

# Pacific Journal of Mathematics

**CHARACTERS AND SCHUR INDICES OF THE UNITARY  
REFLECTION GROUP  $[321]^3$**

MARK BENARD

# CHARACTERS AND SCHUR INDICES OF THE UNITARY REFLECTION GROUP $[3\ 2\ 1]^3$

MARK BENARD

The group  $G$  considered in this paper is the finite irreducible unitary reflection group in 6 dimensions denoted by  $[3\ 2\ 1]^3$  in the notation of H. S. M. Coxeter. A character table for  $G$  is constructed. There are 169 irreducible characters and it is shown that each character has Schur index 1 over  $Q$ . Furthermore, each character has values in the field  $Q(\sqrt{-3})$  and this field is a splitting field for  $G$ .

The concept of a reflection in Euclidean space was generalized by G. C. Shephard [9], who first wrote  $[2\ 1; 3]^3$  for  $[3\ 2\ 1]^3$ . A reflection in unitary space is a linear transformation of finite period with the property that all but one of its characteristic values are equal to 1. While reflections in Euclidean space must have period 2, a reflection in unitary space may have period  $m$  for any integer  $m > 1$ . Shephard and J. A. Todd [10] studied the finite groups generated by unitary reflections and classified the irreducible groups. The ordinary finite reflection groups, studied extensively by H. S. M. Coxeter (see [2] for references), are special cases of the finite unitary reflection groups. The group  $G$  considered in this paper is the largest finite primitive irreducible unitary reflection group which is not a Euclidean group. It is generated by reflections of period 2 and has order  $2^9 \cdot 3^7 \cdot 5 \cdot 7$ . The characters of all the other finite irreducible unitary reflection groups will be studied in a forthcoming paper by this author.

**1. General remarks.** The group  $G$  has a center of order 6 consisting of scalar transformations. The central quotient group is the collineation group in 5 dimensions which was studied by H. H. Mitchell [7], C. M. Hamill [6], and Todd [11, 12]. This group is denoted by  $G_1$  throughout this paper. The element  $z$  denotes a generator of the center of  $G$ , fixed throughout. The groups  $G_2$  and  $G_3$  denote the quotient groups of  $G$  by  $\langle z^2 \rangle$  and  $\langle z^3 \rangle$ , respectively.

The commutator subgroup  $G'$  of  $G$  has index 2 in  $G$  and contains the center of  $G$ . The quotient group  $G'_1$  is the simple group  $PSU_4(3)$  of order  $9 \cdot 9!$ .

The group  $G$  contains a reflection subgroup  $H$  of type  $[2\ 2\ 1]^3$  (see [10]). The elements of  $H$  leave invariant a 1-dimensional subspace of the 6-dimensional unitary space on which  $G$  operates and so  $H$  intersects the center of  $G$  trivially. Furthermore,  $H$  is isomorphic to a

direct product of the simple group of order 25920 and the group of order 2. The character table for the simple group of order 25920 was given by J. S. Frame [5].

The commutator subgroup of  $G$  consists of those elements which can be expressed as a product of an even number of reflections. The conjugacy classes falling inside  $G'$  are called even classes, while the others are called odd classes. In the tables, all the even classes are listed first. If  $\chi$  is an (irreducible) character of  $G$  which does not vanish on all odd classes, then there is another (irreducible) character  $\chi^*$  such that  $\chi^*(g) = -\chi(g)$  if  $g$  belongs to an odd class and  $\chi^*(g) = \chi(g)$  otherwise. The character  $\chi^*$  is called the associate of  $\chi$ . Only one character of each pair of associates is listed in the character tables. The principal character 1 of  $G$  has the associate  $1^*$ , the linear character with kernel  $G'$ . Also, for any  $\chi$ ,  $\chi^* = \chi 1^*$ .

The conjugacy classes of  $G_1$  were determined by Hamill. There are 34 classes, 18 of which are even. Each class of  $G_1$  may give rise to up to 6 classes of  $G$ . The classes of  $G$  are listed in cliques  $C_1, \dots, C_{31}$  corresponding to the classes of the quotient group  $G_1$ . This numbering of cliques coincides with the numbering of the classes of  $G_1$  assigned by Hamill. Each clique is represented in the tables by a class containing elements of smallest order. Since representatives of the other classes can be found by multiplying by appropriate elements of the center, then the orders and character values of all elements are easily deduced from the information provided in the tables. The conjugacy class consisting of the 126 reflections of  $G$  is contained in the clique  $C_2$ . The pairs  $\{C_{9a}, C_{9b}\}$ ,  $\{C_{18a}, C_{18b}\}$ , and  $\{C_{24a}, C_{24b}\}$  correspond to classes of  $G_1$  which are pair-wise algebraically conjugate under complex conjugation. Hence the classes of  $C_{9a}$  ( $C_{18a}$ ,  $C_{24a}$ , respectively) are algebraically conjugate to the classes of  $C_{9b}$  ( $C_{18b}$ ,  $C_{24b}$ , respectively). Only  $C_{9a}$ ,  $C_{18a}$ , and  $C_{24a}$  are listed in the tables.

The automorphism of  $G$  which sends each transformation into its adjoint interchanges  $z$  and  $z^{-1}$ . Hence there are essentially 2 ways in which  $G$  may be represented as a unitary reflection group in 6 dimensions. The character  $\psi$  (or  $\psi_6$ ) is the character of that representation such that  $\psi(z) = -6\omega$ , where  $\omega$  is the primitive cube root of unity satisfying  $\omega - \bar{\omega} = \sqrt{-3}$ . The values of  $\psi$  and the characteristic values for the elements in the underlying representation are given by Todd [11]. The characteristic polynomial for an element of smallest order in each clique is listed in Table I. Obviously the orders of elements are determined by their characteristic values.

