Pacific Journal of Mathematics

CHARACTERIZING LOCAL CONNECTEDNESS IN INVERSE LIMITS

GEORGE RUDOLPH GORDH, JR. AND SIBE MARDESIC

Vol. 58, No. 2 April 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

CHARACTERIZING LOCAL CONNECTEDNESS IN INVERSE LIMITS

G. R. GORDH, JR. AND SIBE MARDEŠIĆ

Let X denote the limit of an inverse system $\underset{\sim}{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$ of locally connected Hausdorff continua. The main purpose of this paper is to define a notion of local connectedness for inverse systems, and to prove that if $\underset{\sim}{X}$ is locally connected, then so is the limit X. If the bonding maps $p_{\alpha\alpha'}$ are surjections, then X is locally connected if and only if $\underset{\sim}{X}$ is. The following corollaries are obtained. (1) If $\underset{\sim}{X}$ is σ-directed and surjective, then X is locally connected. (2) If $\underset{\sim}{X}$ is well-ordered, surjective, and weight $(X_\alpha) \leqq \lambda$ for each α in A, then either weight $(X) \leqq \lambda$, or X is locally connected. (3) If $\underset{\sim}{X}$ is σ-directed and the factor spaces X_α are trees (generalized arcs), then X is a tree (generalized arc). (4) If $\underset{\sim}{X}$ is well-ordered and the factor spaces X_α are dendrites (arcs), then either X is metrizable, or X is a tree (generalized arc).

1. Introduction. By a continuum we mean a compact connected Hausdorff space. Let X denote the limit of an inverse system $\underset{\sim}{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$ where the factor spaces X_α are locally connected continua, and A is an arbitrary directed set. It is well-known that every continuum X can be obtained as the limit of such a system where the factor spaces are polyhedra (see Theorem 10.1, p. 284, [2]). Hence local connectedness of the factor spaces X_α does not imply local connectedness of the limit X. It is the main purpose of this paper to introduce a notion of *local connectedness* for inverse systems, and to prove that for such systems $\underset{\sim}{X}$ the limit space X is locally connected (see Theorem 1). The converse holds if $\underset{\sim}{X}$ is a surjective system, i.e., if the bonding maps $p_{\alpha\alpha'}$ are surjections. An immediate corollary is the known result that if $\underset{\sim}{X}$ is a monotone inverse system, then X is locally connected [1].

In §3 the main theorem is applied to well-ordered and σ-directed inverse systems, i.e., systems in which every countable subset of the index set is bounded above. The following somewhat surprising results are obtained. (1) If the inverse system $\underset{\sim}{X}$ is σ-directed and surjective, then the limit X is locally connected. (2) If $\underset{\sim}{X}$ is well-ordered, surjective, and weight $(X_\alpha) \leqq \lambda$ for each α in A, then weight $(X) \leqq \lambda$ or X is locally connected.

Section 4 contains similar results about well-ordered and σ-directed inverse systems of trees (i.e., locally connected, hereditarily unicoherent continua [9]) and generalized arcs (i.e., ordered continua).

For example, the limit of a σ-directed inverse system of trees (generalized arcs) is a tree (generalized arc).[1]

The problem of characterizing locally connected inverse limits has been studied from a different point of view in [3].

The reader is referred to [1] for basic results concerning inverse limits of compact Hausdorff spaces.

2. Locally connected inverse systems. A continuum X has *property S* if given any open cover $\mathcal{U}$ of X, there exists a finite cover $\mathcal{C}$ of X which refines $\mathcal{U}$ and consists of connected subsets of X. A continuum is locally connected if and only if it has property S (e.g., Chapter IV, Theorem 3.7, p. 106, [11]).

DEFINITION Let $f: X \to Y$ be a mapping of locally connected continua, and let $F \subset U \subset Y$ where F is closed and U is open. We define the *splitting number* $s(f, U, F)$ of the triple (f, U, F) to be the number of components of $f^{-1}(U)$ which meet $f^{-1}(F)$.

LEMMA 1. *The splitting number $s(f, U, F)$ is finite.*

Proof. Since X is locally connected, the components of $f^{-1}(U)$ are open sets. By compactness, only finitely many components of $f^{-1}(U)$ can meet the closed set $f^{-1}(F)$.

