Pacific Journal of Mathematics

ALMOST PERIODIC HOMEOMORPHISMS OF E^2 ARE PERIODIC

BEVERLY L. BRECHNER

Vol. 59, No. 2 June 1975

FACIFIC JOURNAL CF MATHEMATICS
Vol. 59, No. 2, 1975

ALMOST PERIODIC HOMEOMORPHISMS OF E^2 ARE PERIODIC

BEVERLY L. BRECHNER

In this paper we show that every almost periodic homeomorphism of the plane onto itself must be periodic. This improves well-known results.

1. Introduction. In [3] Foland showed that every almost periodic homeomorphism of a *disk* onto itself is topologically either a reflection in a diameter or a rotation. Hemmingsen [7] studies homeomorphisms on compact subsets of E^2, with equicontinuous families of iterates, and shows that if such a compact set has an interior point of infinite order, then the compact set is a disk or annulus. If it is a disk, then the homeomorphism is a rotation or reflection. Kerékjártó [8, pp. 224–226] showed that every periodic homeomorphism of a disk onto itself is a conjugate of either a rotation or a reflection. It was brought to my attention by S. Kinoshita that Kerékjártó in [9] obtains a characterization of those homeomorphisms of S^2 onto itself which are regular; that is, homeomorphisms h such that $\{h^n\}_{n\in I}$ forms an equicontinuous family. It is known [4] that almost periodic homeomorphisms on compact metric spaces satisfy this property, so that our theorem for E^2 would follow from the theorem for S^2.

However, our proof of the main theorem uses Bing's ε-growth technique [6] to obtain an invariant disk, and thus *re*-does a portion of [2], [7], and [9] in a particularly nice way.

Montgomery began a study of almost periodic transformation groups in [13], with the main results for E^3. One very nice theorem states that if G is a one-parameter almost periodic transformation group (a.p.t.g.) of E^3 whose minimal closed invariant sets are one-dimensional, and whose orbits are uniformly bounded, then G is the identity. Our theorem may be regarded as something of an analogue to this theorem for E^2. That is, our theorem shows that if $G = \{h^n\}_{n\in I}$ is an a.p.t.g. of E^2, $h \neq e$, then the orbits are not uniformly bounded.

2. Preliminaries. The definitions used here of the following are as in [4] and [6]: *Relatively dense* subsets of the integers; homeomorphisms *almost periodic at a point*, *pointwise almost periodic* (p.a.p.), and *almost periodic* (a.p.) on the space; *invariant set*; and *minimal set* are defined in [4]. *Property S*, *ε-growth*, and *ε-sequential growth* are defined in [6]. The *orbit* of x in the space X is the set

$\{h^n(x) \mid n \in I\}$, and is denoted by $0(x)$.

We will use the following known results.

PROPOSITION 2.1. [6, pg. 212]. *Let K be a subset of a metric space X. If K has property S, then K is locally connected.*

PROPOSITION 2.2. [6, pg. 215]. *If K is a subset of a metric space X and K has property S, then $\bar{K}$ has property S. Thus, if K has property S, then $\bar{K}$ is locally connected.*

PROPOSITION 2.3. [6, pg. 216]. *Let X be a metric space with property S, H and K subsets of X, and $\varepsilon > 0$. If K is an ε-sequential growth of H, then K has property S and is open in X.*

NOTE. The double arrow in $f: A \twoheadrightarrow B$ denotes an *onto* function.

3. Obtaining invariant disks. In this section we use the concept of an ε-sequential growth to enable us to obtain E^2 as the union of an increasing tower of invariant disks for any a.p. homeomorphism of the plane onto itself.