The characteristic values corresponding to an element  $gz$  can be obtained by multiplying the characteristic values of  $g$  by  $-\omega$ . Since conjugate elements must have the same characteristic values, it is clear that 24 of the 34 cliques consist of 6 conjugacy classes

each. Furthermore, the cliques  $C_{11}$ ,  $C_{16}$ , and  $C_{29}$  each contains either 2 or 6 classes of  $G$ , and  $C_{12}$ ,  $C_{31}$ ,  $C_5$ ,  $C_{25}$ , and  $C_{26}$  each contains either 3 or 6 classes. In §§3 and 4 it is shown that each of these 8 cliques contains exactly the minimum number of cliques listed and that the cliques  $C_{22}$  and  $C_{23}$  each contains 2 classes.

The irreducible characters of  $G$  are found in this paper by decomposing reducible characters constructed from known irreducible characters of  $G$  and  $H$ . The orthogonality relations are the primary tools, but Schur's method for decomposing Kronecker powers of characters is also used in 2 instances. Schur's method provides a technique by which the  $n$ th power of an irreducible character can be partitioned into constituents with the use of the characters of the symmetric group of degree  $n$ . For the full general linear group the constituents produced are irreducible (see [8]), but for arbitrary groups the constituents may be reducible. Given a character  $\chi$  of an arbitrary group and a positive integer  $n$ , there exists characters  $\chi_i$ , one for each irreducible character  $\alpha_i$  of  $\text{Sym}(n)$ , such that for each group element  $g$  and for each  $\sigma \in \text{Sym}(n)$  with cycle structure  $\{1^{m_1}, 2^{m_2}, \dots, n^{m_n}\}$ ,

$$\chi(g)^{m_1} \chi(g^2)^{m_2} \cdots \chi(g^n)^{m_n} = \sum_i \alpha_i(\sigma) \chi_i(g).$$

The system of equations, one for each conjugacy class of  $\text{Sym}(n)$ , can be solved for the  $\chi_i$  to get

$$\chi_i(g) = \sum_{\sigma} \frac{1}{c(\sigma)} \alpha_i(\sigma) \chi(g)^{m_1} \cdots \chi(g^n)^{m_n}$$

where the sum ranges over representatives  $\sigma$  of all conjugacy classes of  $\text{Sym}(n)$  and  $c(\sigma)$  is the order of the centralizer of  $\sigma$ .

**2. Schur indices and splitting fields.** For any irreducible character  $\chi$  of a group and any field  $F$  (of characteristic 0),  $F(\chi)$  denotes the field generated over  $F$  by the values of  $\chi$ . The Schur index  $m_F(\chi)$  of  $\chi$  over  $F$  is the smallest positive integer  $m$  such that  $m\chi$  is afforded by a  $F(\chi)$ -representation. A field  $F$  is a splitting field for a group if each character of the group is afforded by an  $F$ -representation. Hence  $Q(\sqrt{-3})$  is a splitting field for  $G$  if each irreducible character has values in  $Q(\sqrt{-3})$  and Schur index 1 over  $Q$ .

If  $\chi$  is an irreducible character of a group and  $\eta$  is any character afforded by an  $F$ -representation, then  $m_F(\chi)$  divides  $(\chi, \eta)$ . This result may be found in §11 of [4]; other results on Schur indices and splitting fields may be found in Chapter X of [3].

If  $\chi_1$  and  $\chi_2$  are 2 characters afforded by representations  $\mathcal{R}_1$  and  $\mathcal{R}_2$ , respectively, then the character  $\chi_1\chi_2$  is afforded by  $\mathcal{R}_1 \times \mathcal{R}_2$ . Hence if  $\chi_1$  and  $\chi_2$  are afforded by  $F$ -representations, then so is  $\chi_1\chi_2$ . In particular,  $m_F(\chi^*) = m_F(\chi)$  for any irreducible character  $\chi$  of  $G$ .

The characters of  $G_1$  were given by Todd [12]. The irreducible characters are denoted by the symbols  $\phi_n$ , where  $n$  is the degree. The nonassociate characters of degree 35 are denoted by  $\phi_{35}$  and  $\phi'_{35}$ . Similar notation is used for multiple characters of other degrees. It is easy to show that  $Q(\sqrt{-3})$  is a splitting field for  $G_1$ . Each irreducible character of  $G_1$ , with the exception of the 3 characters of degree 560, has multiplicity 1 in some permutation character given by Todd. Thus each of these characters has Schur index 1 over  $Q$ . Each of the other 3 characters has multiplicity 2 in a permutation character, so each has Schur index  $\leq 2$ . Furthermore,  $\phi_{729}$  is rational-valued and thus is afforded by a rational representation, and each character of degree 560 has multiplicity 45 in  $(\phi_{729})^2$ . Hence each of the characters of degree 560 has Schur index 1 over  $Q$ . Since each character of  $G_1$  has values in  $Q(\sqrt{-3})$ , then  $Q(\sqrt{-3})$  is a splitting field for  $G_1$ .

The character  $\psi$  of  $G$  mentioned previously is afforded by a  $Q(\sqrt{-3})$ -representation. Such a representation is given by Shephard and Todd ([10], p. 298).

Each character of  $H$  has Schur index 1 over  $Q$ . The simple group of order 25920 is isomorphic to the commutator subgroup  $W'$  of the Weyl group  $W$  of type  $E_6$ . Each of the characters of  $W$  can be afforded by a rational representation ([1]). Since  $|W:W'| = 2$ , then for each irreducible character  $\chi$  of  $W'$  there exists a character  $\eta$  of  $W$  such that  $(\chi, \eta_{W'}) = 1$ . Thus  $m_Q(\chi) = 1$  for each irreducible character of the simple group of order 25920. Since each character of a group of order 2 has a rational representation, then each character of  $H$  has Schur index 1 over  $Q$ .

