SOME NEW RESULTS ON ODD PERFECT NUMBERS

G. G. Dandapat, John L. Hunsucker and Carl Pomerance

SOME NEW RESULTS ON ODD PERFECT NUMBERS

G. G. Dandapat, J. L. Hunsucker and Carl Pomerance

If m is a multiply perfect number ($\sigma(m) = tm$ for some integer t), we ask if there is a prime p with $m = p^a n$, $(p^a, n) = 1$, $\sigma(n) = p^a$, and $\sigma(p^a) = tn$. We prove that the only multiply perfect numbers with this property are the even perfect numbers and 672. Hence we settle a problem raised by Suryanarayana who asked if odd perfect numbers necessarily had such a prime factor. The methods of the proof allow us also to say something about odd solutions to the equation $\sigma(\sigma(n)) = 2n$.

1. **Introduction.** In this paper we answer a question on odd perfect numbers posed by Suryanarayana [17]. It is known that if m is an odd perfect number, then $m = p^a k^2$ where p is a prime, $p \nmid k$, and $p \equiv a \equiv 1 \pmod 4$. Suryanarayana asked if it necessarily followed that

$$(1) \qquad \sigma(k^2) = p^a , \qquad \sigma(p^a) = 2k^2 .$$

Here, σ is the sum of the divisors function. We answer this question in the negative by showing that no odd perfect number satisfies (1).

We actually consider a more general question. If m is multiply perfect ($\sigma(m) = tm$ for some integer t), we say m has *property S* if there is a prime p with $m = p^a n$, $(p^a, n) = 1$, and the equations

$$(2) \qquad \sigma(n) = p^a , \qquad \sigma(p^a) = tn$$

hold. Note that if n, p, a, t is a solution of (2) with p prime, then $1 = (p^a, \sigma(p^a)) = (p^a, n)$, so that $\sigma(p^a n) = tp^a n$; that is $p^a n$ is multiply perfect. Hence the multiply perfect numbers with property S are in one-to-one correspondence with the solutions of (2). We shall prove:

THEOREM 1. *If p is a prime, n, a, t are positive integers, and* (2) *holds, then either*

$$(3) \qquad n = 21 , \quad p = 2 , \quad a = 5 , \quad t = 3$$

or

$$(4) \qquad n = 2^k , \quad p = 2^{k+1} - 1 , \quad a = 1 , \quad t = 2 .$$

COROLLARY. *If m is a multiply perfect number with property*

S, *then* $m = 672$ *or* m *is an even perfect number. In particular, no odd perfect number has property S.*

Write the odd perfect number $m = p^a k^2$ as a product of primes $p^a p_1^{2a_1} \cdots p_\nu^{2a_\nu}$. (Note that Pomerance [12] and Robbins [14] have shown that $\nu \geq 6$.) Let $N(m)$ be the number of subscripts i for which there is a subscript j such that $(\sigma(p_i^{2a_i} p_j^{2a_j}), p_i p_j) > 1$. Then $0 \leq N(m) \leq \nu$. It is not difficult to see that Suryanarayana's equations (1) are equivalent to the odd perfect m satisfying $N(m) = 0$. Hence the above corollary implies $N(m) > 0$. We show however that $N(m)$ is not even close to 0, but more nearly ν.

THOREM 2. *If* m *is an odd perfect number, then*

$$(5) \qquad \nu + 1 - [\log (\nu + 1)/\log 2] \leq N(m) \leq \nu .$$

Several authors (Kanold [8], Niederreiter [11], Suryanarayana [16], [18]) have considered the equation

$$(6) \qquad \sigma(\sigma(n)) = 2n ,$$

calling the solutions n super perfect. The even super perfects have been completely classified, but it is not known if any odd super perfects exist. The methods we develop to consider (1), (2), and (5) allow us also to get some results on odd solutions of (6). We shall prove:

THEOREM 3. *If* n *is an odd super perfect number, then neither* n *nor* $\sigma(n)$ *is a prime power and either* n *or* $\sigma(n)$ *is divisible by at least 3 distinct primes.*

Note that Suryanarayana [18] has already shown that n is not a prime power, but we give a new proof here for completeness. We (the second and third authors) have actually been able to prove much more than Theorem 3, but we do not give the details in this paper. (We have proved that if n is an odd super perfect number, then $n > 7 \cdot 10^{24}$, $\omega(n\sigma(n)) \geq 5$, and $\omega(n) + \omega(\sigma(n)) \geq 7$. Here $\omega(n)$ is the number of distinct prime factors of n.)

The main tool of this paper (Theorem A in §2) has the remarkable distinction of having been proved independently nine times.

In the research for this paper, the first author worked separately from the other authors.