LEMMA 3.1. *Let X be a compact metric space and let $\{f_n\}$ be an equicontinuous collection of functions on X. Then for each $\varepsilon > 0$, there is a $\delta > 0$ such that* diam $(f_n(\delta\text{-set})) < \varepsilon$, *for all $n \in I$.*

Proof. Let $\varepsilon > 0$. For each $x \in X$, there exists $\gamma > 0$ such that diam $(f_n(\gamma\text{-nbd of } x)) < \varepsilon$ for all $n \in I$, since $\{f_n\}$ is equicontinuous. Choose such a neighborhood for each $x \in X$. This forms a cover of X and therefore some finite subcollection covers X. Let δ be a Lebesgue number for this subcover. Then diam $(f_n(\delta\text{-set})) < \varepsilon$ for all $n \in I$.

LEMMA 3.2. *Let h be a homeomorphism of S^2 onto itself such that $h(p) = p$ where p is the north pole of S^2, and let X be a locally connected continuum in S^2, containing p, such that $h(X) = X$. Let $\varepsilon = \operatorname{diam} S^2$, and by uniform continuity of h, let $\delta > 0$ such that* diam $(h(\delta\text{-set})) < \varepsilon/2$. *Then if* diam $X < \delta$ *and U is the component of $S^2 - X$ containing the south pole, we have $h(U) = U$.*

Proof. We first show that each component of $S^2 - X$ must go onto some component of $S^2 - X$. Let V be a component of $S^2 - X$, and suppose there exist points x and $y \in V$ such that $h(x) \in W_1$, $h(y) \in W_2$, where $W_1 \neq W_2$ are components of $S^2 - X$. Let A be an arc from x to y in V. Since A misses X, and $h(X) = X$, $h(A)$ misses X. But $h(A)$ is connected and contains points of different components of $S^2 - X$, and therefore must contain a point of X. This is a contradiction. Therefore $h(V)$ is a subset of a component of $S^2 - X$. The same argument applied to h^{-1}, shows h^{-1} (componet) $\subseteq$ some component of $S^2 - X$, so that $h(V)$ is a component of $S^2 - X$. We next show that $h(U) = U$. Suppose $h(U) \neq U$. Then there is a component $W (\neq U)$ of $S^2 - X$, such that $h(W) = U$. Now diam $W < \delta$, and therefore diam $h(W) < \varepsilon/2$. Therefore $h(W) \neq U$. This is a contradiction. Thus $h(U) = U$.

LEMMA 3.3. *Let h be an almost periodic homeomorphism of E^2 onto E^2 and let $\varphi\colon E^2 \to S^2$ be the inverse of the stereographic projection. Let p be the north pole of S^2. Let $g\colon S^2 \twoheadrightarrow S^2$ be defined by* $g(x) = \begin{cases} \varphi h \varphi^{-1}(x), & \text{for } x \in S^2 - \{p\} \\ p, & \text{for } x = p \end{cases}$. *Then g is an* a.p. *homeomorphism of S^2 onto S^2.*

Proof. Let $\varepsilon > 0$. We must show that there exists a relatively dense subset A of I such that $d(x, g^n(x)) < \varepsilon$ for all $x \in S^2$ and all $n \in A$. Now we know there exists a relatively dense subset A of I such that $d(x, h^n(x)) < \varepsilon$ for all $x \in E^2$ and all $n \in A$. Also, it follows from pg. 20 of [1] that φ has the property that $d(y, y') \geqq d(\varphi(y), \varphi(y'))$ for all $y, y' \in E^2$. Now since $d(y, h^n(y)) < \varepsilon$ for all $y \in E^2$ and all $n \in A$, $d(\varphi^{-1}(x), h^n\varphi^{-1}(x)) < \varepsilon$ for all $x \neq p \in S^2$, all $n \in A$. Thus $d(\varphi\varphi^{-1}(x), \varphi h^n \varphi^{-1}(x)) < \varepsilon$ and $d(x, \varphi h^n \varphi^{-1}(x)) < \varepsilon$ for all $x \in S^2$, all $n \in A$. It follows that $d(x, g(x)) < \varepsilon$ for all $x \in S^2$, all $n \in A$, and g is a.p.

