

Pacific Journal of Mathematics

**FIELDS OF AUTOMORPHISMS AND DERIVATIONS OF
 C^* -ALGEBRAS**

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Let A be a C^* -algebra with identity; the group of star automorphisms of A leaving its center elementwise fixed is represented as the set of all global sections of a field of groups whose base space is the spectrum of the center of A and whose stalks are groups of automorphisms of a family of quotient C^* -algebras. The derivations of A are similarly represented in a field of Lie algebras.

Introduction. The representation of C^* -algebras as sections in bundles was used by Tomiyama and Takesaki [7] and others to study the automorphisms of these algebras. On the other hand, Teleman [6] applied representations of W^* -algebras as sections in a sheaf to obtain information about their Lie algebra of derivations. For biregular rings a description of the group of automorphisms has also been achieved by Dauns and Hofmann in sheaf theoretical terms [1]. An analytic version of some of these results is presented here via the representation of a C^* -algebra as sections in a uniform field.

Given a C^* -algebra A with identity, there is a bundle or sheaf-like structure called field and denoted by (E, π, T) , whose ingredients are: The maximal ideal space T of the center of A equipped with the hull-kernel topology and a disjoint union E of a family of quotient C^* -algebras $\{A/I_t \mid t \in T\}$ topologized in such a way that the projection $\pi: E \rightarrow T$ and the cross sections $\hat{a}: t \mapsto a + I_t: T \rightarrow E$ are continuous for every $a \in A$. The existence of such a topology depends on the upper semicontinuity of the functions $t \mapsto \|a + I_t\|: T \rightarrow \mathbb{R}_+$. If the primitive ideal space $\text{Prim } A$ is Hausdorff, T is homeomorphic to $\text{Prim } A$ and for the stalk above $P \in \text{Prim } A$, one can take the simple algebra A/P . More generally, T is the Stone-Ćech compactification of $\text{Prim } A$ and I_t is the closed two sided ideal generated by t . In [3] 4.7 Halpern showed that if A is a W^* -algebra, I_t is a primitive ideal of A for every $t \in T$, and that $\{I_t \mid t \in T\}$ (which is homeomorphic to T) is dense in $\text{Prim } A$. The set of all continuous bounded global sections of this field is denoted by $\Gamma(\pi)$, and is naturally endowed with a C^* -algebra structure with respect to point-wise defined operations and the norm given by $\|\sigma\| = \{\|\sigma(t)\| \mid t \in T\}$. The non commutative version of the Gelfand Naimark theorem ([2], Chap. III, 8.14) says that A is isometrically star isomorphic to $\Gamma(\pi)$. One can also decompose A over the spectrum

of any closed subalgebra of the center of A containing the identity [8].

The duality between C^* -algebras and fields of C^* -algebras [8] indicates that all the information relative to the group of automorphisms of a C^* -algebra A can be found in the canonical field (E, π, T) associated to A .

A description of the group N of star automorphisms of A leaving the center elementwise fixed is given in terms of the groups of automorphisms of the fibers of (E, π, T) ; to this effect a field of groups is constructed, and N is realized as the group of all global sections. The derivations of A are similarly described in terms of the derivations of the stalks of the field (E, π, T) . This paper is part of the author's dissertation at Tulane University. I would like to take this opportunity of expressing my sincere gratitude to professors J. Dauns and K. H. Hofmann my research supervisors, for their advice and encouragement.

1. Terminology. For the foundation of the theory of sectional representation we refer to [2,4]. A few definitions follow to establish notation and terminology.

DEFINITION 1.1. Let E and T be topological spaces and $\pi: E \rightarrow T$ a surjective continuous map, then (E, π, T) is called a *field or bundle of topological spaces*, T is the *base space* and $\pi^{-1}(t)$ the *stalk or fiber above t* .

DEFINITION 1.2. A *local selection* is a map $\sigma: Q \rightarrow E$ whose domain Q is an open subset of T and $\pi(\sigma(t)) = t$ for every $t \in Q$. If $Q = T$, then σ is a *global selection*. A continuous selection is called *section*. The set of all global sections is denoted by $\Gamma(\pi)$.

DEFINITION 1.3. Let (E, π, T) be a field of topological spaces and \mathcal{U} a filter on $E \vee E = \{(x, y) \in E \times E \mid \pi(x) = \pi(y)\}$ such that the filter $[\mathcal{U}]$ generated by \mathcal{U} on $E \times E$ is a uniform structure. (E, π, T) is a *uniform field* if the neighborhood filter of every point $x \in E$ has a basis given by the sets $U(\sigma) = \{y \in E \mid \pi(y) \in \text{dom } \sigma \text{ and } (\sigma(\pi(y)), y) \in U\}$ where σ is a local section such that $\sigma(\pi(x)) = x$, $\text{dom } \sigma$ is the domain of σ and U is an element of the uniformity \mathcal{U} .

DEFINITION 1.4. Let (E, π, T) be a uniform field whose stalks are normed spaces and whose uniformity is given by the fundamental entourages $U = \{(x, y) \in E \vee E \mid \|x - y\| < \varepsilon\}$ with $\varepsilon > 0$, then (E, π, T) is a *field or bundle of normed spaces* if the following functions

are continuous:

$$(x, y) \longmapsto x + y: E \vee E \longrightarrow E \text{ and } (\alpha, x) \longmapsto \alpha x: C \times E \longrightarrow E.$$

If the stalks of a field of normed spaces are C^* -algebras and if the functions $(x, y) \mapsto xy: E \vee E \rightarrow E$ and $x \mapsto x^*: E \rightarrow E$ are continuous, then (E, π, T) is a *field or bundle of C^* -algebras*. A field of C^* -algebras such that for every $t \in T$ the fiber E_t above t has an identity $\zeta(t)$ is a *field of C^* -algebras with identity* provided the selection $t \mapsto \zeta(t): T \rightarrow E$ is continuous. The set of all bounded global sections of (E, π, T) is then a C^* -algebra with respect to pointwise defined operations, by taking the sup-norm $\|\sigma\| = \sup\{\|\sigma(t)\| \mid t \in T\}$ and is denoted by $\Gamma^b(\pi)$.

DEFINITION 1.5. A set Σ of local (resp. global) selections of a field of topological spaces is called *full*, if for every $x \in E$ there exists $\tau \in \Sigma$ such that $\tau(\pi(x)) = x$.

DEFINITION 1.6. Let σ be a local section in a field of normed spaces (E, π, T) , then $\{x \in E \mid \pi(x) \in \text{dom } \sigma \text{ and } \|x - \sigma(\pi(x))\| < \varepsilon\}$ is called the ε -*tube around* σ and is denoted by $\mathcal{T}(\sigma, \varepsilon)$.

LEMMA 1.7. Let (E, π, T) be a field of normed spaces. The following two statements are equivalent:

(1) For every local section σ , the function $t \mapsto \|\sigma(t)\|: \text{dom } \sigma \rightarrow R_+$ is upper semicontinuous.

(2) For every local section σ and every $\varepsilon > 0$, the ε -tube around σ , $\mathcal{T}(\sigma, \varepsilon)$ is an open set in E .

