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TRANSVERSALS OF LATIN SQUARES AND THEIR GENERALIZATIONS

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The main theme in this paper is the existence of a transversal with many distinct elements in an array more general than a latin square.

A transversal of a latin square of order n is any set of n cells such that no two come from the same column or same row. There has been a good deal of effort spent on establishing the existence of a transversal that has many distinct elements, e.g. [4, 5]. A close inspection of the argument in [5] reveals that the results there apply in a context far more general than that explicitly considered. Indeed, the assumptions that there are no duplications in a row or column can in some cases be dropped.

A variety of conjectures conclude the paper.

1. **Definitions.** An n -square is an n by n array of n^2 cells in each of which one of the symbols $1, 2, 3, \dots$ appears. An n -square in which each symbol from 1 to n appears n times is called an equi- n -square.

If $m < n$, an (m, n) -rectangle is an m by n array of mn cells in each of which one of the symbols $1, 2, 3, \dots$ appears. There are m rows and n columns.

A *transversal* of an n -square or an (m, n) -rectangle is a set of cells, one from each row and no two from the same column. A *partial transversal* is a subset of a transversal. A transversal is *latin* if no two cells have the same symbol. Since a latin transversal need not contain all the symbols in the array, we do not use the traditional term, "complete". A row (or column) of an n -square is *latin* if no two of its cells contain the same symbol.

Note that a usual latin square can be described as an equi- n -square for which each row and each column is latin. Observe that a latin square is an equi- n -square.

2. **Survey of results.** Ryser in [10] conjectured that a latin square of odd order n has a latin transversal. Koksma [4] proved that a latin square of order n has a transversal with at least $(2n + 1)/3$ distinct symbols. Lindner and Perry, in a mimeographed publication [5], proved that the average number of distinct symbols in transversals of a latin n -square (taken over all transversals) is precisely

$$n\left(1 - \frac{1}{2!} + \frac{1}{3!} - \dots \pm \frac{1}{n!}\right).$$

From this it follows that there is a transversal with at least $\lfloor(1 - 1/e)n\rfloor = .63n$ elements. Because Koksma's result is stronger, [5] was not formally published.

This paper utilizes the technique of Lindner and Perry, which might be called "existence by averaging", to establish, for instance, that an equi- n -square has a transversal with a least $\lfloor(1 - 1/e)n\rfloor$ distinct symbols (Corollary 3.3). Koksma's technique, on the other hand, using all his assumptions, does not seem to be easily generalized.

Bruck proved that the Cayley table of a group of odd order has a transversal (namely the main diagonal). This follows from the fact that such a group is the union of cyclic groups of odd order and hence every element is the square of some element. Paige [7] proved that any finite abelian group that is not of the form $C(2^n) \times H$, where $C(2^n)$ is the cyclic group of order 2^n , $n \geq 1$, and H has odd order, possesses a latin transversal. In [3] Hall generalized this result.

3 Transversals of n -squares. For a subset X of the n^2 cells of an n -square, let $t(X)$ denote the number of transversals that meet X . This number, examined in the context of determinants of matrices with 0-entries in X , has been the subject of some study (see Netto ([6], p. 73)). In the case where X is itself a transversal or a subset of a transversal a formula for $t(X)$ is known (see [6], [9]). It is given in the following lemma, which is another version of the "hatcheck problem".

LEMMA 3.1. *Let X be a set of q cells in an n -square such that no two lie in the same column or in the same row. Then*

$$t(X) = n! \left(\frac{q}{n} - \binom{q}{2} \frac{1}{n(n-1)} + \binom{q}{3} \frac{1}{n(n-1)(n-2)} - \dots \pm \binom{q}{q} \frac{1}{n(n-1) \dots (n-q+1)} \right).$$

The next lemma implies that a set that is not a partial transversal meets at least as many transversals as does a partial transversal of the same cardinality.

LEMMA 3.2. *Let X be a set consisting of q cells, $q \leq n$, in an n -square. Then $t(X) \geq t(Z)$, where Z is a set of q cells in an n -*

square that form a partial transversal.