THEOREM 3.1. *Let h be an* a.p. *homeomorphism on S^2 such that h keeps the north pole p fixed. Then for each $\eta > 0$, there exists an η-disk E which is invariant under h* (in fact $h(E) = E$), *and contains p in its interior.*

Proof. Let γ be the diameter of S^2. Then there exists $\delta > 0$ such that diam ($h(\delta$-set)) $< \gamma/2$, by uniform continuity of h. Let $0 < \varepsilon < \min\{\eta, \delta, \gamma\}$, and let $\{\varepsilon_i\}$ be a decreasing sequence of positive numbers such that $\sum \varepsilon_i < \varepsilon \leqq \eta$. We will obtain E as an ε-sequential growth of the set $\{p\}$.

Let $D_1 = \{p\}$. The set $\{h^n\}_{n \in I}$ is equicontinuous [4, pg. 341], and $\varepsilon_1 > 0$. Thus by Lemma 3.1, there exists $\delta_1 > 0$ such that diam ($h^n(\delta_1$-set)) $< \varepsilon_1$ for all $n \in I$. Let $\mathscr{U}_1 = \{U_{11}\}$ be a cover of D_1 by an open connected set of S^2 such that $\mu(\mathscr{U}_1) < \min\{\delta_1, \varepsilon_1\}$. Let $D_2 = \bigcup_{n \in I} h^n(U_{11})$ and note that D_2 is invariant. We show that D_2 is an ε_1-growth of D_1. We must show parts (i) and (ii) of the definition of ε-growth.

Proof of (i). If $x \in D_2 - D_1$, then there exists an integer n such that $x \in h^n(U_{11})$. But $h^n(U_{11})$ is connected and $\operatorname{diam}(h^n(U_{11})) < \varepsilon_1$. Also, $h^n(U_{11})$ contains p and so meets D_1.

Proof of (ii). U_{11} is an open set containing the compact set D_1. $S^2 - U_{11}$ is compact, and disjoint from D_1 which is compact. Thus $d(D_1, S^2 - U_{11}) = 2\alpha_1$ for some $\alpha_1 > 0$, and it follows that the α_1-nbd. of D_1 is a subset of D_2. Thus (i) and (ii) hold and D_2 is an ε_1-growth of D_1.

We now wish to obtain an ε_2-growth of D_2. We note that since D_2 is invariant, so is $\bar{D}_2$. Now for $\varepsilon_2 > 0$, there exists $\delta_2 > 0$ such that $\operatorname{diam}(h^n(\delta_2\text{-set})) < \varepsilon_2$ for all n. Again this is possible by Lemma 3.1. Let $\mathscr{U}_2$: $U_{2,1}, U_{2,2}, \cdots, U_{2,k_2}$ be a finite cover of $\bar{D}_2$ by open connected subsets of S^2 of diameter $< \min\{\delta_2, \varepsilon_2\}$ and let

$$D_3 = \bigcup_{n \in I} h^n\left(\bigcup_{i=1}^{k_2} U_{2,i}\right).$$

Then D_3 is invariant.

We show that D_3 is an ε_2-growth of D_2. We prove parts (i) and (ii) of the definitions of ε-growth.

Proof of (i). Let $x \in D_3 - D_2$. Then $x \in h^n(U_{2,i})$ for some pair n, i. But $h^n(U_{2,i})$ is connected, meets D_2, and has diameter $< \varepsilon_2$.

Proof of (ii). $\bar{D}_2$ and $S^2 - \bigcup_{n \in I} h^n(\bigcup_{i=1}^{k_2} U_{2,i})$ are disjoint compact subsets of S^2 and thus are a positive distance apart, say $2\alpha_2$. Then the α_2-nbd. of $\bar{D}_2$, and therefore the α_2-nbd. of D_2, is a subset of D_3.