Proof. If the function $t \mapsto \|\sigma(t)\|$ is upper semicontinuous, then for every $\varepsilon > 0$, the set $\{t \in \text{dom } \sigma \mid \|\sigma(t)\| < \varepsilon\}$ is open in T . Let $x \in E$ such that $\|\sigma(\pi(x)) - x\| < \varepsilon$, then there is a δ , $0 < \delta < \varepsilon$ such that $\|\sigma(\pi(x)) - x\| < \varepsilon - \delta$. If τ is a local section for which $\tau(\pi(x)) = x$, the $Q = \{t \in T \mid \|\sigma(t) - \tau(t)\| < \varepsilon - \delta\}$ is an open neighborhood of $\pi(x)$. The restriction τ_Q of τ to Q is then a local section such that $\tau_Q(\pi(x)) = x$ and $\text{Im } \tau_Q$ is contained in $\mathcal{T}(\sigma, \varepsilon - \delta)$. Therefore $\mathcal{T}(\tau_Q, \delta)$ is a neighborhood of x and $\mathcal{T}(\tau_Q, \delta) \subset \mathcal{T}(\sigma, \varepsilon)$. Then $\mathcal{T}(\sigma, \varepsilon)$ is open in E .

The converse follows from [2], Chapter I, 1.30.

REMARK 1.8. If T is assumed to be uniformizable, the above lemma can be formulated in term of global sections, since the existence of a full set of local sections implies the existence of a full set of global sections, indeed if τ is a local section through

$x \in E$ and $f: T \rightarrow [0, 1]$ is continuous function such that $f(\pi(x)) = 1$ and $f(s) = 0$ for $s \in T \setminus \text{dom } \sigma$, then σ defined by $\sigma(s) = f(s)\tau(s)$ if $s \in \text{dom } \tau$ and $\sigma(s) = 0$ if $s \in T \setminus \text{dom } \tau$, is a global section through x .

REMARK 1.9. Let (E, π, T) be a field of normed spaces over a quasicompact space T . If the function $t \mapsto \|\sigma(t)\|: T \rightarrow \mathbf{R}_+$ is upper semicontinuous for every global section σ , then σ is bounded.

Proof. The semicontinuity of $t \mapsto \|\sigma(t)\|$ for $\sigma \in \Gamma(\pi)$ implies that the sets $\{t \in T \mid \|\sigma(t)\| < n\}$, $n = 1, 2, \dots$ form an open covering of T and since T is quasicompact, then σ is bounded.

CONVENTION 1.10. From now on we will assume that $t \mapsto \|\sigma(t)\|$ is upper semicontinuous for every global section in a field (E, π, T) of normed spaces, unless otherwise specified.

2. Fields of linear maps.

2.1 Let (E, π, T) be a field of normed spaces. Denote by \mathcal{B}_t the algebra of all bounded linear transformations of $E_t = \pi^{-1}(t)$. Let \mathcal{D}_0 be the disjoint union of the family $\{\mathcal{B}_t \mid t \in T\}$ and $\rho: \mathcal{D}_0 \rightarrow T$ defined by $\rho(u) = t$ if $u \in \mathcal{B}_t$. Let Σ be a normed space consisting of a full set of bounded global sections of (E, π, T) .

Our aim is to define a uniform field (\mathcal{D}, ρ, T) over the same topological space T , with $\mathcal{D} \subset \mathcal{D}_0$ appropriately chosen. Later in § 3, we will consider a subfield of (\mathcal{D}, ρ, T) whose stalks are groups of isometries, and in § 6 another subfield whose stalks are Lie algebras of derivations will be studied.

2.2. We endow \mathcal{D}_0 with the coarsest topology making continuous the functions

$$\begin{aligned} u &\longmapsto \rho(u): \mathcal{D}_0 \longrightarrow T \quad \text{and} \\ u &\longmapsto u(\sigma(\rho(u))): \mathcal{D}_0 \longrightarrow E \quad \text{for every } \sigma \in \Sigma. \end{aligned}$$

2.3. Let u be an arbitrary element of \mathcal{D}_0 , denote by Q an open neighborhood of $t = \rho(u)$, choose sections σ and τ in Σ such that $u\sigma(t) = \tau(t)$ and take $\varepsilon > 0$.

Define $M(u) = M(u, \sigma, \tau, Q, \varepsilon)$ as follows

$$M(u) = \{v \in \mathcal{D}_0 \mid \rho(v) \in Q \text{ and } v(\sigma(\rho(v))) \in \mathcal{I}(\tau, \varepsilon)\}.$$

Then the filter of neighborhoods of u is generated by the sets of the form $M(u)$.

2.4. For a local selection λ in (\mathcal{D}_0, ρ, T) the following two statements are equivalent:

(1) $t \mapsto \lambda(t): \text{dom } \lambda \rightarrow \mathcal{D}_0$ is a local section.

(2) $t \mapsto \lambda(t)\sigma(t): \text{dom } \lambda \rightarrow E$ is a local section in the field (E, π, T) for every $\sigma \in \Sigma$.

2.5. Let \mathcal{D} be the subset of \mathcal{D}_0 consisting of the points through which there is a global section. Let σ be an element of Σ and $\varepsilon > 0$.

Set $\mathcal{D} \vee \mathcal{D} = \{(u, v) \in \mathcal{D} \times \mathcal{D} \mid \rho(u) = \rho(v)\}$ and define $U(\sigma, \varepsilon)$ as follows: $U(\sigma, \varepsilon) = \{(u, v) \in \mathcal{D} \vee \mathcal{D} \mid \|u\sigma(\rho(u)) - v\sigma(\rho(v))\| < \varepsilon\}$. It is immediate that the finite intersections of sets of the form $U(\sigma, \varepsilon)$ are a fundamental system of entourages of a uniformity \mathcal{U} for ρ .

REMARK. One should observe, that \mathcal{D} may consist of the identity section only

2.6. The uniformity \mathcal{U} for ρ induces the topology defined in \mathcal{D} and hence (\mathcal{D}, ρ, T) is a uniform field.

Proof. Let u be an element of \mathcal{D} and ξ a global section in (\mathcal{D}, ρ, T) such that $\xi(\rho(u)) = u$, denote by ξ_Q the restriction of ξ to an open neighborhood Q of $\rho(u)$, then

$$U(\sigma, \varepsilon)(\xi_Q) = \{v \in \mathcal{D} \mid \rho(v) \in Q \text{ and } \|\xi(\rho(v))\sigma(\rho(v)) - v\sigma(\rho(v))\| < \varepsilon\}$$

is a neighborhood of u in \mathcal{D} because by 2.4

$$t \longmapsto \xi(t)\sigma(t): T \longrightarrow E \text{ is a global section in } (E, \pi, T).$$

Conversely, given a neighborhood of u of the form $M(u, \sigma, \tau, Q, \varepsilon)$, let λ be a global section in (\mathcal{D}, ρ, T) such that $\lambda(\rho(u)) = u$. If we set $t = \rho(u)$, then $u\sigma(t) = \tau(t)$ and $\lambda(t)\sigma(t) = \tau(t)$, hence there is a neighborhood P of t contained in Q such that for every s in P , $\|\lambda(s)\sigma(s) - \tau(s)\| < \varepsilon/2$. If λ_P is the restriction of λ to P , then

$$U(\sigma, \varepsilon/2)(\lambda_P) = \{v \in \mathcal{D} \mid \rho(v) \in P \text{ \& } \|\lambda(\rho(v))\sigma(\rho(v)) - v\sigma(\rho(v))\| < \varepsilon/2\}$$

is contained in $M(u, \sigma, \tau, Q, \varepsilon)$. Thus the topology induced by the uniformity coincides with the topology defined in \mathcal{D} .