**3. The group  $G_2$ .** The faithful irreducible characters of  $G_2$ , the quotient group of  $G$  by  $\langle z^2 \rangle$ , are denoted by  $\theta_n$ , with the same conventions as for the characters  $\phi_n$ . Schur's method of partition characters applied to  $\psi^3$  yields 3 characters of degrees 20, 56, and 70. In particular,  $\psi^3 = \theta_{20} + \theta_{56} + 2\theta_{70}$ , where

$$\theta_{20}(g) = \frac{1}{6}(\psi(g)^3 - 3\psi(g)\psi(g^2) + 2\psi(g^2))$$

$$\theta_{56}(g) = \frac{1}{6}(\psi(g)^3 + 3\psi(g)\psi(g^2) + 2\psi(g^2))$$

$$\theta_{70}(g) = \frac{1}{3}(\psi(g)^3 - \psi(g^3))$$

for all  $g \in G$ . The orthogonality relations show that these 3 characters

are irreducible. Only  $\theta_{56}$  is rational-valued, but  $\bar{\theta}_{20} = \theta_{20}^*$ . However,  $\bar{\theta}_{70} \neq \theta_{70}^*$ , so this provides another irreducible character.

The subgroup  $H$  of  $G$  has an isomorphic image  $H_2$  in  $G_2$  since  $H \cap \langle z^2 \rangle = \langle 1 \rangle$ . Let  $\alpha$  denote the character of  $G_2$  induced from the principal character of  $H_2$ .  $H_2$  has a faithful irreducible character  $\rho$  of degree 6 (see [5]), and  $\beta$  denotes the character of  $G_2$  induced by  $\rho$ . More irreducible characters of  $G_2$  are found from the following equations.

$$\begin{aligned} \theta'_{70} &= \alpha - \phi_1 - \phi_{35} - \phi_{90} - \theta'_{56} \\ \theta_{420} &= \theta_{20}\phi_{21} \\ \theta_{540} &= \theta_{20}\phi_{35} - \theta_{20} - \theta_{70} - \overline{\theta'_{70}} \\ \theta_{140} &= \theta_{20}\phi'_{35} - \theta_{20} - \theta_{540} \\ \theta_{1280} &= \theta_{20}\phi_{140} - \theta_{20}^* - \theta_{420} - 2\theta_{540}^* \\ \theta_{560} &= \theta_{70}\phi'_{35} - \bar{\theta}_{70}^* - \theta_{540} - \theta_{1280} \\ \theta'_{56} &= \beta - \phi_{21}^* - \phi'_{315} - \phi_{420}^* - \theta_{140} - \theta_{560} \\ \theta_{504} &= \theta_{56}\phi_{21} - \theta_{56} - \theta_{56}^* - \theta_{560}^* \\ \theta_{896} &= \theta_{70}\phi_{21} - \theta'_{70} - \theta_{504}^* \\ \theta'_{504} &= \theta_{56}\phi'_{35} - \theta_{560} - \theta_{896}^* \\ \theta_{630} &= \theta'_{70}\phi_{21} - \theta_{70} - \bar{\theta}_{70} - \theta_{140} - \theta_{560}^* \end{aligned}$$

Since the sum of the squares of the degrees of all irreducible characters must equal  $|G_2|$ , then the remaining characters must have degrees whose squares total 28800. The character  $\theta_{56}\phi_{140}$ , after subtracting the irreducible constituents which are already known (see decomposition below), yields a reducible character  $\zeta$ . The character  $\zeta$  has degree 240 and is the sum of 2 distinct irreducible characters. Because of degree limitations, then these characters must each have degree 120. This completes the list of characters for  $G_2$ ; there are 63 irreducible characters. The clique  $C_{22}$  contains either 2 or 6 classes of  $G$ , so it must yield 2 classes of  $G_2$ . Since all but the last 2 faithful characters vanish on  $C_{22}$ , then the last 2 must have nonzero values there and must be associates. The values of  $\theta_{120}$  are found by applying the orthogonality relations. All faithful characters of  $G_2$  vanish on the cliques  $C_{12}$ ,  $C_{31}$ ,  $C_5$ ,  $C_{25}$ , and  $C_{26}$ , so each of these provides one conjugacy class of  $G_2$ .

All faithful irreducible characters of  $G_2$  are rational-valued except for the characters  $\theta_{20}$ ,  $\theta_{70}$ ,  $\bar{\theta}_{70}$ ,  $\theta_{540}$ , and their associates. These 8

characters all have values in  $Q(\sqrt{-3})$ . Since  $\psi$  is afforded by a  $Q(\sqrt{-3})$ -representation and since  $\theta_{20}$  has multiplicity 1 in  $\psi^3$ , then  $\theta_{20}$  and  $\theta_{20}^*$  each has Schur index 1 over  $Q$ . Since  $\theta_{70}$  and  $\theta_{540}$  each has multiplicity 1 in  $\theta_{20}\phi_{35}$ , then each of these characters (along with each associate and complex conjugate) has Schur index 1 over  $Q$ .