Thus (i) and (ii) hold, and D_3 is an ε_2-growth of D_2.

It is clear that we may continue the process inductively, obtaining at the ith stage, a connected open set D_i which is an ε_{i-1}-growth of D_{i-1}. Let $E' = \bigcup_{i=1}^{\infty} D_i$. Then by Proposition 2.3, E' is open and has property S. Thus $\bar{E}'$ is a locally connected continuum, by Proposition 2.2. Further $\bar{E}'$ is invariant. We show that $\bar{E}'$ has no cut points. Note that E' has no cut points since it is open (and connected). Thus any cut point of $\bar{E}'$ would be in $\bar{E}' - E'$, so that there would exist a component of $\bar{E}'$ containing points of $\bar{E}' - E'$ only. But these are all limit points of E'. This is a contradiction, and it follows that $\bar{E}'$ has no cut points.

Thus $\bar{E}'$ is a locally connected continuum with no cut points, and from Theorem 9 of [11] it follows that the boundary of each of its complementary domains is a simple closed curve. Now one of its complementary domains, say F, contains the open southern hemisphere, and therefore has diameter $\geqq \gamma$, while each of the other complementary domains has diameter less than ε, since $\operatorname{diam} \bar{E}' < \varepsilon$. Thus by Lemma 3.2, F is invariant, and $h(F) = F$. Let $E = S^2 - F$.

Then diam $E < \varepsilon$, $h(E) = E$, and E is a disk, by the Jordan-Schoenflies theorem [6, pg. 257], since it's a continuum not separating S^2 and has a simple closed curve as its boundary. Clearly E contains p in its interior. Then E is the desired 2-cell.

COROLLARY 3.1.1. *Let h be an* a.p. *homeomorphism of E^2 onto itself. Then E^2 is the union of an increasing sequence of disks $\{B_i\}_{i=1}^{\infty}$ such that*

(1) $B_1 \subsetneqq B_2^0 \subsetneqq B_2 \subsetneqq B_3^0 \subsetneqq B_3 \subsetneqq \cdots \subsetneqq B_n^0 \subsetneqq B_n \subsetneqq \cdots$ *and*

(2) $h(B_n) = B_n$ *for all* n.

Proof. Let $\{\varepsilon_i\}$ be a decreasing sequence of positive numbers. By Theorem 3.1, there exist disks K_i' on S^2 such that (1) diam $K_i' < \varepsilon_i$, (2) $h(K_i') = K_i'$ and (3) K_i' contains p, the north pole of S^2. Let $K_1 = K_1'$, $K_2 =$ first K_i' such that $K_i' \subsetneqq (K_1)^0$, $K_3 =$ first K_i' such that $K_i' \subsetneqq (K_2)^0$, etc. Let $\varphi: S^2 \to E^2$ be the stereographic projection. Then $\{B_i\} = \{\varphi(K_i)\}$ is the desired sequence.

4. The main theorem.

In this section we prove the main theorem of this paper.

LEMMA 4.1. *Let B_1 and B_2 be 2-cells in E^2 such that $B_1 \subsetneqq B_2^0$. Let h be a homeomorphism of B_2 onto itself such that*

(1) $h(B_1) = B_1$,

(2) $h = \varphi^{-1} r \varphi$ *for some rotation r on the disk D_2 with center at the origin and radius 2, where $\varphi: B_2 \twoheadrightarrow D_2$ is a homeomorphism, and*

(3) $\varphi(\text{Bd}\, B_1)$ *is a circle centered at the origin. Then there exists a homeomorphism $g: B_2 \twoheadrightarrow D_2$ such that*

(1) $g(\text{Bd}\, B_1)$ *is the unit circle, and*

(2) $h = g^{-1} r g$.

Proof. We first make a definition. We call a homeomorphism $f: D_2 \twoheadrightarrow D_2$ *radial* iff f takes each radius onto itself, and is such that circles centered at the origin go onto circles centered at the origin.

