c \mid x, y; z) \\ &= [cx - az] + [cy - bz] - \frac{1}{2}ac - \frac{1}{2}bc + \frac{1}{2}a + \frac{1}{2}b + 1 \\ &\quad + B(a, b; c \mid x, y; z)\,. \end{aligned} \tag{4.1}$$

Let

$$R = A(a, b; c \mid x, y; z) + A(b, c; a \mid y, z; x) + A(c, a; b \mid z, x; y)\,. \tag{4.2}$$

Then by (4.1) and (3.13) we get

$$\begin{aligned} R &= [cx - az] + [cy - bz] + [ay - bx] + [az - cx] \\ &\quad + [bz - cy] + [bx - ay] - (bc + ca + ab) \\ &\quad + (a + b + c) + 3 + S\,. \end{aligned} \tag{4.3}$$

Applying Lemma 2, this reduces to

$$R = (a-1)(b-1)(c-1) + 1 \qquad \text{(Case 1)}\,. \tag{4.4}$$

As for Case 2, let r_0, s_0 denote the exceptional pair, that is,

$$\xi_0 = \frac{r_0 + x}{a} = \frac{s_0 + y}{b} = \eta_0\,.$$

$$\begin{aligned} A(b, c; a \mid y, z; x) &= \sum_{r=0}^{a-1}\left[\frac{b(r+x)}{a} - y\right]\left[\frac{c(r+x)}{a} - z\right] \\ &= \sum_{r=0}^{a-1}\left\{1 + \left[y - \frac{b(r+x)}{a}\right]\right\}\left\{1 + \left[z - \frac{c(r+x)}{a}\right]\right\} \\ &\quad - \left\{1 + \left[z - \frac{c(r_0+x)}{a}\right]\right\}\,. \end{aligned} \tag{4.5}$$

Then, as above, it follows that

$$A(b, c; a \mid y, z; x) = B(b, c; a \mid y, z; x) + [ay - bx] + [az - cx] + \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}ab - \frac{1}{2}ac - [z - c\xi_0] . \tag{4.6}$$

Similarly

$$A(c, a; b \mid z, x; y) = B(c, a; b \mid z, x; y) + [bx - ay] + [bz - cy] + \frac{1}{2}a + \frac{1}{2}c - \frac{1}{2}ab - \frac{1}{2}bc - [z - c\eta_0] . \tag{4.7}$$

However (4.1) remains unchanged.

Thus

$$R = S - (bc + ca + ab) + (a + b + c) - 1 - 2[z - c\eta_0] .$$

Therefore, by (3.15), we have

$$R = (a - 1)(b - 1)(c - 1) - [z - c\eta_0] \quad \text{(Case 2)} . \tag{4.8}$$

Finally, for Case 3, let r_0, s_0, t_0, be the exceptional triple:

$$\frac{r_0 + x}{a} = \frac{s_0 + y}{b} = \frac{t_0 + z}{c} \quad (\xi_0 = \eta_0 = \zeta_0) . \tag{4.9}$$

We now have

$$A(b, c; a \mid y, z; x) = \sum_{r=0}^{a-1} \left\{1 + \left[y - \frac{b(r + x)}{a}\right]\right\} \left\{1 + \left[z - \frac{c(r + x)}{a}\right]\right\} - 1 - [y - b\xi_0] - [z - b\xi_0] .$$

Thus in place of (4.6) we get

$$A(b, c; a \mid y, z; x) = B(b, c; a \mid y, z; x) + [ay - bx] + [az - cx] + \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}ab - \frac{1}{2}ac - [y - b\xi_0] - [z - c\xi_0] . \tag{4.10}$$

There are similar formulas for $A(c, a; b \mid z, x; y)$ and $A(a, b; c \mid x, y; z)$. Since

$$[cy - bz] + [bz - cx] = [az - cx] + [cx - az] = [bx - ay] + [ay - bx] = 0 ,$$

it follows that

$$\begin{aligned} R &= S + a + b + c - bc - ca - ab - [y - b\xi_0] - [z - c\xi_0] \\ &\quad - [z - c\eta_0] - [x - a\eta_0] - [x - a\zeta_0] - [y - b\zeta_0] \\ &= S + a + b + c - bc - ca - ab \\ &\quad - 2[x - a\xi_0] - 2[y - b\eta_0] - 2[z - c\zeta_0] . \end{aligned}$$

Making use of (3.16) we get

$$R = (a-1)(b-1)(c-1) - [x - a\xi_0] - [y - b\eta_0] - [z - c\zeta_0] \quad \text{(Case 3)} \tag{4.11}$$

or, if we prefer,

$$R = (a-1)(b-1)(c-1) + r_0 + s_0 + t_0 \quad \text{(Case 3)} . \tag{4.11}'$$

This completes the proof of

THEOREM 2. *Let* $(b, c) = (c, a) = (a, b) = 1$ *and*

$$0 \leqq x < 1 , \quad 0 \leqq y < 1 , \quad 0 \leqq x < 1 .$$

Then

$$R = A(a, b; c \mid x, y; z) + A(b, c; a \mid y, z; x) + A(c, a; b \mid z, x; y)$$

is evaluated by (4.4), (4.8) *and* (4.11).

In particular, for $x = y = z$, we have

COROLLARY 2.

$$\begin{aligned} &\sum_{r=0}^{a-1} \left[\frac{b(r+x)}{a} - x \right] \left[\frac{c(r+x)}{a} - x \right] \\ &+ \sum_{s=0}^{b-1} \left[\frac{c(s+x)}{b} - x \right] \left[\frac{a(s+x)}{b} - x \right] \\ &+ \sum_{t=1}^{c-1} \left[\frac{a(t+x)}{c} - x \right] \left[\frac{b(t+x)}{c} - x \right] = (a-1)(b-1)(c-1) . \end{aligned} \tag{4.12}$$

REFERENCES

1. L. Carlitz, *Some sums containing the greatest integer function*, Revue Roumaine de Mathématiques pures et appliquées, to appear.
2. ———, *A three-term relation for the Dedekind-Rademacher sums*, Publications Mathematicae, **14** (1967), 119-124.
3. H. Rademacher, *Generalization of the reciprocity theorem for Dedekind sums*, Duke Math. J., **21** (1954), 391-398.
4. ———, *Some remarks on certain generalized Dedekind sums*, Acta Arithmetica, **9** (1964), 97-105.
5. H. Rademacher and E. Grosswald, *Dedekind Sums*, The Mathematical Association of America, 1972.

Received October 31, 1974. Supported in part by NSF grant GP-37924XI.

DUKE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 57, No. 2 February, 1975