3. Field of automorphisms.

DEFINITION 3.1. Let (G, p, T) be a uniform field such that the following conditions are satisfied.

(a) All the stalks are groups and the functions $(u, v) \mapsto uv$:

$G \vee G \mapsto G$ and $u \mapsto u^{-1}: G \rightarrow G$ are continuous, where $G \vee G = \{(u, v) \in G \times G \mid p(u) = p(v)\}$.

(b) The uniform structure induced by the field uniformity \mathcal{U} is coarser than the left (resp. right) uniform structure of the stalk induced by (a).

(c) For every entourage $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that

$$V \subset \{(u, v) \in G \vee G \mid (\zeta(p(u)), u^{-1}v) \in U\}$$

(resp. $V \subset \{(u, v) \in G \vee G \mid (\zeta(p(u)), vu^{-1}) \in U\}$), where $\zeta(t)$ is the unit elements of the group $G_t = p^{-1}(t)$.

(d) For every entourage $U \in \mathcal{U}$ and every global section λ there exists an entourage $V \in \mathcal{U}$ such that $(\lambda(t)u\lambda(t)^{-1}, \lambda(t)v\lambda(t)^{-1}) \in U$ if $(u, v) \in V$ and $t = p(u) = p(v)$.

Then (G, p, T) is called a *left* (resp. *right*) *uniform field of topological groups*.

The set $\Gamma(p)$ of all global sections is a group with respect to pointwise defined operations and is naturally equipped with a uniform structure $\tilde{\mathcal{U}}$ with fundamental entourages of the form $\tilde{U} = \{(\lambda, \mu) \in \Gamma(p) \times \Gamma(p) \mid (\forall t \in T)(\lambda(t), \mu(t)) \in U\}$ with $U \in \mathcal{U}$. The neighborhood filter of the identity ζ of $\Gamma(p)$ for the topology determined by $\tilde{\mathcal{U}}$ is given by the sets

$$\tilde{U}(\zeta) = \{\lambda \in \Gamma(p) \mid (\zeta, \lambda) \in \tilde{U}\}.$$

REMARK 3.2. Conditions (b) and (c) of Definition 3.1 imply that the field uniformity \mathcal{U} of (G, p, T) induces on each stalk the left (resp. right) uniform structure of the stalk induced by (a).

REMARK 3.3. The following condition implies the conditions (b) and (c) of Definition 3.1.

(b') A fundamental system of entourages for the field uniformity \mathcal{U} of (G, p, T) is given by the sets of the form

$$\{(u, v) \in G \vee G \mid u^{-1}v \in W(\zeta)\} \text{ (resp. } \{(u, v) \in G \vee G \mid vu^{-1} \in W(\zeta)\} \text{)}.$$

where ζ is the unit element of $\Gamma(p)$ and W ranges through \mathcal{U} .

3.4. We recall that if N is a group and \mathcal{S} is a filter on N satisfying the Properties I and II listed below, then there is a unique topology on N with respect to which N is a topological group having \mathcal{S} as filter of neighborhoods of the identity.

(I) Given $U \in \mathcal{S}$ there exists $V \in \mathcal{S}$ such that $V^{-1}V \subset U$.

(II) For all $v \in N$ and $V \in \mathcal{S}$ we have $vVv^{-1} \in \mathcal{S}$.

3.5. If (G, p, T) is a left uniform field of topological groups, then $\Gamma(p)$ is a topological group.

Proof. (I) Given a neighborhood of ζ of the form $\tilde{U}(\zeta)$ by 3.1. (c) we can find $V \in \mathcal{U}$ such that $V \subset \{(u, v) \mid u^{-1}v \in U(\zeta)\}$.

Let W a symmetric entourage such that $W \circ W \subset V$, then $\lambda \in \tilde{W}(\zeta)$ and $\mu \in \tilde{W}(\zeta)$ imply $(\lambda, \mu) \in \tilde{V}$ and so $\lambda^{-1}\mu \in \tilde{U}(\zeta)$.

(II) Given $U \in \mathcal{U}$ and $\lambda \in \Gamma(p)$ there exists $V \in \mathcal{U}$ such that $\lambda \tilde{V}(\zeta) \lambda^{-1} \subset \tilde{U}(\lambda)$: Take the entourage V corresponding to U and λ condition (d) of Definition 3.1, then $\mu \in \tilde{V}(\zeta)$ implies $\lambda \mu \lambda^{-1} \in \tilde{U}(\zeta)$. Hence $\Gamma(p)$ is a topological group whose left uniform structure is coarser than $\tilde{\mathcal{U}}$.

EXAMPLE 3.6. Let T the discrete space $N \setminus \{0\}$. Endow \mathbf{Q} with the p -adic topology and $G = T \times \mathbf{Q}$ with the product topology. Take as uniformity \mathcal{U} for $\pi: G \rightarrow T$ the one given by the fundamental entourages

$$U(n) = \{(x, y) \in G \mid h_n(\pi(x), \sup(|x|_p, |y|_p)) \mid x - y|_p < 1/n\}$$

for $n = 1, 2, \dots$, where $h_n: T \times \mathbf{R}_+ \rightarrow \{0, 1\}$ is the function defined by

$$h_n(t, r) = \begin{cases} 1 & \text{if } t \leq n \\ 0 & \text{if } t > n \text{ and } r < 1 + 1/n \\ 1 & \text{if } t > n \text{ and } r \geq 1 + 1/n. \end{cases}$$

Then (G, p, T) is a uniform field of topological groups, and the left uniform structure of $\Gamma(\pi)$ is strictly coarser than $\tilde{\mathcal{U}}$.

3.7. Let A be an involutive normed algebra, a bijection $\lambda: A \rightarrow A$ is a *star automorphism* of A provided $\lambda(x + y) = \lambda x + \lambda y$, $\lambda(xy) = (\lambda x)(\lambda y)$, $\lambda(\alpha x) = \alpha \lambda x$ and $\lambda x^* = (\lambda x)^*$ for every $x, y \in A$ and $\alpha \in \mathbf{C}$. If $\|\lambda x\| = \|x\|$ for every $x \in A$, λ is called an *isometric star automorphism* of A .

3.8. Let (E, π, T) be a field of involutive normed algebras. Denote by $\text{Aut } E_t$ the group of isometric star automorphisms of $E_t = \pi^{-1}(t)$ and let G_0 be the disjoint union of all the groups $\text{Aut } E_t$. Let \mathcal{S} be an involutive normed algebra consisting of a full set of bounded global sections of (E, π, T) and consider the uniform field (\mathcal{S}, ρ, T) defined in § 2.

If $G = \mathcal{S} \cap G_0$ is furnished with the topology and uniform structure induced by the field topology and field uniformity of \mathcal{S} and if p is the restriction of $\rho: \mathcal{S} \rightarrow T$ to G , then (G, p, T) is also

a uniform field.

3.9. Suppose that Σ is the algebra $\Gamma^b(\pi)$ of all bounded global sections of (E, π, T) . If λ and μ are global sections in (G, p, T) , then $u \mapsto (\lambda\mu)(t) = \lambda(t)\mu(t)$ is also a global section in (G, p, T) .