Now let $\Psi: D_2 \twoheadrightarrow D_2$ be a radial homeomorphism of D_2 onto itself such that $\Psi(\varphi(\text{Bd}\, B_1))$ is the unit circle. Then $\Psi\varphi$ is a homeomorphism of B_2 onto D_2 such that $\Psi\varphi(\text{Bd}\, B_1)$ is the unit circle. Further, for any rotation r, since $\Psi^{-1} r \Psi = r$, $\varphi^{-1} r \varphi = \varphi^{-1}(\Psi^{-1} r \Psi)\varphi = \varphi^{-1}\Psi^{-1} r \Psi \varphi = (\Psi\varphi)^{-1} r (\Psi\varphi)$. Thus we let $g = \Psi\varphi$ and g is the desired homeomorphism.

LEMMA 4.2. *Let B_1 and B_2 be 2-cells in E^2 such that $B_1 \subsetneqq B_2^0$. Let $h: B_2 \twoheadrightarrow B_2$ be a homeomorphism such that*

(1) $h(B_1) = B_1$

(2) *there exists a homeomorphism* $\varphi_1: B_1 \twoheadrightarrow$ *unit disk such that* $h | B_1 = \varphi_1^{-1} r_1 \varphi_1$, *for some rotation* $r_1: E^2 \twoheadrightarrow E^2$, *and*

(3) *there exists a homeomorphism* $\varphi_2: B_2 \twoheadrightarrow D_2$, *where* D_2 *is the disk of radius* 2 *about the origin, such that*

(a) $h = \varphi_2^{-1} r_2 \varphi_2$, *for some rotation* r_2 *of* E^2 *onto itself, and*

(b) $\varphi_2(\mathrm{Bd}\, B_1) =$ *unit circle.*

Then there exists a homeomorphism $g: B_2 \twoheadrightarrow D_2$ *such that*

(1) $g | B_1 = \varphi_1$,

(2) $g(\mathrm{Bd}\, B_1) =$ *unit circle, and*

(3) $h = g^{-1} r_1 g$.

Proof. Let the annulus between Bd D_1 and Bd D_2 be decomposed into the continuous collection $\mathscr{A}$ of arcs which are the intersections of the radii of D_2 with the annulus. Note that $\varphi_1 \varphi_2^{-1}: D_1 \twoheadrightarrow D_1$ is a homeomorphism that takes $\varphi_2(x)$ to $\varphi_1(x)$ for each $x \in B_1$. We extend $\varphi_1 \varphi_2^{-1}$ to a homeomorphism $\Psi: D_2 \twoheadrightarrow D_2$ by taking each element $A \in \mathscr{A}$ with endpoint $\varphi_2(x) \in \mathrm{Bd}\, D_1$ to the element $A' \in \mathscr{A}$ with endpoint $\varphi_1(x)$ in Bd D_1, in such a way that distance along the segments A and A' are preserved. Thus Ψ is a homeomorphism of D_2 onto itself such that $\Psi | D_1 = \varphi_1 \varphi_2^{-1}$.

Now let $g = \Psi \varphi_2$. We show that g is the required homeomorphism. Since $\Psi = \varphi_1 \varphi_2^{-1}$ on D_1, $\Psi \varphi_2 = (\varphi_1 \varphi_2^{-1}) \varphi_2 = \varphi_1$ on B_1, so g is an extension of φ_1. Also $g(\mathrm{Bd}\, B_1) =$ unit circle. It remains to show that $h = g^{-1} r_1 g$ on $B_2 - B_1$.