To compute the Schur indices for the rational-valued characters, the following 6 reducible characters are needed.

$$\begin{aligned}\alpha &= \phi_1 + \phi_{35} + \phi_{90} + \theta_{56} + \theta'_{70} \\ \beta &= \phi_{21}^* + \phi'_{315} + \phi_{420}^* + \theta'_{56} + \theta_{140} + \theta_{560} \\ \theta_{56}\phi_{140} &= \theta'_{56} + \theta_{120} + \theta_{120}^* + \theta_{504} + \theta'_{504} + \theta'_{504} + 3\theta_{560}^* + 2\theta_{896} + 2\theta_{1280} \\ \theta'_{70}\phi_{21} &= \theta_{70} + \bar{\theta}_{70} + \theta_{140} + \theta_{560}^* + \theta_{630} \\ \theta'_{70}\phi_{35} &= \theta'_{56} + \theta_{70}^* + \theta_{420} + \theta'_{504} + \theta_{504}^* + \theta_{896}^* \\ \theta_{120}\phi_{21} &= \theta_{120} + \theta_{560} + \theta_{560}^* + \theta_{1280}.\end{aligned}$$

Each irreducible character of  $H_2$  has Schur index 1 over  $Q$  and  $\rho$  is rational-valued, so  $\rho$  is afforded by a rational representation. Hence  $\alpha$  and  $\beta$  are both afforded by rational representations. From the decompositions of  $\alpha$  and  $\beta$ , each of the irreducible characters  $\theta_{56}$ ,  $\theta'_{70}$ ,  $\theta'_{56}$ ,  $\theta_{140}$ , and  $\theta_{560}$  has Schur index 1 over  $Q$ . Therefore each of the next 3 reducible characters listed has a rational representation. So each of the characters  $\theta_{120}$ ,  $\theta_{504}$ ,  $\theta'_{504}$ ,  $\theta_{630}$ ,  $\theta_{420}$ , and  $\theta_{896}$  has Schur index 1 over  $Q$ . Finally, then, the character  $\theta_{120}\phi_{21}$  has a rational representation so  $\theta_{1280}$  has Schur index 1 over  $Q$ . Therefore each irreducible character of  $G_2$  has Schur index 1 over  $Q$ .

**4. The group  $G_3$ .** An irreducible character of  $G_3$  of degree  $n$  with the property that its value at  $z$  is  $n\omega$  is denoted by  $\lambda_n$ , with the same conventions as used for the characters of  $G_1$  and  $G_2$ . The partitioning of the character  $\bar{\psi}^2$  yield irreducible characters  $\lambda_{15}$  and  $\lambda_{21}$ , where

$$\lambda_{15}(g) = \frac{1}{2}(\overline{\psi(g)^2} - \overline{\psi(g^2)})$$

$$\lambda_{21}(g) = \frac{1}{2}(\overline{\psi(g)^2} + \overline{\psi(g^2)})$$

for all  $g \in G$ . Other irreducible characters are found as follows.

$$\lambda_{315} = \lambda_{15}\phi_{21}$$

$$\lambda_{105} = \psi\theta_{20} - \lambda_{15}^*$$

$$\lambda'_{105} = \psi\theta'_{70} - \lambda_{315}$$

$$\lambda_{210} = \psi\theta_{56} - \lambda_{21} - \lambda'_{105}$$

$$\lambda''_{105} = \psi\theta_{70} - \lambda_{105} - \lambda_{210}$$

$$\lambda_{384} = \psi\bar{\theta}_{70} - \lambda_{15} - \lambda_{21}$$

$$\lambda_{336} = \psi\theta'_{56}$$

$$\lambda_{720} = \psi\theta_{120}$$

$$\lambda_{420} = \lambda_{15}^*\phi'_{35} - \lambda_{105}$$

$$\lambda_{630} = \lambda_{21}\phi'_{35} - \lambda_{105}''$$

$$\lambda_{756} = \lambda_{15}\theta_{90} - \lambda_{384} - \lambda_{210}$$

$$\lambda_{729} = \lambda_{21}\phi_{90} - \lambda_{21} - \lambda'_{105} - \lambda''_{105} - \lambda_{210} - \lambda_{336}^* - \lambda_{384}$$

$$\lambda_{945} = \lambda_{15}\phi_{140} - \lambda_{105} - \lambda_{420} - \lambda_{630}^*$$

These irreducible characters, together with their associates and complex conjugates, are all the faithful irreducible characters of  $G_3$ .

All faithful irreducible characters of  $G_3$  have values in  $Q(\sqrt{-3})$  and none are rational-valued. Since  $\psi$  is afforded by a  $Q(\sqrt{-3})$ -representation and since  $\bar{\psi}^2 = \lambda_{15} + \lambda_{21}$ , then each of the characters  $\lambda_{15}$  and  $\lambda_{21}$  has Schur index 1 over  $Q$ . Furthermore, as can be seen from the equations given above, each of the other faithful irreducible characters of  $G_3$  has multiplicity 1 in some character afforded by a  $Q(\sqrt{-3})$ -representation, so each irreducible character of  $G_3$  has Schur index 1 over  $Q$ .

The faithful irreducible characters of  $G_3$  vanish on the cliques  $C_{11}$ ,  $C_{16}$ ,  $C_{29}$ ,  $C_{22}$ , and  $C_{23}$ , so each of these cliques yields only one class of  $G_3$ . Since each of these cliques provides 2 classes of  $G_2$ , then each contains exactly 2 classes of  $G$ . Also, each of the cliques  $C_{12}$ ,  $C_{31}$ ,  $C_5$ ,  $C_{25}$ , and  $C_{26}$  provides 1 class of  $G_2$  and 3 classes of  $G_3$ , so each contains exactly 3 classes of  $G$ . Hence  $G$  has 169 conjugacy classes.

**5. The faithful characters.** An irreducible character of  $G$  of degree  $n$  with the property that its value at  $z$  is  $-n\omega$  is denoted by the symbol  $\psi_n$ . The character  $\psi_6$  has already been mentioned. Other faithful irreducible characters are found as follows



$$\begin{aligned}
\psi_{126} &= \psi_6 \phi_{21} \\
\psi_{210} &= \psi_6 \phi'_{35} \\
\psi_{84} &= \theta_{20} \lambda_{15} - \psi_6 - \psi_{210} \\
\psi_{336} &= \theta_{20} \lambda_{21} - \psi_{84} \\
\psi_{840} &= \theta'_{56} \lambda_{15} \\
\psi_{120} &= \psi_6 \phi_{35} - \psi_6 - \psi_{84} \\
\psi_{420} &= \psi_6 \phi_{90} - \psi_{120} \\
\psi_{630} &= \psi_6 \phi_{140} - \psi_{210}^* \\
\psi'_{840} &= \psi_6 \phi_{210} - \psi_{420} \\
\psi_{384} &= \psi_6 \phi'_{280} - \psi_{120} - \psi_{336}^* - \psi'_{840} \\
\psi_{540} &= \theta'_{70} \lambda_{15} - \psi_{126} - \psi_{384} \\
\psi_{1260} &= \theta_{120} \lambda_{15} - \psi_{540}
\end{aligned}$$

These characters, together with their associates and complex conjugates, form the complete set of faithful irreducible characters of  $G$ .