Proof. For every $\sigma \in \Sigma$, $t \mapsto \mu(t)\sigma(t)$ belongs to Σ by 2.2 and then $t \mapsto \lambda(t)(\mu(t)\sigma(t))$ is also in Σ , then invoking 2.2 again, $\lambda\mu$ is a global section in (G, p, T) .

3.10. The function $u \mapsto u^{-1}: G \rightarrow G$ is continuous when G is furnished with its field topology.

Proof. Let u be an element of G and set $t = p(u)$. Given a neighborhood of u^{-1} in G of the form $M(u^{-1}, \sigma, \tau, Q, \varepsilon)$ where $\sigma, \tau \in \Sigma$ and $u^{-1}\sigma(t) = \tau(t)$, then $u\tau(t) = \sigma(t)$ and so $M(u, \tau, \sigma, Q, \varepsilon)$ is a neighborhood of u in G . If $v \in M(u, \tau, \sigma, Q, \varepsilon)$, then $p(v^{-1}) = p(v)$ belongs to Q and $\|v\tau(p(v)) - \sigma(p(v))\| = \|v^{-1}\sigma(p(v)) - \tau(p(v))\| < \varepsilon$, hence $v^{-1} \in M(u^{-1}, \sigma, \tau, Q, \varepsilon)$, therefore $u \mapsto u^{-1}: G \rightarrow G$ is continuous.

3.11. The function $(u, v) \mapsto uv: G \vee G \rightarrow G$ is continuous if $\Sigma = \Gamma^b(\pi)$.

Proof. Given an element (u_0, v_0) of $G \vee G$, choose any neighborhood $U(\sigma, \varepsilon)(\gamma)$ and set $t_0 = p(u_0v_0)$. Without loss of generality γ can be taken of the form $\gamma = \xi\eta$ where ξ and η are local sections in (G, p, T) such that $\xi(p(u_0)) = u_0$ and $\eta(p(v_0)) = v_0$. We can consider η as the restriction of a global section $\bar{\eta}$ to an open subset of T . Since σ belongs to Σ , then $\bar{\eta}\sigma$ belongs to Σ too, and if u is in $U(\bar{\eta}\sigma, \varepsilon/2)(\xi)$ and v is in $U(\sigma, \varepsilon/2)(\eta)$, then uv is in $U(\sigma, \varepsilon)(\gamma)$.

3.12. Condition (b') of 3.3 is satisfied by (G, p, T) , that is, a fundamental system of entourages for the field uniformity \mathcal{U} of (G, p, T) is given by the sets of the form $\{(u, v) \in G \vee G \mid u^{-1}v \in W(\zeta)\}$ where ζ is the unit element of $\Gamma(p)$ and W range through \mathcal{U} .

Proof. For a fundamental entourage of \mathcal{U} one has

$$\begin{aligned} U(\sigma, \varepsilon) &= \{(u, v) \in G \vee G \mid \|u\sigma(p(u)) - v\sigma(p(v))\| < \varepsilon\} \\ &= \{(u, v) \in G \vee G \mid \|\sigma(p(u)) - u^{-1}v\sigma(p(v))\| < \varepsilon\} \\ &= \{(u, v) \in G \vee G \mid u^{-1}v \in U(\sigma, \varepsilon)(\zeta)\} . \end{aligned}$$

PROPOSITION 3.13. If $\Sigma = \Gamma^b(\pi)$, then (G, p, T) is a left uniform

field of topological groups and hence $\Gamma(p)$ is a topological group equipped with the standard topology derived from (G, p, T) .

Proof. Condition (a) of Definition 3.1 was proved in 3.10 and 3.11, condition (b') of Remark 3.3 was proved in 3.12; we are going to verify condition (d) of Definition 3.1.

Given an entourage of the form $U(\sigma, \varepsilon)$ and a global section λ of (G, p, T) , we take $V = U(\lambda^{-1}\sigma, \varepsilon)$, then $(u, v) \in V$ implies $(\lambda(t)u\lambda(t)^{-1}, \lambda(t)v\lambda(t)^{-1}) \in U(\sigma, \varepsilon)$, with $t = p(u) = p(v)$. Hence (G, p, T) is a left uniform field of topological groups.

4. Automorphisms leaving center fixed.

4.1. Let (E, π, T) be a field of involutive normed algebras and $\Sigma = \Gamma^b(\pi)$, then an isometric star automorphism λ of Σ is $C^b(T)$ -linear if and only if it leaves setwise invariant the ideals of the form $I_t = \{\sigma \in \Sigma \mid \sigma(t) = 0\}$, i.e. $\lambda I_t \subset I_t$ for each $t \in T$. In particular, $\lambda I_t \subset I_t$ implies $\lambda^{-1} I_t \subset I_t$ and hence $\lambda I_t = I_t$.

Proof. This is a direct consequence of [8] 2.3.

THEOREM 4.2. *Let (E, π, T) be a field of involutive normed algebras and $\Sigma = \Gamma^b(\pi)$. The group N of isometric star automorphisms of Σ leaving setwise invariant the ideals of the form $I_t = \{\sigma \in \Sigma \mid \sigma(t) = 0\}$ is isomorphic to the group $\Gamma(p)$ of all global sections of the field of isometries (G, p, T) associated with (E, π, T) .*

Proof. Let $\lambda \in N$ and $t \in T$. Define λ_t as follows $x \mapsto \lambda_t(x) = (\lambda\sigma_x)(t)$: $E_t \rightarrow E_t$ where $\sigma_x \in \Sigma$ and $\sigma_x(t) = x$. The definition of $\lambda_t(x)$ is clearly independent of the particular σ_x taken through x , and λ_t is a star automorphism of E_t whose inverse is $(\lambda^{-1})_t$. Furthermore, by [8] 2.3, $\|\lambda_t(x)\| = \|(\lambda\sigma_x)(t)\| \leq \|\lambda\| \|\sigma_x(t)\| = \|\sigma_x(t)\| = \|x\|$ and $\|\lambda_t^{-1}(x)\| = \|(\lambda^{-1}\sigma_x)(t)\| \leq \|\lambda^{-1}\| \|\sigma_x(t)\| = \|\sigma_x(t)\|$, if $x = \sigma(t)$ for each $x \in E_t = \pi^{-1}(t)$, then $\|x\| = \|\lambda_t^{-1}(\lambda_t(x))\| \leq \|\lambda_t(x)\|$, hence λ_t is an isometry of E_t .