It is sufficient to show that $r_2 = r_1$ on Bd D_1. Now $\varphi_1^{-1} r_1 \varphi_1 = \varphi_2^{-1} r_2 \varphi_2$ on Bd B_1, so $r_2 = (\varphi_1 \varphi_2^{-1})^{-1} r_1 (\varphi_1 \varphi_2^{-1})$ is a conjugate of a rotation. But it follows from [12] that the rotations are characterized by numbers in $1 - 1$ correspondence with $0 \leqq x < 1$, and any conjugate $f^{-1} r f$ of a rotation is characterized (even though not necessarily a rotation) by the same number as the number for the rotation r. Thus the characterizing number for a conjugate of r_1 is the same as for r_1. It follows that $r_1 = r_2$, and $h = g^{-1} r_1 g$ on B_2.

THEOREM 4.1. *Let* h *be an almost periodic homeomorphism of* E^2 *onto itself. Then* h *is periodic.*

Proof. Let $\{B_i\}$ be an increasing tower of 2-cells of E^2 such that $B_1 \subseteqq B_2^0 \subseteqq B_2 \subseteqq B_3^0 \subseteqq B_3 \subseteqq \cdots \subseteqq B_n^0 \subseteqq B_n \subseteqq \cdots$, $\bigcup B_i = E^2$, and $h(B_i) = B_i$. This sequence exists by Corollary 3.1.1.

Case (i). *h is orientation preserving.* Since B_i is invariant, $h | B_i$ is a.p. on B_i and orientation preserving, and therefore is a conjugate of a rotation on the 2-cell D_i centered at the origin and of radius i. (This follows from [3].) That is, there exists a rotation $r_i: D_i \twoheadrightarrow D_i$ and a homeomorphism $\varphi_i: B_i \twoheadrightarrow D_i$ such that $h|B_i = \varphi_i^{-1} r_i \varphi_i$.

We will show that each r_i must be rational. Suppose by way of contradiction that r_i is an irrational rotation, for some i (the first such i). Then since $r_i = \varphi_i(h|B_i)\varphi_i^{-1}$ and B_{i-1} is invariant, $\varphi_i(B_{i-1})$ is invariant under r_i and in fact Bd $(\varphi_i(B_{i-1}))$ is invariant under r_i. Let x be any point in Bd $(\varphi_i(B_{i-1}))$. Since r_i is an invariant rotation $\overline{0(x)}$ under r_i will contain the circle C_x of radius $|x|$. Thus $C_x \subseteq \varphi_i(\text{Bd}\,(B_{i-1}))$. But $\varphi_i(\text{Bd}\,(B_{i-1}))$ is a simple closed curve, and it follows that $C_x = \varphi_i(\text{Bd}\,(B_{i-1}))$. By Lemma 4.1, we may assume that $\varphi_i(B_{i-1}) =$ radius (i $-$ 1) disk, and by Lemma 4.2, we may assume that $\varphi_i|B_{i-1} = \varphi_{i-1}$, (that is φ_i is an extension of φ_{i-1}) and further that $r_i = r_{i-1}$. Thus since r_i is the first irrational rotation, $i = 1$.

Clearly this process may be continued inductively, obtaining $r_i = r_{i-1}$, for all i. But then h would be the conjugate of an irrational rotation on E^2. However, such a rotation is not a.p. since $d(x, h^n(x)) \longrightarrow \infty$ as $x \longrightarrow \infty$ (for fixed n). This is a contradiction. It follows that each r_i is a rational rotation.

Now since each r_i is a rational rotation, it is of finite order, say n_i. But since $h|B_i$ is a conjugate of a periodic homeomorphism of order n_i on the disk D_i, each point of B_i (except the "center") has the same order, namely n_i. Thus each point of B_{i-1} has order n_i under $h|B_i$ and therefore under $h|B_{i-1}$. We may backtrack inductively until $i = 2$, so that each of $\{h|B_1, h|B_2, \cdots, h|B_i\}$ makes each point of B_i (except the "center") a point of order n_i; that is, the orbit consists of n_i points. It follows that for any $j > i$, the points of B_j must all have order n_j and therefore $n_j = n_i$. Thus h must be periodic on $\cup B_i = E^2$. But a periodic orientation preserving homeomorphism on E^2 is a conjugate of a rotation on E^2 [8, 2, 14]. Thus h is a conjugate of a rotation.