All faithful irreducible characters have values in  $Q(\sqrt{-3})$  and none are rational-valued. Also, as can be seen from the equations above, each of these characters has multiplicity 1 in some character afforded by a  $Q(\sqrt{-3})$  representation. Thus each of these characters has Schur index 1 over  $Q$ , and in particular,  $Q(\sqrt{-3})$  is a splitting field for  $G$ .

**6. The tables.** There are 169 conjugacy classes of  $G$ . Table I lists one class for each of the 34 cliques except  $C_{18b}$ ,  $C_{9b}$ , and  $C_{24b}$ . The elements of  $C_{18b}$ ,  $C_{9b}$ , and  $C_{24b}$  are inverses of the elements of  $C_{18a}$ ,  $C_{9a}$ , and  $C_{24a}$ , respectively. The even classes are listed first, and except where noted otherwise, each clique contains 6 classes. The order and characteristic polynomial given for each clique is for an element of smallest order. Representatives for other classes in a clique can be found by multiplying by appropriate powers of  $z$ , where  $z$  has characteristic polynomial  $(x + \omega)^6$ .

The character table for  $G$  is given in 4 parts. Table II gives the characters  $\phi_n$  of the central quotient group  $G_1$ . The 34 irreducible characters of  $G_1$  are the 16 given characters plus 1,  $\bar{\phi}_{280}$ , and the 16 possible associates. The 29 faithful irreducible characters of  $G_2$  are the 15 characters  $\theta_n$  given in Table III plus  $\theta_{70}$  and the 13 possible associates. The characters  $\lambda_n$  given in Table IV satisfy  $\lambda_n(z) = n\omega$ . The 58 faithful irreducible characters of  $G_3$  are the 15 given

characters plus their complex conjugates and the 28 possible associates. The characters  $\psi_n$  given in Table V are faithful characters of  $G$  satisfying  $\psi_n(z) = -n\omega$ . The final 48 irreducible characters are the 13 given characters plus their complex conjugates and the 22 possible associates.

The following notation is used in the character table to simplify the display:  $a = 1 + 2\omega$ ,  $b = 1 + 3\omega$ ,  $c = 1 + 4\omega$ ,  $d = 1 + 6\omega$ .

TABLE I. CONJUGACY CLASSES OF  $G$

Clique	Smallest Order	Class Size	Centralizer	Characteristic Polynomial
<b>(Even classes)</b>				
1	1	1	$2^9 \cdot 3^7 \cdot 5 \cdot 7$	$(x - 1)^6$
3	2	2835	$2^9 \cdot 3^3$	$(x - 1)^4(x + 1)^2$
4	3	3360	$2^4 \cdot 3^6$	$(x - 1)^3(x^3 - 1)$
8	3	560	$2^5 \cdot 3^7$	$(x^2 + x + 1)^3$
10	6	90720	$2^4 \cdot 3^3$	$(x^2 - 1)^2(x^2 + x + 1)$
13	6	45360	$2^5 \cdot 3^3$	$(x^2 + x + 1)(x^4 + x^2 + 1)$
14	5	653184	$2^2 \cdot 3 \cdot 5$	$(x - 1)(x^5 - 1)$
15	4	34020	$2^7 \cdot 3^2$	$(x - 1)^2(x^2 + 1)^2$
17	6	90720	$2^4 \cdot 3^3$	$(x - 1)^2(x^4 + x^2 + 1)$
18a	9	120960	$2^2 \cdot 3^4$	$(x - 1)^2(x - \omega)(x^3 - \bar{\omega})$
28	12	272160	$2^4 \cdot 3^2$	$(x^2 - x + 1)(x^4 - x^2 + 1)$
30	7	933120	$2 \cdot 3 \cdot 7$	$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
11 (2 classes)	3	120960	$2^2 \cdot 3^4$	$(x^3 - 1)^2$
12 (3 classes)	4	408240	$2^5 \cdot 3$	$(x^2 - 1)(x^4 - 1)$
16 (2 classes)	3	10080	$2^4 \cdot 3^5$	$(x^3 - 1)^2$
29 (2 classes)	9	725760	$2 \cdot 3^3$	$x^6 + x^3 + 1$
31 (3 classes)	8	816480	$2^4 \cdot 3$	$(x^2 + 1)(x^4 + 1)$
<b>(Odd classes)</b>				
2	2	126	$2^8 \cdot 3^5 \cdot 5$	$(x - 1)^2(x + 1)$
6	6	60480	$2^3 \cdot 3^4$	$(x - 1)(x^2 - 1)(x^3 - 1)$
7	4	68040	$2^6 \cdot 3^2$	$(x^2 - 1)(x^4 - 1)$
9a	6	5040	$2^5 \cdot 3^5$	$(x - \bar{\omega})^2(x^2 - \bar{\omega})(x^2 + x + 1)$
19	6	30240	$2^4 \cdot 3^4$	$(x - 1)^2(x - \omega)^2(x^2 - \omega)$
20	12	544320	$2^3 \cdot 3^2$	$(x^2 + x + 1)(x^4 - 1)$
21	10	653184	$2^2 \cdot 3 \cdot 5$	$(x + 1)(x^5 - 1)$
24a	18	362880	$2^2 \cdot 3^3$	$(x - \bar{\omega})(x^2 - 1)(x^3 - \omega)$
27	12	272160	$2^4 \cdot 3^2$	$(x - \omega)(x + \bar{\omega})(x^4 - x^2 + 1)$
5 (3 classes)	2	11340	$2^7 \cdot 3^3$	$(x^2 - 1)^3$
22 (2 classes)	6	1088640	$2^2 \cdot 3^2$	$x^6 - 1$
23 (2 classes)	6	272160	$2^4 \cdot 3^2$	$x^6 - 1$
25 (3 classes)	8	816480	$2^4 \cdot 3$	$(x^2 - 1)(x^4 + 1)$
26 (3 classes)	4	68040	$2^6 \cdot 3^2$	$(x^2 + 1)(x^4 - 1)$