Consider the selection $\varphi(\lambda)$ of (G, p, T) given by $t \mapsto \lambda_t$: $T \rightarrow G$. For every $\sigma \in \Sigma$ the selection $t \mapsto \lambda_t(\sigma(t))$: $T \rightarrow E$ is continuous, hence by 2.4 $\varphi(\lambda)$ is a global section. Thus φ is a group homomorphism from N into $\Gamma(p)$. On the other hand, given a global section η in the field of automorphisms (G, p, T) and a section $\sigma \in \Sigma$, the map $t \mapsto \eta(t)\sigma(t)$: $T \rightarrow E$ is a section $\eta\sigma \in \Sigma$, and since $\|(\eta\sigma)(t)\| = \|\sigma(t)\|$ for every $t \in T$, then $\alpha(\eta)$: $\sigma \mapsto \eta\sigma$ is an isometric star automorphism of Σ leaving setwise invariant the ideals $\{I_t \mid t \in T\}$ and α is a group

homomorphism whose inverse is φ , indeed, for every $\lambda \in N$, $\sigma \in \Sigma$ and $t \in T$, $(\alpha(\varphi(\lambda))\sigma)(t) = \varphi(\lambda)(t)\sigma(t) = (\lambda\sigma)(t)$, then $\alpha(\varphi(\lambda)) = \lambda$, that is, $\alpha \circ \varphi = \text{id}_N$. On the other hand, for every $\eta \in \Gamma(p)$, every $t \in T$ and every $x \in E_t$, if $\sigma \in \Sigma$ and $\sigma(t) = x$, then $\varphi(\alpha(\eta))(t)x = \varphi(\alpha(\eta))(t)\sigma(t) = (\alpha(\eta)\sigma)(t) = \eta(t)\sigma(t) = \eta(t)x$, then $\alpha(\varphi(\eta))(t) = \eta(t)$, therefore $\varphi(\alpha(\eta)) = \eta$, that is, $\varphi \circ \alpha = \text{id}_{\Gamma(p)}$.

REMARK 4.3. Consider the group homomorphism $\varepsilon_t: \eta \mapsto \eta(t): \Gamma(p) \rightarrow \text{Aut } \pi^{-1}(t)$, then $\text{Im } \varepsilon_t = p^{-1}(t)$ because G was defined as the subset of G_0 consisting of the points u such that there is a global section $\eta: T \rightarrow G_0$ with $\eta(p(u)) = u$. But

$$\begin{aligned} \ker \varepsilon_t &= \{\eta \in \Gamma(p) \mid (\forall \sigma \in \Sigma)(\eta(t)\sigma(t) = \sigma(t))\} \\ &= \{\eta \in \Gamma(p) \mid (\forall \sigma \in \Sigma)(\alpha(\eta)\sigma - \sigma \in I_t)\}, \end{aligned}$$

then $p^{-1}(t)$ is isomorphic to the quotient group N/R_t where $R_t = \{\lambda \in N \mid (\forall \sigma \in \Sigma)(\lambda\sigma - \sigma \in I_t)\}$.

4.4. Let A be a C^* -algebra with identity and B a sub- C^* -algebra of A contained in its center and containing the identity, then the group N of star automorphisms of A leaving setwise invariant the closed two sided ideals generated by a maximal ideal of B is equal to the group of star automorphisms of A leaving B elementwise fixed.

Proof. This follows from 4.1 and [8], Theorem 3.5.

DEFINITION 4.5. Let (E, π, T) be a field of involutive normed algebras, an automorphism (φ, A) of (E, π, T) ([8], 4.3) is called *isometric star automorphism* if φ is the identity map of T and $A: E \rightarrow E$ is a homeomorphism such that $\pi \circ A = \pi$ and $x \mapsto Ax: E_t \rightarrow E_t$ is an isometric star automorphism.

4.6. Let (E, π, T) be a field of involutive normed algebras and $\Sigma = \Gamma^b(\pi)$, then the group N of isometric star automorphisms of Σ leaving setwise invariant the ideals of the form $I_t = \{\sigma \in \Sigma \mid \sigma(t) = 0\}$ is anti-isomorphic to the group $\text{Isom}(E, \pi, T)$ of isometric star automorphisms of the field (E, π, T) .

Proof. Let ξ be an element of N . Define $\theta(\xi)$ by $x \mapsto \theta(\xi)x = (\xi\sigma)(\pi(x)): E \rightarrow E$ with $\sigma \in \Sigma$ such that $\sigma(\pi(x)) = x$. By [8] 2.3, $\|\theta(\xi)x\| = \|(\xi\sigma)(\pi(x))\| \leq \|\xi\| \|\sigma(\pi(x))\| = \|x\|$.

Similarly $\|\theta(\xi^{-1})x\| \leq \|x\|$ for every $x \in E$ and since $\theta(\xi^{-1}) = \theta(\xi)^{-1}$, then $\|x\| = \|\theta(\xi^{-1})\theta(\xi)x\| \leq \|\theta(\xi)x\|$, that is $\theta(\xi)$ is an isometric

star automorphism in each stalk for every $\xi \in N$.

Thus $(1_T, \theta(\xi))$ belongs to $\text{Isom}(E, \pi, T)$.

Conversely, if $(1_T, A)$ is an isometric star automorphism of (E, π, T) and σ is an element of Σ , then $A \circ \sigma$ is also an element of Σ . Define $\gamma(A)$ as follows $\sigma \mapsto \gamma(A)\sigma = A \circ \sigma: \Sigma \rightarrow \Sigma$, this is an isometric star automorphism of Σ and leaves the ideals $\{I_t \mid t \in T\}$ invariant. Thus $\gamma: (1_T, A) \mapsto \gamma(A)$ is a group homomorphism from $\text{Isom}(E, \pi, T)$ into the opposite group of N . Furthermore, for every $x \in E$, if $\sigma \in \Sigma$ and $\sigma(\pi(x)) = x$, then $\theta(\gamma(A))x = (\gamma(A)\sigma)(\pi(x)) = (A \circ \sigma)(\pi(x)) = A(\sigma(\pi(x))) = Ax$, hence $\theta(\gamma(A)) = A$ for every $(1_T, A) \in \text{Isom}(E, \pi, T)$.

Similarly for every $t \in T$, $\sigma \in \Sigma$ and $\xi \in N$, $\gamma(\theta(\xi)\sigma)(t) = (\theta(\xi) \circ \sigma)(t) = \theta(\xi)(\sigma(t)) = (\xi\sigma)(t)$, then $\gamma(\theta(\xi)) = \xi$ for every $\xi \in N$. Thus γ is an anti-isomorphism between $\text{Isom}(E, \pi, T)$ and N whose inverse is θ .

4.7. For a C^* -algebra with identity, we collect the preceding results in the following

THEOREM. *Let A be a C^* -algebra with identity and $\text{Aut } A$ its group of star automorphisms. There is an exact sequence*

$$1 \longrightarrow N \xrightarrow{j} \text{Aut } A \xrightarrow{k} H$$

where H is the group of homeomorphisms of the center Z of A , and k assigns to a star automorphism of A the homeomorphism of T associated with the inverse of its restriction to Z .

For N one can take of the following groups:

The group N_1 of star automorphisms of A leaving the center elementwise fixed.

The group N_2 of star automorphisms of A leaving setwise invariant the closed two sided ideals generated by a maximal ideal of the center Z of A .

The groups N_3 of all global sections of the field (G, p, T) obtained from the canonical field (E, π, T) , over the spectrum T of Z , associated A .

The group N_4 opposite to the group of star automorphisms of the canonical field (E, π, T) of the form $(1_T, A)$ where 1_T is the identity map of the spectrum T of the center of A .

Proof. This follows from 4.2, 4.4, 4.6 and [8] Theorem 3.5.

4.8. Let (E, π, T) be a field of C^* -algebras with identity, $\text{Aut}(E, \pi, T)$ the group of star automorphisms of (E, π, T) and

$\text{Aut}_T(E, \pi, T)$ its group of star automorphisms of the form $(1_T, A)$, then we have the exact sequence

$$\text{Aut}_T(E, \pi, T) \xrightarrow{\chi} \text{Aut}(E, \pi, T) \xrightarrow{\kappa} H$$

where χ is the inclusion map, H is the group of homeomorphisms of T and $\kappa(\angle, A) = \angle$.