2. **Preliminaries.** If x, y are integers, we shall write $x \,\|\, y$ if $x \mid y$ and $(x, y/x) = 1$. If p, q are distinct primes, we shall denote by $\text{ord}_q (p)$ the exponent p belongs to mod q, that is, the smallest

natural number d for which $p^d \equiv 1 \pmod{q}$. We denote by $a_q(p)$ the integer e such that $q^e \| p^d - 1$, where $d = \mathrm{ord}_q(p)$. Clearly $\mathrm{ord}_q(p) \mid q - 1$ and $a_q(p) \geqq 1$.

From Theorems 94 and 95 in Nagell [10] and the fact that $\sigma(p^x) = (p^{x+1} - 1)/(p - 1)$, we have:

LEMMA 1. *Suppose p, q are distinct primes with $q \neq 2$ and b, c are natural numbers. Then*
   (i)   *if $p \equiv 1 \pmod{q}$, then $q^b \| \sigma(p^c)$ if and only if $q^b \| c + 1$,*
   (ii)   *if $p \not\equiv 1 \pmod{q}$, then $q^b \| \sigma(p^c)$ if and only if $b \geqq a_q(p)$, $\mathrm{ord}_q(p) \mid c + 1$, and $q^{b - a_q(p)} \| c + 1$.*

LEMMA 2. *Suppose p, q are distinct primes, x, y, b, c are natural numbers, $\sigma(q^x) = p^y$ and $q^b \| \sigma(p^c)$. Assume $q \neq 2$. Then*
   (i)   *if $p \equiv 1 \pmod{q}$, then $q^b \| c + 1$,*
   (ii)   *if $p \not\equiv 1 \pmod{q}$, then $\mathrm{ord}_q(p) \mid c + 1$ and $q^{b-1} \| c + 1$.*

Proof. Now (i) follows from (i) of Lemma 1. Also (ii) will follow from (ii) of Lemma 1 provided we show $a_q(p) = 1$. Now $p^y = \sigma(q^x) = 1 + q + \cdots + q^x$, so that $p^y - 1 \equiv q \pmod{q^2}$. Then since $p \not\equiv 1 \pmod{q}$, we have $q \| (p^y - 1)/(p - 1) = \sigma(p^{y-1})$. Lemma 1 now implies $a_q(p) = 1$.

There is a well-known result about expressions of the form $(a^b - 1)/(a - 1)$ (see Bang [2], Zsigmondy [20], Sylvester [19], Birkhoff and Vandiver [3], Dickson [4], Kanold [7], Artin [1], Leopoldt [9], Richter [13]), which implies the following:

THEOREM A. *If p is a prime, x is a natural number, and $1 < d \mid x + 1$, then there is a prime $q \mid \sigma(p^x)$ with $\mathrm{ord}_q(p) = d$, unless*
   (i)   *$p = 2$ and $d = 6$,*
   (ii)   *p is a Mersenne prime (that is, of the form $2^k - 1$) and $d = 2$.*

3. **The main results.** In this section we prove Theorems 1 and 2.

Proof of Theorem 1. We first consider the case $p = 2$. From the equation $\sigma(n) = 2^a$ and Theorem A, we see that n is a product of distinct Mersenne primes (cf. Sierpiński [15]); say $n = p_1 p_2 \cdots p_s$ where each $p_i = 2^{k_i} - 1$, k_i is prime and $k_1 < k_2 < \cdots < k_s$. Then $a = \sum k_i$. Now $tn = \sigma(2^a) = 2^{1 + \sum k_i} - 1$. Hence for $1 \leqq j \leqq s$, we have $2^{k_j} - 1 \mid 2^{1 + \sum k_i} - 1$, so that $k_j \mid \sum k_i$. Since the k_j are distinct primes, we have

$$(7) \qquad\qquad \prod_{i=1}^{s} k_i \,\Big|\, 1 + \sum_{i=1}^{s} k_i \,.$$

Then $s \geqq 2$. Now the expression $\prod k_i - 1 - \sum k_i$ increases separately in each of the s "variables" $k_1, k_2, \cdots, k_s$. If $s = 2$, $k_1 = 2$, $k_2 = 3$, we have $2 \cdot 3 \mid 1 + 2 + 3$. This gives the solution (3). If $s = 2$ and $k_2 \geqq 5$, then $k_1 k_2 - 1 - k_1 - k_2 \geqq 2 \cdot 5 - 1 - 2 - 5 > 0$, so that (7) fails. Also if $s \geqq 3$, $\prod k_i - 1 - \sum k_i > 2^s - 1 - 2s > 0$, so again (7) fails.