DEFINITION. Let $\underset{\sim}{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$ be an inverse system of continua over an arbitrary directed set A. We say that the system $\underset{\sim}{X}$ is *locally connected* if (1) the factor spaces X_α are locally connected; and (2) whenever $F_\alpha \subset U_\alpha \subset X_\alpha$, where F_α is closed and U_α is open, there exists an $\alpha' \geqq \alpha$ in A such that the splitting number $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$ agrees with $s(p_{\alpha\alpha''}, U_\alpha, F_\alpha)$ for every $\alpha'' \geqq \alpha'$.

THEOREM. 1. *The limit of a locally connected inverse system is locally connected.*

Proof. Let $\underset{\sim}{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$ be a locally connected inverse system with limit X and projections $p_\alpha: X \to X_\alpha$. We shall prove that X has property S. Let $\mathcal{U}$ be any open cover of X. There exists an $\alpha \in A$ and a finite open cover $\mathcal{U}_\alpha = (U_1, \cdots, U_n)$ of X_α such that $\{p_\alpha^{-1}(U_i)\}_{i=1}^n$ refines $\mathcal{U}$ (e.g., Lemma 3.7, p. 263, [2]). Choose open covers $\mathcal{U}'_\alpha = (U'_1, \cdots, U'_n)$ and $\mathcal{U}''_\alpha = (U''_1, \cdots, U''_n)$ of X_α such that $U''_i \subset \mathrm{cl}(U''_i) \subset U'_i \subset \mathrm{cl}(U'_i) \subset U_i$. Let $F_i = \mathrm{cl}(U''_i)$ and consider the pairs (U'_i, F_i). Since the system $\underset{\sim}{X}$ is locally connected, there exists an $\alpha' \in A$ such that for $\alpha'' \geqq \alpha'$ we have $s(p_{\alpha\alpha'}, U'_i, F_i) = s(p_{\alpha\alpha''}, U'_i, F_i)$ for $1 \leqq i \leqq n$. Let s_i denote the splitting number $s(p_{\alpha\alpha'}, U'_i, F_i)$. For $\alpha' \in A$ as above, let

[1] M. Smith has announced results similar to Corollary 5 and Theorem 6 at the Topology Conference held at the University of North Carolina at Charlotte, March, 1974.

$\{V^i_{\alpha' j}\}_{j=1}^{s_i}$ denote the collection of components of $p_{\alpha\alpha'}^{-1}(U'_i)$ which intersect $p_{\alpha\alpha'}^{-1}(F_i)$. For $\alpha'' \geqq \alpha'$ there are also s_i components of $p_{\alpha\alpha'}^{-1}(U'_i)$ which intersect $p_{\alpha\alpha'}^{-1}(F_i)$. Denote these components by $\{V^i_{\alpha'' j}\}_{j=1}^{s_i}$, and assume that they are labelled so that $p_{\alpha'\alpha''}(V^i_{\alpha'' j}) \subset V^i_{\alpha' j}$. Define $C^i_{\alpha'' j} = \mathrm{cl}(V^i_{\alpha'' j})$ for all $\alpha'' \geqq \alpha'$, and let

$$C^i_j = \operatorname{inv\,lim}\{C^i_{\alpha'' j};\ \alpha'' \geqq \alpha'\}.$$

Since $\{F_i\}$ covers X_α, it follows that $\{C^i_{\alpha'' j}\}$ covers $X_{\alpha''}$ for each $\alpha'' \geqq \alpha'$. To every $x \in X$ one can assign a pair (i, j) such that $p_{\alpha''}(x) \in C^i_{\alpha'' j}$. Since i and j vary through a finite set, some pair (i, j) occurs cofinally often; and consequently $x \in C^i_j$. Consequently, $\{C^i_j\}_{i,j}$ covers X and refines $\{p_\alpha^{-1}(U_i)\}_{i=1}^n$ which refines $\mathcal{U}$. Since each C^i_j is a subcontinuum of X, it follows that X has property S.

The next theorem provides a converse to Theorem 1 for inverse systems with surjective bounding maps.