Furthermore, if (E, π, T) is a central field, that is, if the center of $\Gamma(\pi)$ is $C(T)$, then $\text{Aut}(E, \pi, T)$ is anti-isomorphic to $\text{Aut } \Gamma(\pi)$.

Proof. Every star automorphism of $\Gamma(\pi)$ leaves its center invariant, then if (E, π, T) is central it determines a star automorphism of the pair $(C(T), \Gamma(\pi))$ and hence a star automorphism of (E, π, T) , conversely every star automorphism of (E, π, T) determines a star automorphism of $(C(T), \Gamma(\pi))$ and hence a star automorphism of $\Gamma(\pi)$ ([8], 4.6).

5. Fields of derivations.

DEFINITION 5.1. Let A be a complex algebra and $\delta: A \rightarrow A$ be a linear map, δ is a *derivation* of A provided $\delta(xy) = \delta(x)y + x\delta(y)$ for every $x, y \in A$. Given $a \in A$, the map $x \mapsto ax - xa$ is an *inner derivation*. If δ and ρ are derivations of A and z is a complex number, then $x \mapsto \delta(x) + \rho(x)$, $x \mapsto z\delta(x)$ and $x \mapsto \delta(\rho(x)) - \rho(\delta(x))$ are derivations of A denoted by $\delta + \rho$, $z\delta$ and $[\delta, \rho]$ respectively. The set J of all derivations of A is a Lie Algebra with respect to these operations. In particular the bracket product is a bilinear map such that $[\lambda, \mu] = -[\mu, \lambda]$ and $[[\lambda, \mu], \nu] + [[\mu, \nu], \lambda] + [[\nu, \lambda], \mu] = 0$. If δ is a derivation of a C^* -algebra, then δ is norm continuous and its restriction to the center of A is identically zero.

5.2. If A is a C^* -algebra and I is a closed two sided ideal of A , then $\delta(I) \subset I$ for every derivation δ of A ([5], 2.2).

Proof. $I^2 = I$, where I^2 denotes the linear subspace generated by the set of all products xy , with $x \in I$ and $y \in I$. Since $\delta(xy) = x\delta(y) + \delta(x)y$ belongs to I , for every $x, y \in I$, then $\delta(I) \subset I$.

5.3. Let (E, π, T) be a field of normed algebras. Denote by $\text{Der } E_t$ the Lie algebra of all derivations of $E_t = \pi^{-1}(t)$ and let L_0 be the disjoint union of the family $\{\text{Der } E_t \mid t \in T\}$. Let Σ be a normed algebra consisting of a full set of bounded global sections of (E, π, T) and consider again the uniform field (\mathcal{D}, ρ, T) defined in § 2, associated with Σ .

If $L = \mathcal{D} \cap L_0$ is equipped with the topology and uniform structure induced by the field topology and field uniformity of \mathcal{D} and if q is the restriction of $\rho: \mathcal{D} \rightarrow T$ to L , then (L, q, T) is also a uniform field.

5.4. Suppose that Σ consists of all global sections with quasicompact support in (E, π, T) and that λ and μ are global sections in (L, q, T) , then $t \mapsto (z\lambda)(t) = z\lambda(t)$ with z a complex number, $t \mapsto (\lambda + \mu)(t) = \lambda(t) + \mu(t)$ and $t \mapsto [\lambda, \mu](t) = [\lambda(t), \mu(t)]$ are also global sections of (L, q, T) .

Proof. If $\sigma \in \Sigma$, then by 2.4 $t \mapsto \lambda(t)\sigma(t)$ and $t \mapsto \mu(t)\sigma(t)$ are global sections of (E, π, T) and their supports are still quasicompact. The maps $t \mapsto (z\lambda(t))(\sigma(t))$, $t \mapsto (\lambda(t) + \mu(t))(\sigma(t))$ and

$$t \longmapsto \lambda(t)(\mu(t)\sigma(t)) - \mu(t)(\lambda(t)\sigma(t))$$

also belong to Σ , thus by 2.4, $z\lambda$, $\lambda + \mu$ and $[\lambda, \mu]$ are global sections in the field (L, q, T) .

REMARK. Recall that if σ is a global section with quasicompact support the upper semicontinuity of the function $t \mapsto \|\sigma(t)\|: t \rightarrow \mathbf{R}_+$ implies that σ is bounded (1.9).

5.5. In the field of derivations (L, q, T) the sum $(u, v) \mapsto u + v: L \vee L \rightarrow L$ is continuous.

Proof. Let $(u_0, v_0) \in L \vee L$, $t = q(u_0) = q(v_0)$ and $M(u_0 + v_0, \sigma, \tau, Q, \varepsilon)$ a fundamental neighborhood of $u_0 + v_0$. Without loss of generality τ can be taken of the form $\tau = \tau_1 + \tau_2$ where τ_1 and τ_2 are global sections such that $\tau_1(t) = u_0\sigma(t)$ and $\tau_2(t) = v_0\sigma(t)$. If $(u, v) \in L \vee L$, $u \in M(u_0, \sigma, \tau_1, Q, \varepsilon/2)$ and $v \in M(v_0, \sigma, \tau_2, Q, \varepsilon/2)$, the $u + v \in M(u_0 + v_0, \sigma, \tau, Q, \varepsilon)$. Hence the sum is continuous.

5.6. Let (E, π, T) be a field of normed algebras, $\Gamma_c(\pi)$ the algebra of all global sections with quasicompact support of (E, π, T) and J the Lie algebra of derivations of $\Gamma_c(\pi)$ leaving setwise invariant the ideals of the form $I_t = \{\sigma \in \Gamma_c(\pi) \mid \sigma(t) = 0\}$, then J is isomorphic to the algebra $\Gamma(q)$ of all global sections of the field (L, q, T) .

Proof. Let $\lambda \in J$ and $t \in T$. Define λ_t as follows $\lambda_t: x \mapsto (\lambda\sigma_x)(t): E_t \rightarrow E_t$ where $\sigma_x \in \Gamma_c(\pi)$ and $\sigma_x(t) = x$. This function is well defined, indeed, if $\sigma, \tau \in \Gamma_c(\pi)$ and $\sigma(t) = \tau(t) = x$, then $\lambda(\sigma - \tau) \in I_t$

and $(\lambda\sigma)(t) = (\lambda\tau)(t)$. Let $\psi(\lambda): t \mapsto \lambda_t: T \rightarrow L$, if $\sigma \in \Gamma_c(\pi)$, then $t \mapsto \lambda_t(\sigma(t)): T \rightarrow E$ is a section, hence by 2.4 $\psi(\lambda)$ is an isomorphism from the Lie algebra J into the Lie algebra of all global sections of (L, q, T) .

On the other hand, given a global section ξ in the field (L, q, T) and a global section $\sigma \in \Gamma_c(\pi)$, $\beta(\xi)\sigma: t \mapsto \xi(t)\sigma(t): T \rightarrow E$ belongs to $\Gamma_c(\pi)$. Thus $\beta: \xi \mapsto \beta(\xi): \Gamma(q) \rightarrow J$ is an isomorphism of Lie algebras whose inverse is ψ .