We now consider the case $p > 2$. Since $\sigma(n) = p^a$ is odd, we have $n = 2^k p_1^{2a_1} \cdots p_r^{2a_r}$ where $k \geqq 0$, $r \geqq 0$, and $p_1, \cdots, p_r$ distinct odd primes. Suppose $r = 0$, so that $n = 2^k$. Then $\sigma(n) = 2^{k+1} - 1 = p^a$. Suppose $a > 1$. By Theorem A, there is a prime $q \mid \sigma(p^{2a-1})$ with $\mathrm{ord}_q (p) = 2a$. Then $q \mid (p^{2a} - 1)/(p^a - 1) = p^a + 1 = 2^{k+1}$, an impossibility since q is odd (cf. Gerono [6]). Hence $a = 1$ and we have solution (4). Thus we may assume $r \geqq 1$. Now for $1 \leqq i \leqq r$, we have $\sigma(p_i^{2a_i}) \mid p^a$ and $p_i^{2a_i} \mid \sigma(p^a)$. Lemma 2 then implies $p_i \mid a + 1$, so that $p_1 p_2 \cdots p_r \mid a + 1$. Theorem A implies there is a prime $q \mid \sigma(p^a)$ with $\mathrm{ord}_q (p) = p_1 p_2 \cdots p_r$. Then $q \neq 2, p_1, \cdots, p_r$, and since $q \mid tn$, we have $q \mid t$. Hence

$$p_1 p_2 \cdots p_r < q \leqq t = \frac{\sigma(p^a)}{n} = \frac{\sigma(p^a)}{p^a} \cdot \frac{\sigma(n)}{n}$$

$$= \frac{p^{a+1} - 1}{p^a(p - 1)} \cdot \frac{2^{k+1} - 1}{2^k} \cdot \prod \frac{p_i^{2a_i+1} - 1}{p_i^{2a_i}(p_i - 1)}$$

$$< \frac{p}{p - 1} \cdot 2 \cdot \prod \frac{p_i}{p_i - 1} \,,$$

so that

$$1 < \frac{2p}{p - 1} \cdot \prod \frac{1}{(p_i - 1)} \leqq \frac{2p}{(p - 1)(p_1 - 1)} \leqq \frac{2 \cdot 3}{2 \cdot 4} < 1 \,,$$

a contradiction.

Proof of Theorem 2. If i is such that $1 \leqq i \leqq \nu$ and $(\sigma(p_i^{2a_i} p_j^{2a_j}), p_i p_j) = 1$ for all j, $1 \leqq j \leqq \nu$, then $p_i^{2a_i} \mid \sigma(p^a)$ and $\sigma(p_i^{2a_i}) \mid p^a$. Let Ω be the set of such subscripts i, and let ω be the cardinality of Ω. Lemma 2 implies that $\prod_\Omega p_i \mid a + 1$. Since also $2 \mid a + 1$, we have at least $2^{\omega+1} - 1$ divisors d of $a + 1$ with $d > 1$. Since p is not a Mersenne prime (we have $p \equiv 1 \pmod{4}$), Theorem A implies for each such d, there is a prime $r = r_d \mid \sigma(p^a)$ with $\mathrm{ord}_r (p) = d$. Then each r_d is odd, and since m is perfect, we have $r_d \in \{p_1, p_2, \cdots, p_\nu\}$. Hence $2^{\omega+1} - 1 \leqq \nu$, so that $\omega \leqq [\log (\nu + 1)/\log 2] - 1$.

4. Super perfect numbers.

LEMMA 3. *Let n be an odd super perfect number. Then*

(i) *n is a square,*

(ii) $\sigma(n)$ *is odd,*

(iii) *the prime factorization of* $\sigma(n)$ *is* $p^a p_1^{2a_1} \cdots p_\nu^{2a_\nu}$ *where* $p \equiv a \equiv 1 \pmod 4$ *and* $\nu \geqq 0$.

Proof. Kanold [8] proved (i) and (ii). Then $m = \sigma(n)$ is an odd integer for which $2 \,\|\, \sigma(m)$. Then such an odd integer must have the prime factorization indicated in (iii) (cf. Euler [5]).

Proof of Theorem 3. Suppose $\sigma(n)$ is the prime power p^a. Then $\sigma(p^a) = \sigma(\sigma(n)) = 2n$, so that Theorem 1 implies $p^a n$ is even, contradicting Lemma 3.

Suppose n is the prime power q^b. Then, in the notation of Lemma 3, we have just proved that $\nu \geqq 1$, so that for $1 \leqq i \leqq \nu$ we have $p_i^{2a_i} \,|\, \sigma(q^b)$ and $\sigma(p_i^{2a_i}) \,|\, q^b$. Say $r = \max \{p_1, p_2, \cdots, p_\nu\}$. Now Lemma 2 implies either $r^2 \,|\, b + 1$ or $r \cdot \mathrm{ord}_r(q) \,|\, b + 1$ in which case $\mathrm{ord}_r(q) > 1$. In the first case $b + 1$ has the 2 divisors r and r^2 which are multiples of r. In the second case, $b + 1$ has the 2 divisors r and $r \cdot \mathrm{ord}_r(q)$ which are multiples of r. Since q is odd, in either case Theorem A implies there are 2 distinct primes dividing $\sigma(q^b)$ which are $1 \pmod r$. This contradicts (iii) of Lemma 3 and the choice of r.