THEOREM 2. *Let $X = \operatorname{inv\,lim} \underset{\sim}{X}$ where $\underset{\sim}{X}$ is a surjective inverse system of continua. If X is locally connected, then the system $\underset{\sim}{X}$ is locally connected.*

The proof of Theorem 2 depends on two simple lemmas.

LEMMA 2. *Let X_1, X_2 and Y be locally connected continua and suppose that $f_i\colon X_i \to Y$ $(i = 1, 2)$ and $g\colon X_2 \to X_1$ are continuous surjections such that $f_2 = f_1 g$. Let $F \subset U \subset Y$ where F is closed and U is open. Then $s(f_1, U, f) \leqq s(f_2, U, F)$.*

Proof. Let $s_1 = s(f_1, U, F)$, and let $V_1, \cdots, V_{s_1}$ denoted the components of $f_1^{-1}(U)$ which meet $f_1^{-1}(F)$. For each $i \leqq s_1$, at least one component of $g^{-1}(V_i)$ meets $g^{-1}(f_1^{-1}(F)) = f_2^{-1}(F)$. Since each component of $g^{-1}(V_i)$ is a component of $f_2^{-1}(U)$, at least s_1 components of $f_2^{-1}(U)$ meet $f_2^{-1}(F)$. Thus $s_1 \leqq s(f_2, U, F)$.

LEMMA 3. *Let A be a directed set and N the set of natural numbers. If $\pi\colon A \to N$ is an order preserving bounded function, then π is eventually constant.*

Proof. Let $m = \max \pi(A)$, and choose $\alpha \in A$ such that $\pi(\alpha) = m$. Thus for $\alpha' \geqq \alpha$, $\pi(\alpha') = m$.

Proof of Theorem 2. Let $\underset{\sim}{X} = \{X_\alpha;\ p_{\alpha\alpha'};\ A\}$ be a surjective system of continua with locally connected limit X and projections $p_\alpha\colon X \to X_\alpha$. Since the projections p_α are surjections (e.g., Theorem 2.6, **[1]**), each factor space X_α is the image of a locally connected continuum; hence each X_α is locally connected (e.g., Theorem 3–22, p. 126, [5]). Given $\alpha \in A$, let $A(\alpha) = \{\alpha' \in A \mid \alpha' \geqq \alpha\}$, and let $F_\alpha \subset U_\alpha \subset X_\alpha$ where F_α is closed and U_α is open. Define $\pi: A(\alpha) \to N$ by $\pi(\alpha') = s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$. Lemma 2 implies that π is order preserving and bounded by $s(p_\alpha, U_\alpha, F_\alpha)$. By Lemma 3, there exists $\alpha' \in A(\alpha)$ such that for all $\alpha'' \geqq \alpha'$, $\pi(\alpha') = \pi(\alpha'')$; i.e., $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha) = s(p_{\alpha\alpha''}, U_\alpha, F_\alpha)$.

COROLLARY 1. *Let $\underset{\sim}{X}$ be a surjective inverse system of locally connected continua with limit X. Then X is locally connected if and only if $\underset{\sim}{X}$ is locally connected.*

A surjective continuous function $f: X \to Y$ between continua is *monotone* if $f^{-1}(y)$ is a continuum for each $y \in Y$. An inverse system of continua is *monotone* if each bonding map is monotone.

COROLLARY 2. (Capel [1]). *The limit of a monotone inverse system of locally connected continua is locally connected.*

Proof. Let $\{X_\alpha;\ p_{\alpha\alpha'};\ A\}$ be a monotone inverse system of locally connected continua. Let $F_\alpha \subset U_\alpha \subset X_\alpha$ where F_α is closed and U_α is open in X_α. If $\alpha' \geqq \alpha$, then since $p_{\alpha\alpha'}$ is monotone, the splitting number $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$ is precisely the number of components of U_α which meet F_α. Thus, for $\alpha' \geqq \alpha$ the splitting number $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$ is independent of α', and so the inverse system is locally connected. By Theorem 1, the limit of the system is locally connected.

3. Well-ordered and σ-directed inverse systems of locally connected continua.

We say that a quasi-ordered set A is *σ-directed* (directed) if every countable (finite) subset of A is bounded above. Thus every bounded quasi-ordered set is σ-directed. Clearly, an unbounded well-ordered set is σ-directed if and only if it contains no cofinal sequence. Another example of a σ-directed set is the collection of all countable subsets of a given set, ordered by inclusion. An inverse system is said to be *σ-directed* (*well-ordered*) if its index set is σ-directed (well-ordered).