5.7. With the same hypothesis and notation of 5.6, let ε_t be the Lie algebra homomorphism $\varepsilon_t: \xi \mapsto \xi(t): \Gamma(q) \rightarrow \text{Der } E_t$, then $\text{Im } \varepsilon_t = q^{-1}(t)$ because L was defined as the subset of L_0 consisting of the points u such that there is a global section $\xi: T \rightarrow L_0$ with $\xi(q(u)) = u$. But

$$\begin{aligned} \ker \varepsilon_t &= \{\xi \in \Gamma(q) \mid (\forall \sigma \in \Gamma_c(\pi))(\xi(t)\sigma(t) = 0)\} \\ &= \{\xi \in \Gamma(q) \mid (\forall \sigma \in \Gamma_c(\pi))(\beta(\xi)\sigma \in I_t)\}, \end{aligned}$$

then the fiber $L_t = q^{-1}(t)$ is isomorphic to the quotient Lie algebra J/K_t where $K_t = \{\lambda \in J \mid (\forall \sigma \in \Gamma_c(\pi))(\lambda\sigma \in I_t)\}$.

DEFINITION 5.8. Let (E, π, T) be a field of normed algebras. A continuous map $\Delta: E \rightarrow E$ is called a *derivation* of (E, π, T) if $\pi \circ \Delta = \pi$ and for every $t \in T$, the map $x \mapsto \Delta x: E_t \rightarrow E_t$ is a derivation of $E_t = \pi^{-1}(t)$.

5.9. Let (E, π, T) be a field of normed algebras and $\Gamma_c(\pi)$ the algebra of all global sections with quasicompact support, then the Lie algebra J of all continuous derivations of $\Gamma_c(\pi)$ leaving setwise invariant the ideals $I_t = \{\sigma \in \Gamma_c(\pi) \mid \sigma(t) = 0\}$ for every $t \in T$ is isomorphic to the Lie algebra $\text{Der}(E, \pi, T)$ of derivations of the field (E, π, T) .

Proof. Let ρ be an element of J . Define $F(\rho)$ as follows $x \mapsto F(\rho)x = (\rho\sigma)(\pi(x)): E \rightarrow E$, where σ is an element of $\Gamma_c(\pi)$ such that $\sigma(\pi(x)) = x$. The function $F(\rho)$ is well defined because if $\tau(\pi(x)) = x$ and $t = \pi(x)$, then $\sigma - \tau \in I_t$ and hence $(\rho\sigma)(\pi(x)) = (\rho\tau)(\pi(x))$.

To prove that $F(\rho)$ is continuous, say at x , let $\mathcal{S}(\tau, \varepsilon) \cap \pi^{-1}(Q)$ be a fundamental neighborhood of $F(\rho)x$, without loss of generality one can take τ of the form $\tau = \rho\sigma$ where σ is an element of $\Gamma_c(\pi)$ such that $\sigma(\pi(x)) = x$.

Take $W = \mathcal{S}(\sigma, \varepsilon(\varepsilon + \|\rho\|)^{-1}) \cap \pi^{-1}(Q)$ as neighborhood of x . By ([8], 2.3), for every $y \in W$,

$$\|F(\rho)y - \tau(\pi(y))\| \leq \|\rho\| \|y - \sigma(\pi(y))\| < \varepsilon,$$

then $F(\rho)y \in \mathcal{S}(\tau, \varepsilon) \cap \pi^{-1}(Q)$ and $F(\rho)$ is continuous at x , thus $F(\rho)$ belongs to $\text{Der}(E, \pi, T)$ and F is a Lie algebra homomorphism. Conversely, if Δ is a derivation of (E, π, T) and $\sigma \in \Gamma_c(\pi)$, then $\Delta \circ \sigma$ is also a global section with quasicompact support in the field (E, π, T) .

Define $G(\Delta)$ by $\sigma \mapsto G(\Delta)\sigma = \Delta \circ \sigma: \Gamma_c(\pi) \rightarrow \Gamma_c(\pi)$, then $G(\Delta)$ is a derivation of $\Gamma_c(\pi)$ leaving invariant the ideals of the form I_t . Furthermore $G(\Delta)$ is a continuous function, indeed, let $\sigma \in \Gamma_c(\pi)$, T_σ the support of σ and $\varepsilon > 0$. Since for every $t \in T$, Δ is continuous at $\sigma(t)$, there is a neighborhood Q_t of t and $\delta_t > 0$ such that $x \in \pi^{-1}(Q_t) \cap \mathcal{S}(\sigma, \delta_t)$ implies $\Delta x \in \mathcal{S}(\Delta\sigma, \varepsilon)$. Without loss of generality Q_t can be supposed to contain the complement of T_σ . Since T_σ is quasicompact, there exist a finite number of Q_t 's covering T . If δ is taken to be the minimum of the corresponding δ 's, then $\Delta\mathcal{S}(\sigma, \delta) \subset \mathcal{S}(\Delta\sigma, \varepsilon)$. It follows that $G(\Delta)$ is continuous and hence it belongs to J .

Therefore G is an isomorphism between $\text{Der}(E, \pi, T)$ and J whose inverse is F .

5.10. Summing up, we have the following

THEOREM. *If A is C^* -algebra with identity and B is a sub- C^* -algebra contained in the center of A and containing its identity, then the following Lie algebras are isomorphic.*

- (i) *The Lie algebra of derivations of A .*
- (ii) *The Lie algebra of derivations of the canonical field (E, π, T) associated with the pair (B, A) .*
- (iii) *The Lie algebra of global sections of the field of derivations (L, q, T) obtained from (E, π, T) .*

Proof. This follows from 5.2, 5.6 and 5.9.

6. The exponential function.

6.1. Let A be a C^* -algebra with identity. We identify A with the algebra $\Gamma(\pi)$ of all global sections of the canonical field (E, π, T) associated with A over the spectrum T of the center of A . As usual I_t denotes the ideal $I_t = \{\sigma \in \Gamma(\pi) \mid \sigma(t) = 0\}$. It was observed in [8] 3.1, that I_t is the closed two sided ideal generated by $t \in T$.

Let J be the Lie algebra of derivations of A and N the group of automorphisms of A leaving the center elementwise fixed. The following diagram is commutative

$$\begin{array}{ccccc}
 \delta & J & \xrightarrow{\exp} & N & \lambda \\
 \downarrow & \downarrow & & \downarrow & \downarrow \\
 \delta_t & \text{Der}(A/I_t) & \xrightarrow{\exp_t} & \text{Aut}(A/I_t) & \lambda_t
 \end{array}$$

where $\delta_t(x) = (\delta\sigma)(t)$, $\lambda_t(x) = (\lambda\sigma)(t)$ for every $x \in A/I_t$ with $\sigma \in \Gamma(\pi)$ such that $\sigma(t) = x$, $(\exp \delta)(a) = a + \delta(a) + \delta^2(a)/2! + \dots$, for every $\delta \in J$ with $a \in A$, and

$$(\exp_t u)(a + I_t) = (a + I_t) + u(a + I_t) + u^2(a + I_t)/2! + \dots,$$

for every $u \in \text{Der}(A/I_t)$ and every $a \in A$.

Proof. δ_t and λ_t are well defined because the ideal I_t is left fixed by $\delta \in J$ and $\lambda \in N$.