Suppose both n and $\sigma(n)$ are divisible by precisely 2 distinct primes. Now if $(n, \sigma(n)) = 1$, then $n\sigma(n)$ is divisible by precisely 4 distinct primes and $\sigma(n\sigma(n)) = \sigma(n)\sigma(\sigma(n)) = 2n\sigma(n)$. Then Lemma 3 implies $n\sigma(n)$ is an odd perfect number. This contradicts the previously stated result ([12], [14]) that every odd perfect number is divisible by at least 7 distinct primes. Hence $(n, \sigma(n)) > 1$. Hence from Lemma 3 we have the prime factorizations

$$n = q^{2b} r^{2c}$$

$$\sigma(n) = q^\alpha s^\beta .$$

Now $\sigma(q^{2b}) \,|\, s^\beta$ and since $n \,|\, \sigma(\sigma(n))$, we have $q^{2b} \,|\, \sigma(s^\beta)$. Then, as in the above paragraph, there are at least 2 distinct primes dividing $\sigma(s^\beta)$ which are $1 \pmod q$. This contradicts $\sigma(s^\beta) \,|\, 2n$.

REFERENCES

1. E. Artin, *The orders of the linear groups*, Comm. Pure Appl. Math., **VIII** (1955), 355–366.

2. A. S. Bang, *Taltheoretiske Undersøgelser*, Tidsskrift Math. 5 **IV** (1886), 70–80 and 130–137.

3. G. D. Birkhoff and H. S. Vandiver, *On the integral divisors of $a^n - b^n$*, Ann. of Math., **5** (1904), 173-180.

4. L. E. Dickson, *On the cyclotomic function*, Amer. Math. Monthly, **12** (1905), 86-89.

5. L. Euler, *Tractatus de Numerorum Doctrina*, Commentationes Arithmeticae Collectae, **2** (1849), 514.

6. C. G. Gerono, *Note sur la résolution en nombres entiers et positifs de l'équation $x^m = y^n + 1$*, Nouv. Ann. Math., (2) **9** (1870), 469-471, **10** (1871), 204-206.

7. H.-J. Kanold, *Satze über Kreisteilungspolynome und ihre Anwendungen auf einige zahlentheoretische Probleme, I*, J. Reine Angew. Math., **187** (1950), 169-182.

8. ———, *Über "Super perfect numbers"*, Elem. Math., **24** (1969), 61-62.

9. H. W. Leopoldt, *Lösung einer Aufgabe von Kostrikhin*, J. Reine Angew. Math., **221** (1966), 160-161.

10. T. Nagell, *Introduction to Number Theory*, Chelsea Publ. Co., New York, 1964.

11. H. G. Niederreiter, *Solution of Aufgabe 601*, Elem. Math., **25** (1970), 66-67.

12. C. Pomerance, *Odd perfect numbers are divisible by at least seven distinct primes*, Acta Arith., **25** (1974), 265-300.

13. B. Richter, *Die Primfaktorzerlegung der Werte der Kreisteilungspolynome*, J. Reine Angew. Math., **254** (1972), 123-132.

14. N. Robbins, *The non-existence of odd perfect numbers with less than seven distinct prime factors*, doctoral dissertation at the Polytechnic Institute of Brooklyn, June, 1972.

15. W. Sierpiński, *Sur les nombres dont la somme des diviseurs est un puissance du nombre 2*, The Golden Jubile Commemoration Volume (1958-9), Calcutta Math. Soc., 7-9.

16. D. Suryanarayana, *Super perfect numbers*, Elem. Math., **24** (1969), 16-17.

17. ———, *Problems in theory of numbers*, Bull. Amer. Math. Soc., **76** (1970), 977.

18. ———, *There is no odd super perfect number of the form $p^{2\alpha}$*, Elem. Math., **28** (1973), 148-150.

19. J. J. Sylvester, *On the divisors of the sum of a geometrical series whose first term is unity and common ratio any positive or negative integer*, Nature **XXXVII** (1888), 417-418.

20. K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatshefte Math. Phys., **3** (1892), 265-284.

INDIAN INSTITUTE OF TECHNOLOGY
AND
UNIVERSITY OF GEORGIA

PACIFIC JOURNAL OF MATHEMATICS

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39.** All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

Pacific Journal of Mathematics
Vol. 57, No. 2   February, 1975