Case (ii). *h is orientation reversing.* Since B_i is invariant, $h|B_i$ is a.p. on B_i and orientation reversing, and hence a conjugate of a reflection [3]. Thus the fixed point set of $h|B_i$ is a "diameter" of B_i, and every other point of B_i has order 2 (its orbit consists of 2 points). Thus $h|B_i$ is of order 2, also. By induction $\{h|B_1, h|B_2, h|B_3, \cdots, h|B_i\}$ are each of order 2. For any $j > i$, the order of $h|B_j =$ order of $h|B_i$, by the same argument. Thus h is of period 2 on $\cup B_i = E^2$, and also is orientation reversing. It follows that h is a conjugate of a reflection [8, 2].

REMARK. It is clear that we have also proved that if h is an almost periodic homeomorphism of S^2 onto itself which is orientation preserving, and therefore keeps at least one point fixed [1, pg. 237], then h is a conjugate of a rotation. How are arbitrary almost periodic homeomorphisms of S^2 onto itself characterized? There are many nonconjugate fixed point free, almost periodic homeomorphisms of S^2 onto itself. For example, let f be a reflection of S^2 thru the equator, and let r be any rotation of S^2 thru the axis containing the north and south poles. Then rf is fixed point free, no two are conjugate, and each of these is almost periodic. (Note that if r is the 180° rotation, then rf is the antipodal map.) Are conjugates of these maps the only fixed point free homeomorphisms on S^2? Gerhard Ritter has just informed me that he will answer this question in the affirmative, in a forthcoming paper.

REFERENCES

1. Lars V. Ahlfors, *Complex Analysis*, 2nd edition, 1966, McGraw-Hill Book Co., N.Y.
2. Samuel Eilenberg, *Sur les transformations pèriodiques de la surface de sphere*, Fund. Math., **22** (1934), 28-41.
3. N. E. Foland, *A characterization of the almost periodic homeomorphisms on the closed 2-cell*, Proc. of Amer. Math. Soc., **16** (1965), 1031-1034.
4. W. H. Gottschalk, *Minimal Sets: An introduction to topological dynamics*, Bull. of Amer. Math. Soc., **64** (1958), 336-351.
5. Gottschalk and Hedlund, *Topological Dynamics*, American Mathematical Society Colloquium Publication, Vol. **36**, 1955.
6. Hall and Spencer, *Elementary Topology*, John Wiley and Sons, Inc., New York, London, 1955, 6th Printing 1964.
7. E. Hemmingsen, *Plane continua admitting non-periodic autohomeomorphisms with equicontinuous iterates*, Math. Scand., **2** (1954), 119-141.
8. B. v. Kerékjártó, *Vorlesungen über Topologie*, Verlag von Julius Springer, Berlin, 1923.
9. B. v. Kerékjártó, *Topologische Characterisierung der linearen Abbildungen*, Acta Sci. Math. Szeged, **6** (1934), 235-262.
10. K. Kuratowski, *Topologie*, vol. I, Academic Press, New York and London, 1966. (Theorem 3, pg. 185).
11. R. L. Moore, *Concerning the common boundary of two domains*, Fund. Math., **6** (1924), 203-213.
12. E. van Kampen, *The topological transformations of a simple closed curve into itself*, American J. Math., **57** (1935), 142-152.
13. Deane Montgomery, *Almost periodic transformation groups*, TAMS, **42** (1937), 322-332.
14. A. P. Wu, *Orientation Preserving Periodic Homeomorphisms on the Plane*, Masters' Thesis, University of Fla., Aug. 1971.

Received September 20, 1974.

UNIVERSITY OF FLORIDA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 59, No. 2 June, 1975