TABLE II. CHARACTERS OF  $G_1$ 

Clique

1	21	35	35	90	140	189	210	280	315	315	420	560	729	896	560
3	5	3	3	10	12	-3	2	-8	11	11	4	-16	9	0	-16
4	3	8	-1	9	-4	0	3	10	18	-9	6	2	0	-4	2
8	-6	8	8	9	5	27	21	10	-9	-9	-39	-34	0	32	20
10	-1	0	3	1	0	0	-1	-2	2	-1	-2	2	0	0	2
13	2	0	0	1	-3	3	5	-2	-1	-1	1	2	0	0	-4
14	1	0	0	0	0	-1	0	0	0	0	0	0	-1	1	0
15	1	3	3	-2	4	5	-2	0	-1	-1	4	0	-3	0	0
17	-1	3	0	1	0	0	-1	1	-1	2	-2	2	0	0	-4
18a	0	2	-1	0	-1	0	0	b	0	0	0	-1	0	-1	2
28	-2	0	0	1	1	-1	1	0	-1	-1	1	0	0	0	0
30	0	0	0	-1	0	0	0	0	0	0	0	0	1	0	0
11	3	-1	-1	0	5	0	3	1	0	0	-3	2	0	-4	2
12	1	-1	-1	2	0	1	-2	0	-1	-1	0	0	1	0	0
16	3	-1	8	9	-4	0	3	1	-9	18	6	2	0	-4	20
29	0	-1	2	0	-1	0	0	1	0	0	0	-1	0	-1	-1
31	-1	-1	-1	0	0	1	0	0	1	1	0	0	-1	0	0
2	9	15	-5	30	20	9	30	40	75	-15	60	0	81	64	0
6	3	0	1	3	2	0	3	-2	0	-3	0	0	0	-2	0
7	1	3	-1	2	0	-3	-2	0	3	1	0	0	-3	0	0
9a	0	6	4	3	-7	9	3	-2d	3	3	-3	6a	0	-8	0
19	-3	3	-2	3	2	0	3	1	-3	0	-6	0	0	4	0
20	1	0	-1	-1	0	0	1	0	0	1	0	0	0	0	0
21	-1	0	0	0	0	-1	0	0	0	0	0	0	1	-1	0
24a	0	0	1	0	-1	0	0	$\omega$	0	0	0	a	0	1	0
27	0	2	0	-1	1	1	-1	0	-1	-1	1	0	0	0	0
5	1	-1	-5	6	4	9	-10	-8	3	9	-4	0	9	0	0
22	1	-1	1	0	1	0	-1	1	0	0	-1	0	0	0	0
23	1	-1	-2	3	-2	0	-1	1	-3	0	2	0	0	0	0
25	-1	-1	-1	0	0	1	0	0	1	-1	0	0	-1	0	0
26	-3	-1	3	2	4	1	2	0	-1	5	4	0	-3	0	0

TABLE III. FAITHFUL CHARACTERS OF  $G_2$

Clique	20	56	56	70	70	120	504	504	540	560	630	896	140	420	1280
1	20	56	56	70	70	120	504	504	540	560	630	896	140	420	1280
3	-4	8	8	2	2	8	8	8	-12	16	-14	0	4	-20	0
4	2	11	2	7	7	-6	18	-9	0	2	9	-4	-4	6	-16
8	-7	2	2	-11	16	12	18	18	-27	-34	-18	32	-22	42	-16
10	2	-1	2	-1	-1	2	2	-1	0	-2	1	0	4	-2	0
13	-1	2	2	-1	-4	-4	2	2	3	-2	-2	0	-2	-2	0
14	0	1	1	0	0	0	-1	-1	0	0	0	1	0	0	0
15	4	0	0	2	2	0	0	0	4	0	-6	0	4	4	0
17	2	2	-1	2	-1	2	-1	2	0	-2	1	0	-2	-2	0
18a	-1	2	-1	b	1	0	0	0	0	-1	0	-1	2	0	2
28	-1	0	0	1	-2	0	0	0	-1	0	0	0	2	2	0
30	-1	0	0	0	0	1	0	0	1	0	0	0	0	0	-1
11	2	2	2	-2	-2	3	0	0	0	2	0	-4	-4	6	2
16	2	2	11	-2	7	-6	-9	18	0	2	9	-4	14	6	-16
29	-1	-1	2	1	1	0	0	0	0	-1	0	-1	-1	0	2
2	0	24	-16	20	20	0	96	24	0	-80	60	64	0	0	0
6	0	3	-4	-1	-1	0	0	3	0	-2	-3	-2	0	0	0
7	0	4	0	2	2	0	0	-4	0	0	-2	0	0	0	0
9a	3 $\bar{a}$	6	2	-d	2	0	6	6	9 $\bar{a}$	10	6	-8	0	0	0
19	0	0	-1	-4	5	0	3	0	0	4	-3	4	0	0	0
20	0	1	0	-1	-1	0	0	-1	0	0	1	0	0	0	0
21	0	-1	-1	0	0	0	1	-1	0	0	0	-1	0	0	0
24a	a	0	-1	- $\omega$	-1	0	0	0	0	1	0	1	0	0	0
27	$\bar{a}$	0	0	$\bar{a}$	0	0	0	0	a	0	0	0	0	0	0
22	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0
23	0	0	-3	0	3	0	-3	0	0	0	3	0	0	0	0