Proof. Assume that X has at least two cells in the same row. (A similar argument applies if some column contains at least two cells of X .) Let c be a cell of X in the row mentioned. Let Y be the set of q cells obtained from X by deleting cell c and adjoining a cell c' in a row not meeting X , but in the same column as c . Let X' be the set of cells in X that are not in the row containing c . Let $X'' = X - \{c\}$. Thus $X'' \supset X'$.

Now, $t(X)$ equals:

- the number of transversals that meet X'' , but not c or c'
- + $t(\{c\})$
- + the number of transversals that meet X'' and also c' .

On the other hand, $t(Y)$ equals:

- the number of transversals that meet X'' , but not c or c'
- + $t(\{c'\})$
- + the number of transversals that meet both X'' and c .

To compare these two sums, observe first that the first terms of each are the same and that $t(\{c\}) = t(\{c'\})$. Also,

the number of transversals that meet both X'' and c equals

the number of transversals that meet both X' and c , which equals

the number of transversals that meet both X' and c' .

Since $X'' \supset X'$, it follows by comparison of the third terms of the sums for $t(X)$ and $t(Y)$ that $t(X) \geq t(Y)$. Repeated application of this argument, at most $q - 1$ times, establishes the lemma.

The following theorem and its corollary generalizes the result of Lindner and Perry from latin n -squares to n -squares.

THEOREM 3.2. *In an n -square in which each symbol $1, 2, \dots, s$ appears at least q times, $q \leq n$, there is a transversal that contains at least*

$$s \left[\frac{q}{n} - \binom{q}{2} \frac{1}{n(n-1)} + \binom{q}{3} \frac{1}{n(n-1)(n-2)} - \dots \pm \binom{q}{q} \frac{1}{n(n-1) \dots (n-(q-1))} \right]$$

distinct symbols.

Proof. Let U be the set of ordered pairs (t, i) where symbol i

is contained in transversal t . The cardinality of U is equal to

$$n! \cdot (\text{the average number of distinct symbols in all transversals of the } n\text{-square}).$$

On the other hand, since there are s symbols, U has cardinality

$$s \cdot (\text{the average number of transversals that contain a given symbol}).$$

Let X_i be the set of cells occupied by the symbol i . Since $|X_i| \geq q$, $t(X_i)$ is greater than or equal to the number of transversals that meet q diagonal elements, by Lemma 3.2. Comparison of these two expressions for the cardinality of U together with Lemma 3.1 establishes the theorem.

The case $q = n$ is singled out in the following corollary.

COROLLARY 3.3. *In an equi- n -square there is a transversal that contains at least*

$$n \left(1 - \frac{1}{2!} + \frac{1}{3!} - \dots \pm \frac{1}{n!} \right)$$

distinct symbols.

It is not clear how much Corollary 3.3 can be strengthened. Koksma's argument for $(2n + 1)/3$ does not apply to equi- n -squares, since it makes use in several places of the assumption that each row and each column is latin. Moreover, Ryser's conjecture is not valid for equi- n -squares, where n is odd and at least 3. To see this, consider the equi- n -square whose first $n - 1$ rows each consist of the symbols $1, 2, \dots, n$ in order, and whose n^{th} row is the same set of symbols, in the order $2, 3, \dots, n, 1$. It is a simple matter to show that it does not have a latin transversal. Note, incidentally, that each row of this equi- n -square is latin.

The proofs of the next two theorems, being similar to that of Theorem 3.2, are only sketched.

THEOREM 3.4. *Let n be even and at least 4. Let each of $n^2/2$ symbols appear twice in an n -square. Then there is a transversal that contains n distinct symbols.*

Proof.

$$n! \cdot (\text{the average number of distinct symbols in a transversal})$$

$$= \frac{n^2}{2} \cdot (\text{the average number of transversals that contain a given symbol}).$$

Thus

$n!$ (the average number of distinct symbols in a transversal)

$$\geq \frac{n^2}{2} \cdot n! \left(\frac{2}{n} - \frac{1}{n(n-1)} \right).$$

Hence the average number of distinct symbols

$$\geq n - \frac{1}{2} \cdot \frac{n}{n-1}.$$

If $n \geq 4$, there is consequently a transversal with n distinct symbols.