LEMMA 4. *Let A be a σ-directed set and let N denote the set of natural numbers. If $\pi: A \to N$ is an order preserving function, then π is eventually constant.*

Proof. If π is not eventually constant, then there exists an increasing sequence $\{\alpha_i\}_{i=1}$ in A such that $\{\pi(\alpha_i)\}_{i=1}$ is cofinal in N.

Since A is σ-directed, there exists $\alpha \in A$ such that $\alpha_i \leqq \alpha$ for every $i \in N$. Thus $\pi(\alpha_i) \leqq \pi(\alpha)$ for every i, which is a contradiction.

THEOREM 3. *The limit of a σ-directed surjective inverse system of locally connected continua is locally connected.*

Proof. Let $\underset{\sim}{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$ be a σ-directed surjective inverse system of locally connected continua. According to Theorem 1, it suffices to show that $\underset{\sim}{X}$ is a locally connected system. Let $F_\alpha \subset U_\alpha \subset X_\alpha$ where F_α is closed and U_α is open. Let $A(\alpha) = \{\alpha' \in A \mid \alpha' \geqq \alpha\}$ and note that $A(\alpha)$ is a σ-directed set. We define a function $\pi: A(\alpha) \to N$ by $\pi(\alpha') = s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$. By Lemma 2, π is an increasing function. Thus, by Lemma 4, π is eventually constant, and there exists $\alpha' \in A(\alpha)$ such that $\pi(\alpha') = \pi(\alpha'')$ whenever $\alpha' \leqq \alpha''$. Thus for $\alpha' \leqq \alpha''$ we have $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha) = s(p_{\alpha\alpha''}, U_\alpha, F_\alpha)$, and $\underset{\sim}{X}$ is locally connected.

COROLLARY 3. *If X is the limit of a σ-directed inverse system of hereditarily locally connected continua, then X is hereditarily locally connected.*

Proof. Let $X = \operatorname{inv\,lim}\{X_\alpha; p_{\alpha\alpha'}; A\}$ where A is σ-directed and the factor spaces X_α are hereditarily locally connected continua. Let Y be any subcontinuum of X. Then $\{p_\alpha(Y); p_{\alpha\alpha'} | p_{\alpha'}(Y); A\}$ is a σ-directed surjective inverse system of locally connected continua with limit Y (see [1]). By Theorem 3, Y is locally connected.

The *weight* of a topological space X, denoted $w(X)$, is the smallest cardinal number λ such that X admits a basis for its topology of cardinality λ.

THEOREM 4. *Let X be the limit of a well-ordered surjective inverse system $\underset{\sim}{X}$ of locally connected continua X_α such that $w(X_\alpha) \leqq \lambda$ for each X_α. Then, either $w(X) \leqq \lambda$, or X is locally connected. In particular, if the factor spaces X_α are metrizable, then either X is metrizable, or X is locally connected.*

Proof. Let A denote the well-ordered index set for the system $\underset{\sim}{X}$. If A contains a cofinal sequence, then X is the limit of an inverse sequence of continua X_n such that $w(X_n) \leqq \lambda$; hence $w(X) \leqq \lambda$. Otherwise, A is σ-directed and X is locally connected by Theorem 3.

REMARK. Suppose that the nonmetrizable continuum X is the limit of a well-ordered surjective inverse system of metric continua X_α. If X is non-locally connected, then by Theorem 4 the factor spaces X_α are eventually nonlocally connected as well. This remark applies to all continua of weight $\aleph_1$, since such continua are known to be limits of well-ordered surjective inverse systems of metric continua [7].

COROLLARY 4. *Let X be the limit of a well-ordered inverse system $\underline{X}$ of hereditarily locally connected continua X_α such that $w(X_\alpha) \leqq \lambda$ for each $\alpha \in A$. Then either $w(X) \leqq \lambda$, or X is hereditarily locally connected.*

4. Well-ordered and σ-directed inverse systems of trees and generalized arcs. A continuum X is a *tree* [9] if each pair of points is separated by a third point. A continuum X with precisely two nonseparating points is called a *generalized arc* (or an *ordered continuum*). According to [9], a continuum X is a tree if and only if X is locally connected and hereditarily unicoherent. Clearly every subcontinuum of a tree X is a tree, and consequently X is hereditarily locally connected. It follows immediately from Theorem 4.1(3) of [4] that a tree is a generalized arc if and only if it is atriodic.