TABLE IV. FAITHFUL CHARACTERS OF  $G_3$ 

Clique	1	15	21	105	105	105	210	315	336	384	420	630	729	756	945
-F	1														
3		-1	5	-7	9	9	2	-5	16	0	4	6	9	-12	-15
4		3	6	3	12	3	15	9	6	12	-6	-9	0	0	0
8		6	3	15	-12	15	3	-36	-6	24	33	9	0	27	-27
10		-1	2	-1	0	3	-1	1	-2	0	-2	3	0	0	0
13		2	-1	-1	0	3	-1	4	-2	0	1	-3	0	3	-3
14		0	1	0	0	0	0	0	1	-1	0	0	-1	1	0
15		3	1	5	1	1	-2	3	0	0	4	2	-3	-4	1
17		2	2	2	0	0	2	-2	-2	0	-2	0	0	0	0
18a		$\bar{a}$	$\bar{\omega}a$	$\bar{\omega}\bar{a}$	$\omega\bar{a}$	$\bar{\omega}\bar{a}$	$a$	0	$\omega a$	$\bar{a}$	$\bar{\omega}\bar{a}$	0	0	0	0
28		0	1	-1	-2	1	1	0	0	0	1	-1	0	-1	1
30		1	0	0	0	0	0	0	0	-1	0	0	1	0	0
12		-1	1	1	1	1	-2	-1	0	0	0	-2	1	0	1
31		1	-1	-1	1	1	0	-1	0	0	0	0	-1	0	1
2		5	11	5	35	25	50	45	-64	64	20	-30	81	36	45
6		-1	2	-1	2	1	-1	-3	-4	-2	2	3	0	0	0
7		1	3	1	3	1	2	1	0	0	0	-2	-3	0	-3
9a		$-4\omega$	$\bar{c}-3+8\omega$	$2\omega b$	$-\bar{c}$	$4\bar{\omega}-5$	0	$2\omega b$	$-8\omega$	$3+8\omega$	$3\bar{c}$	0	0	9	-9
19		$2\bar{\omega}$	$2\bar{\omega}$	$2\bar{\omega}$	$-4\bar{\omega}$	$4\bar{\omega}$	$2\bar{\omega}$	$-6\bar{\omega}$	$2\bar{\omega}$	$4\bar{\omega}$	$2\bar{\omega}$	0	0	0	0
20		1	0	1	0	1	-1	1	0	0	0	1	0	0	0
21		0	1	0	0	0	0	0	1	-1	0	0	1	1	0
24a		-1	$-\omega$	$-\omega$	$-\bar{\omega}$	$\omega$	-1	0	$-\bar{\omega}$	1	$-\bar{\omega}$	0	0	0	0
27		-2	-1	1	0	1	1	0	0	0	-1	1	0	1	-1
5		-3	3	-3	3	9	-6	-3	0	0	-12	-6	9	-12	-3
25		-1	1	-1	-1	1	0	1	0	0	0	0	-1	0	1
26		-1	1	-1	-3	-1	2	3	0	0	4	2	3	-4	-5

TABLE V. FAITHFUL CHARACTERS OF  $G$

Clique													
1	6	84	120	126	210	336	384	420	630	840	840	540	1260
3	2	-4	8	10	6	-16	0	12	18	-8	-8	-12	4
4	3	6	15	9	-3	6	12	12	-9	6	-3	0	-18
8	-3	-15	-6	18	-24	-6	24	-21	9	12	-42	54	18
10	-1	2	-1	1	-3	2	0	0	3	-2	1	0	-2
13	-1	-1	2	-2	0	2	0	-3	3	4	-2	-6	-2
14	1	-1	0	1	0	1	-1	0	0	0	0	0	0
15	2	4	0	2	6	0	0	-4	2	0	0	4	-4
17	2	2	2	-2	0	2	0	0	0	-2	-2	0	4
18a	$\omega\bar{a}$	$a$	$\bar{\omega}a$	0	$\omega a$	$-\omega\bar{a}$	$\bar{a}$	$\bar{\omega}\bar{a}$	0	$a$	$\bar{\omega}a$	0	0
28	1	-1	0	-2	0	0	0	1	1	0	0	2	-2
30	-1	0	1	0	0	0	-1	0	0	0	0	1	0
2	4	16	40	36	-20	-16	64	80	60	-80	40	0	0
6	1	-2	1	3	1	2	-2	2	3	4	1	0	0
7	2	0	4	2	-2	0	0	0	-2	0	-4	0	0
9a	$2\bar{\omega} - 72\bar{\omega} - t$		$2\bar{c}$	0	$4\omega b$	$8\omega - 6$	$-8\omega$	$2\omega - 3$	$-3\omega b$	$-8\omega$	$2\bar{c}$	0	0
19	$-2\bar{\omega}$	$-2\bar{\omega}$	$-2\bar{\omega}$	$6\bar{\omega}$	$4\bar{\omega}$	$2\bar{\omega}$	$4\bar{\omega}$	$-4\bar{\omega}$	0	$-2\bar{\omega}$	$-2\bar{\omega}$	0	0
20	-1	0	1	-1	1	0	0	0	1	0	-1	0	0
21	-1	1	0	1	0	-1	-1	0	0	0	0	0	0
24a	$\bar{\omega}$	1	$\omega$	0	$\bar{\omega}$	$-\bar{\omega}$	1	$-\omega$	0	1	$\omega$	0	0
27	$a$	$a$	0	0	0	0	0	$\bar{a}$	$a$	0	0	0	0