The next theorem is a companion of Theorem 3.4.

THEOREM 3.5. *Let q be greater than 2 and let n be a positive multiple of q . Let each of n^2/q symbols appear q times in an n -square. Then some transversal contains more than $n-q/2$ distinct symbols.*

Proof. There is a transversal for which the number of distinct symbols is at least

$$\frac{n^2}{q} \left(\frac{q}{n} \frac{1}{n} - \frac{q \cdot q - 1}{1 \cdot 2} \cdot \frac{1}{n(n-1)} + \frac{q \cdot q - 1 \cdot q - 2}{1 \cdot 2 \cdot 3} \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} + \dots \frac{q(q-1) \dots 1}{1 \cdot 2 \dots q} \frac{1}{n} \frac{1}{n-1} \dots \frac{1}{n-q+1} \right).$$

Hence, there is one with more than

$$\frac{n^2}{q} \left(\frac{q}{n} - \frac{q(q-1)}{2} \frac{1}{n(n-1)} \right)$$

distinct symbols. Since $n \geq q$ the theorem follows.

4. Transversals of (m, n) -rectangles. The method used in Section 3 also applies to (m, n) -rectangles. However, this section will illustrate a different averaging process, much simpler, and only slightly weaker. It employs the notion of a "singular" pair of cells. Two cells in different rows and different columns form a *singular* pair if they contain the same symbol. The method is based on a count of incidences of transversals and singular pairs.

THEOREM 4.1. *Let q divide mn and let each symbol in an (m, n) -rectangle appear q times. Then there is a transversal with at most*

$$\frac{m(q-1)}{2(n-1)}$$

singular pairs.

Proof. Count the set of ordered pairs (t, p) , where t is a transversal and p is a singular pair in t . Counting in both orders yields

$$\begin{aligned} & n(n-1) \cdots (n-m+1) \text{ (average number of singular} \\ & \text{pairs on a transversal)} \\ & \leq \frac{mn}{q} \cdot \frac{q(q-1)}{2} \cdot \text{(average number of transversals on} \\ & \text{a singular pair)} \\ & \leq \frac{mn}{q} \cdot \frac{q(q-1)}{2} (n-2)(n-3) \cdots (n-m+1). \end{aligned}$$

The theorem follows immediately.

The following corollaries are immediate consequences.

COROLLARY 4.2. *If each symbol in an (m, n) -rectangle appears n times, then there is a transversal with at least $m/2$ distinct symbols.*

COROLLARY 4.3. *If each symbol in an (m, n) -rectangle appears q times and if*

$$\frac{m(q-1)}{2(n-1)} < 1,$$

then there is a latin transversal.

A special case of Corollary 4.3 is given by the following.

COROLLARY 4.4. *If each symbol in an (m, n) -rectangle appears m times, and if*

$$n > \frac{m^2 - m + 2}{2},$$

then there is a latin transversal.

The method of Section 3 yields a slightly stronger result, which implies that “ $>$ ” can be replaced by “ \geq ” in Corollary 4.4.

5. **Rows or columns with many distinct symbols.** The “existence by averaging” technique may also be applied to establish the existence of a row or column in an n -square with “many” distinct symbols.

THEOREM 5.1. *Let the cells of an n -square be occupied by the symbols $1, 2, \dots, k$, with i appearing n_i times, $1 \leq i \leq k$. Then some row or column contains at least*

$$\frac{1}{n}(\sqrt{n_1} + \sqrt{n_2} + \dots + \sqrt{n_k})$$

different symbols.