It is known that the limit of a monotone inverse system of trees is a tree (see the proof of Theorem 4.2 in [4]); and that the limit of a monotone inverse system of generalized arcs is a generalized arc (Lemma 4.7 of [1], or [8]). We shall obtain the same conclusions for σ-directed inverse systems of trees and generalized arcs without any assumptions about the bonding maps.

LEMMA 5. *Suppose that X is the limit of an arbitrary inverse system of trees* (generalized arcs). *If X is locally connected, then X is a tree* (generalized arc).

Proof. Since the factor spaces are hereditarily unicoherent, X is also hereditarily unicoherent by a routine application of ((2.9), p. 235, [1]). Consequently, X is a tree. If the factor spaces are generalized arcs, then X is chainable (e.g., [6]). Since chainable continua are atriodic, X is an atriodic tree; i.e., a generalized arc.

REMARK. The proof of Lemma 5 can be modified to show that a locally connected tree-like (arc-like, i.e., chainable) continuum is a tree (generalized arc). If X is tree-like, then X is hereditarily unicoherent. Consequently, if X is locally connected, then X is a tree. If, in addition, X is arc-like, then X is atriodic; hence X is a generalized arc (see [8] for a different proof).

THEOREM 5. *If X is the limit of a σ-directed inverse system of trees* (generalized arcs), *then X is a tree* (generalized arc).

Proof. Apply Corollary 3 and Lemma 5.

THEOREM 6. *Let X be the limit of a well-ordered inverse system of trees* (generalized arcs) *X_α such that $w(X_\alpha) \leqq \lambda$ for each X_α. Then, either $w(X) \leqq \lambda$, or X is a tree* (generalized arc).

Proof. Apply Corollary 4 and Lemma 5.

COROLLARY 5. *Let X be the limit of a well-ordered inverse system of dendrites* (arcs). *Then, either X is metrizable, or X is a tree* (generalized arc).

Proof. A dendrite (arc) is a metrizable tree (generalized arc) (see (1.1), p. 88 and Theorem (6.2), p. 54 of **[10]**). Thus the desired conclusion follows from Theorem 6.

REMARK. The limit of a well-ordered inverse system of arcs need not be metrizable. For example, the long line (p. 55, **[5]**) is the limit of a well-ordered monotone inverse system of arcs.

REFERENCES

1. C. E. Capel, *Inverse limit spaces*, Duke Math. J., **21** (1954), 233–246.
2. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, N.J., 1952.
3. M. K. Fort, Jr. and J. Segal, *Local connectedness of inverse limit spaces*, Duke Math. J., **28** (1961), 253–260.
4. G. R. Gordh, Jr., *Monotone decompositions of irreducible Hausdorff continua*, Pacific J. Math., **36** (1971), 647–658.
5. J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961.
6. S. Mardešic, *Chainable continua and inverse limits*, Glasnik Mat. Fiz. Astr., **14** (1959), 219–232.
7. ———, *On covering dimension and inverse limits of compact spaces*, Illinois J. Math., **4** (1960), 278–291.
8. ———, *Locally connected, ordered and chainable continua*, Rad Jugoslav. Akad. Znan. Umjetn., **319** (1960), 147–166.
9. L. E. Ward, Jr., *Mobs, trees and fixed points*, Proc. Amer. Math. Soc., **8** (1957), 798–804.
10. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications 28, Providence, 1942.
11. R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications 32, Providence, 1949.

Received February 25, 1974. During this research the first author was visiting the University of Zagreb on an exchange program sponsored jointly by the National Academy of Sciences (U.S.A.) and the Yugoslav Academy of Sciences and Arts.

UNIVERSITY OF ZAGREB, ZAGREB, YUGOSLAVIA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $ 72.00 a year (6 Vols., 12 issues). Special rate: $ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Pacific Journal of Mathematics

Vol. 58, No. 2 April, 1975