REFERENCES

1. M. Benard, *On the Schur indices of the exceptional Weyl groups*, Annals of Math., **94** (1971), 89-107.
2. H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, 2nd Edition, Springer-Verlag, New York, 1965.
3. C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience, New York, 1962.
4. W. Feit, *Characters of Finite Groups*, Benjamin, New York, 1967.
5. J. S. Frame, *The simple group of order 25920*, Duke J. Math., **2** (1936), 477-484.
6. C. M. Hamill, *On a finite group of order 6, 531, 840*, Proc. London Math. Soc., (2), **52** (1951), 401-454.
7. H. H. Mitchell, *Determination of all primitive collineation groups in more than four variables which contain homologies*, Amer. J. Math., **36** (1914), 1-12.
8. I. Schur, *Über die rationalen Darstellungen der allgemeinen linearen Gruppe*, Sitzber. Preuss. Akad. Wiss., (1927), 58-75.
9. G. C. Shephard, *Unitary groups generated by reflections*, Canad. J. Math., **5** (1953), 364-383.
10. G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math., **6** (1954), 274-304.
11. J. A. Todd, *The invariants of a finite collineation group in 5 dimensions*, Proc. Camb. Phil. Soc., **46** (1950), 73-90.
12. ———, *The characters of a collineation group in 5 dimensions*, Proc. Royal Soc. of London, (A) **200** (1950), 320-336.

Received April 23, 1974. This research was partially supported by National Science Foundation Grant #GP-29437.



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)

University of California  
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

\* \* \*

AMERICAN MATHEMATICAL SOCIETY

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION  
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1975 Pacific Journal of Mathematics  
All Rights Reserved



Zvi Artstein and John Allen Burns, <i>Integration of compact set-valued functions</i> .....	297
Mark Benard, <i>Characters and Schur indices of the unitary reflection group [321]<sup>3</sup></i> .....	309
Simeon M. Berman, <i>A new characterization of characteristic functions of absolutely continuous distributions</i> .....	323
Monte Boisen and Philip B. Sheldon, <i>Pre-Prüfer rings</i> .....	331
Hans-Heinrich Brungs, <i>Three questions on duo rings</i> .....	345
Iracema M. Bund, <i>Birbaum-Orlicz spaces of functions on groups</i> .....	351
John D. Elwin and Donald R. Short, <i>Branched immersions between 2-manifolds of higher topological type</i> .....	361
Eric Friedlander, <i>Extension functions for rank 2, torsion free abelian groups</i> .....	371
Jon Froemke and Robert Willis Quackenbush, <i>The spectrum of an equational class of groupoids</i> .....	381
Barry J. Gardner, <i>Radicals of supplementary semilattice sums of associative rings</i> .....	387
Shmuel Glasner, <i>Relatively invariant measures</i> .....	393
George Rudolph Gordh, Jr. and Sibe Mardesic, <i>Characterizing local connectedness in inverse limits</i> .....	411
Siegfried Graf, <i>On the existence of strong liftings in second countable topological spaces</i> .....	419
Stanley P. Gudder and D. Strawther, <i>Orthogonally additive and orthogonally increasing functions on vector spaces</i> .....	427
Darald Joe Hartfiel and Carlton James Maxson, <i>A characterization of the maximal monoids and maximal groups in <math>\beta_X</math></i> .....	437
Robert E. Hartwig and S. Brent Morris, <i>The universal flip matrix and the generalized faro-shuffle</i> .....	445
William Emery Haver, <i>Mappings between ANRs that are fine homotopy equivalences</i> .....	457
J. Bockett Hunter, <i>Moment sequences in <math>l^p</math></i> .....	463
Barbara Jeffcott and William Thomas Spears, <i>Semimodularity in the completion of a poset</i> .....	467
Jerry Alan Johnson, <i>A note on Banach spaces of Lipschitz functions</i> .....	475
David W. Jonah and Bertram Manuel Schreiber, <i>Transitive affine transformations on groups</i> .....	483
Karsten Juul, <i>Some three-point subset properties connected with Menger's characterization of boundaries of plane convex sets</i> .....	511
Ronald Brian Kirk, <i>The Haar integral via non-standard analysis</i> .....	517
Justin Thomas Lloyd and William Smiley, <i>On the group of permutations with countable support</i> .....	529
Erwin Lutwak, <i>Dual mixed volumes</i> .....	531
Mark Mahowald, <i>The index of a tangent 2-field</i> .....	539
Keith Miller, <i>Logarithmic convexity results for holomorphic semigroups</i> .....	549
Paul Milnes, <i>Extension of continuous functions on topological semigroups</i> .....	553
Kenneth Clayton Pietz, <i>Cauchy transforms and characteristic functions</i> .....	563
James Ted Rogers Jr., <i>Whitney continua in the hyperspace <math>C(X)</math></i> .....	569
Jean-Marie G. Rolin, <i>The inverse of a continuous additive functional</i> .....	585
William Henry Ruckle, <i>Absolutely divergent series and isomorphism of subspaces</i> .....	605
Rolf Schneider, <i>A measure of convexity for compact sets</i> .....	617
Alan Henry Schoenfeld, <i>Continuous measure-preserving maps onto Peano spaces</i> .....	627
V. Merriline Smith, <i>Strongly superficial elements</i> .....	643
Roger P. Ware, <i>A note on quadratic forms over Pythagorean fields</i> .....	651
Roger Allen Wiegand and Sylvia Wiegand, <i>Finitely generated modules over Bezout rings</i> .....	655
Martin Ziegler, <i>A counterexample in the theory of definable automorphisms</i> .....	665