Proof. Let U be the set of ordered pairs (L, i) , where L is a line (either a row or a column) that contains the symbol i . Since there are $2n$ such lines, U has

$$2n \cdot (\text{average number of distinct symbols in a line}).$$

On the other hand, U has

$$k \cdot (\text{average number of lines that contain a given symbol}).$$

To evaluate the second average, let $L(i)$ be the number of lines that contain the symbol i . Let $R(i)$ be the number of rows and $C(i)$ be the number of columns that contain i . Thus $L(i) = R(i) + C(i)$.

Now, the set of cells occupied by i is contained in the intersection of $R(i)$ rows and $C(i)$ columns. Consequently

$$R(i) \times C(i) \geq n_i.$$

It follows that

$$R(i) + C(i) \geq 2\sqrt{n_i},$$

hence that

$$L(i) \geq 2\sqrt{n_i}.$$

Thus

$$\sum_{i=1}^k L(i) \geq \sum_{i=1}^k 2\sqrt{n_i},$$

from which the theorem follows.

The specialization of Theorem 5.1 to an equi- n -square is described in the next corollary.

COROLLARY 5.2. *In an equi- n -square there is a row or a column that contains at least \sqrt{n} distinct symbols.*

G. D. Chakerian and D. Hickerson have independently shown that Corollary 5.2 is best possible if it is not required the set of cells occupied by a given symbol be topologically connected.

6. Conjectures. The following conjectures, some of which are logically related, may suggest directions for further study.

(1) An equi- n -square has a transversal with at least $n - 1$ distinct symbols.

(2) An n -square in which each symbol appears at most $n - 1$ times has a latin transversal. (It is easy to show by induction, or by either averaging method that if each symbol in an n -square, $n \geq 3$, appears at most two times, the n -square has a latin transversal.)

(3) An $(n - 1, n)$ -rectangle in which each symbol appears at most n times has a latin transversal.

(4) A row-latin $(n - 1, n)$ -square has a latin transversal.

(5) An (m, n) -rectangle in which each symbol appears at most n times has a latin transversal.

Note that Conjectures (3) and (5) are equivalent. Moreover, for $m = 1$, Conjecture (5) is immediate. For $m = 2$, Conjecture (5) is valid with the weaker assumption that each symbol appears at most $2n - 1$ times.

(6) An $(n - 1, n)$ -rectangle in which each symbol appears exactly n times has a latin transversal.

(7) An (m, n) -rectangle in which each symbol appears at most $m + 1$ times has a latin transversal.

REFERENCES

1. L. D. Baumert, et al, *A combinatorial packing problem*, Computers in Algebra and Number Theory, vol. 4, SIAM/AMS Proceedings, 1971.
2. G. D. Chakerian and S. K. Stein, *On measures of symmetry of convex bodies*, *Canad. J. Math.*, **17** (1965), 497-504.
3. M. Hall, *A combinatorial problem on abelian groups*, *Proc. Amer. Math. Soc.*, **3** (1952), 584-587.
4. K. K. Koksma, *A lower bound for the order of a partial transversal of a Latin square*, *J. Combinatorial Theory*, **7** (1969), 94-95.
5. C. C. Lindner and N. C. Perry, *Transversals of Latin squares*, 1970, Auburn University, mimeographed.
6. E. Netto, *Lehrbuch der Combinatorik*, Chelsea, N.Y.
7. L. J. Paige, *A note on finite abelian groups*, *Bull. Amer. Math. Soc.*, **53** (1947), 590-593.

8. ———, *Complete mappings of finite groups*, Pacific J. Math., **1** (1951), 111-116.
9. H. J. Ryser, *Combinatorial Mathematics*, Wiley, 1963.
10. ———, *Neure Probleme der Kombinatorik in Vorträge über Kombinatorik*, Oberwolfach 24-29 Juli, 1967. Mathematischen Forschungsinstitut, Oberwolfach, 1968.
11. S. K. Stein, *The symmetry function in a convex body*, Pacific J. Math., **6** (1956), 145-148.

Received November 27, 1974. For a general survey of transversals in latin n -squares see J. Dénes and A. D. Keedwell, *Latin squares and their applications*, Academic Press, 1974. Incidentally, it is mentioned there that the analog of Conjecture (1) has been proposed for latin n -squares.

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