Let $\delta \in J$, then $(\exp \delta_t)(x) = x + \delta_t(x) + \delta_t^2(x)/2! + \dots = \sigma(t) + (\delta\sigma)(t) + (\delta^2\sigma)(t)/2! + \dots = ((\exp \delta)\sigma)(t) = (\exp \delta)_t(x)$, where $x \in A/I_t$ and $\sigma(t) = x$. Thus the diagram is commutative.

6.2. Now we can define an “exponential function” from the field (L, q, T) of derivations of $A = \Gamma(\pi)$ into the field (G, p, T) of automorphisms of A .

$$EXP: u \longmapsto (\exp)_t(u): L \longrightarrow G \text{ with } t = q(u).$$

Clearly $p(EXP u) = q(u)$ for every $u \in L$, and using the notation of 5.6 from $EXP: L \rightarrow G$ we get back $\exp: J \rightarrow N$ as follows: Consider $\sigma \mapsto (EXP(\psi(\delta)(\cdot)))(\sigma(\cdot))$, with $\delta \in J$ and $\sigma \in \Gamma(\pi)$, then

$$(EXP(\psi(\delta)(t)))(\sigma(t)) = (\exp_t \delta_t)(\sigma(t)) = (\exp \delta)_t(\sigma(t)) = ((\exp \delta)\sigma)(t)$$

for every $t \in T$. Thus $EXP(\psi(\delta)(\cdot))(\sigma(\cdot)) = (\exp \delta)\sigma$.

6.3. $EXP: L \rightarrow G$ is continuous on bounded sets when L and G are endowed with their field topologies.

Proof. Let $L_M = \{w \in L \mid \|w\| \leq M\}$ where $\|w\|$ is the operator norm of w . Let $u \in L_M$, $t = q(u)$, $\varepsilon > 0$ and $\delta \in J$ such that $\delta_t = u$. Since $EXP u = \exp_t u = \exp_t \delta_t = (\exp \delta)_t$, then a fundamental neighborhood of $EXP u$ in G is of the form

$$\{v \in G \mid s = p(v) \in Q \text{ \& \; } \inf_{x \in I_s} \|\lambda a - (\exp \delta)a + x\| < \varepsilon \\
 \text{for } \lambda \in N \text{ with } \lambda_s = v\}$$

where Q is an open neighborhood of $t = p(EXP u)$ and $a \in A$. Let k be a positive integer such that $2M \|a\| \sum_{n=k}^{\infty} M^n/n! < \varepsilon/2$ and consider the following neighborhood of u in L_M .

$$V(u) = \{w \in L_M \mid s = q(w) \text{ \& } \inf_{x \in I_s} \|(\xi - \delta)\delta^j a + x\| < \varepsilon/2 \exp(M+1) \\ \text{for } j = 1, \dots, k \text{ \& } \xi \in J \text{ with } \xi_s = w\}$$

Denote by $\|\cdot\|_s$ the quotient norm in A/I_s , then for every $w = \xi_s \in V(u)$ we have

$$\begin{aligned} & \|(\exp \xi)a - (\exp \delta)a + I_s\|_s \\ & \leq \sum_{n=1}^{\infty} 1/n! \|(\xi^n - \delta^n)a + I_s\|_s \\ & = \sum_{n=1}^{\infty} 1/n! \|\sum_{i=1}^n \xi^{i-1}(\xi - \delta)\delta^{n-i}a + I_s\|_s \\ & \leq \sum_{n=1}^k (M+1)^{n-1}/(n-1)! \sup_{1 \leq i \leq n} \|(\xi - \delta)\delta^{n-i}a + I_s\|_s \\ & \quad + \sum_{n=k+1}^{\infty} nM^{n-1}/n! \|\xi_s - \delta_s\| \|a + I_s\|_s \\ & \leq \varepsilon/2 + 2M \|a\| \sum_{n=k}^{\infty} M^n/n! \leq \varepsilon. \end{aligned}$$

Thus $EXP: L \rightarrow G$ is continuous on bounded sets.

7. One example. Let $M_2(C)$ be the algebra of all 2×2 complex matrices and let A be the C^* -algebra consisting of all sequences $X = (X_n)_n$ of elements of $M_2(C)$ converging to a diagonal matrix

$$\begin{pmatrix} \alpha(X) & 0 \\ 0 & \beta(X) \end{pmatrix}$$

The center Z of A is C^* -isomorphic to the algebra c of all convergent sequences of complex numbers. For every positive integer $k \geq 1$ (resp. for $k = 0$) the ideal c_k of all convergent complex sequences whose k th term is zero (resp. whose limit is zero) is a maximal ideal of c , and these are all its maximal ideals.

The closed two sided ideal generated by c_k with $k \geq 1$ (resp. by c_0) is the ideal of all X such that $X_k = 0$ (resp. $\lim X_k = 0$). By the (noncommutative) Gelfand-Naimark representation theorem ([2], Chapter III, 8.14; [8], 3.5), A is isometrically star isomorphic to the algebra of all global sections of a field of C^* -algebra (E, π, T) whose base space $T = \mathbb{Z}_+ \cup \{\omega\}$ is the one point compactification of the discrete space \mathbb{Z}_+ of position integers, and whose stalks are $\{t\} \times M_2(C)$ for $t \neq \omega$ and $\{t\} \times C^2$ for $t = \omega$. Every star automorphism of $M_2(C)$ is inner given by a unitary matrix, and C^2 has only two automorphisms, the identity and $(z_1, z_2) \mapsto (z_2, z_1): C^2 \mapsto C^2$. The group N of star automorphisms of A leaving the center elementwise fixed is then isomorphic to the group of all global sections of a field (G, p, T) over the same base space and its stalks are the group of star automorphisms of $M_2(C)$ if $t \neq \omega$ and the group with two elements if $t = \omega$. Every section of (G, p, T) is given by a sequence $(U_k)_k$ of 2×2 unitary matrices converging to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in

the strong operator topology on the set of diagonal matrices. Indeed, to a continuous section of (G, p, T) there is associated a sequence $(U_k)_k$ of unitary 2×2 matrices such that if $X = (X_k)_k$ belongs to A and X_ω is its limit, then $\lim U_k X_k U_k^* = \varphi(X_\omega)$ where φ is a star automorphism of C^2 .

But $\lim U_k X_k U_k^* = \lim U_k X_\omega U_k^*$, thus $U_k D U_k$ converges to D or to $\begin{pmatrix} d_2 & 0 \\ 0 & d_1 \end{pmatrix}$ for every diagonal matrix $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$. The sequence $(U_k)_k$ is uniquely determined up to a sequence $(D_k)_k$, $D_k = \begin{pmatrix} d_k & 0 \\ 0 & d_k \end{pmatrix}$ with $|d_k| = 1$.

Now let J be the Lie algebra of derivations of A . Then J is isomorphic to the Lie algebra of all global sections of the field (L, q, T) obtained from (E, π, T) .

Every derivation of $M_2(C)$ is inner and C^2 being commutative has no nonzero derivations. Then every global section of (L, q, T) is given by a sequence $(V_k)_k$ of complex 2×2 matrices converging to zero in the strong operator topology on the set of diagonal matrices, i.e. $V_k D - D V_k$ converges to zero for every diagonal matrix D .

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Received October 9, 1974 and in revised form March 25, 1975. Research with partial support of COLCIENCIAS.

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