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ON HOMOGENEOUS ALGEBRAS

LOWELL G. SWEET

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If A is an algebra over a field K let $\text{Aut}(A)$ denote the group of algebra automorphisms of A . Then A is said to be **extremely homogeneous** if $\text{Aut}(A)$ act transitively on $A \setminus \{0\}$. Also A is said to be **homogeneous** if $\text{Aut}(A)$ acts transitively on the one-dimensional subspaces of A . The purpose of this paper is to investigate some of the basic properties of homogeneous algebras. In particular, the alternative homogeneous algebras and the homogeneous algebras of dimension 2 are classified.

All algebras are assumed to be finite dimensional and not necessarily associative.

We now include a brief historical account of this topic. The concept of an extremely homogeneous algebra arose from a particular problem in the structure of certain finite p -groups as studied by Boen, Rothaus and Thompson [1]. Extremely homogeneous algebras have been investigated by Kostrikin [4]. Homogeneous algebras over finite fields other than $GF(2)$ have been investigated by Shult [6], [7], and his results completed the work on the related p -groups. The case of homogeneous algebras over $GF(2)$ was considered by Gross [3]. Swierczkowski classified all real homogeneous Lie algebras [9] and finally Dykovic classified all real homogeneous algebras [2]. A homogeneous algebra A is said to be nontrivial if $A^2 \neq 0$ and $\dim A > 1$. The author has shown that there are no nontrivial homogeneous algebras over an algebraically closed field [8].

The paper is divided into five sections: arbitrary homogeneous algebras, alternative homogeneous algebras, power-associative homogeneous algebras, homogeneous quasi-division algebras and finally homogeneous algebras of dimension 2.

I. Arbitrary homogeneous algebras. Let A be an arbitrary algebra over a field K . Then left multiplication by a fixed element $a \in A$ induces a linear map on A which is denoted by L_a . Similarly right multiplication by a induces a linear map on A denoted by R_a . We do not distinguish between the map L_a and its matrix representation relative to some fixed basis. By $\text{End}(A)$ we indicate the vector space of all linear maps on A . By L we indicate the subspace of $\text{End}(A)$ consisting of all L_x as x runs through A and similarly for R . An algebra A is said to be nonzero if $A^2 \neq 0$.

THEOREM 1. *Let A be a nonzero homogeneous algebra over a*

field K . Then

- (i) $\dim L = \dim R = \dim A$
- (ii) If $a, b \in A \setminus \{0\}$ then L_a and L_b are projectively similar and similarly for R_a and R_b ,
- (iii) $\text{Aut}(A)$ acts as a transitive group of collineations on the points of the projective geometry $P(A)$.

Proof. (1) Let $a \in A \setminus \{0\}$. Then if $aA = 0$ the homogeneity condition implies that $A^2 = 0$ which is a contradiction. This fact implies that the map $\phi: x \mapsto L_x$ is a linear isomorphism and so $\dim L = \dim A$. Similarly it can be shown that $\dim R = \dim A$.

(2) The proof is a simple generalization of a related result found in the introduction of the paper by Boen, Rothaus, and Thompson [1].

(3) This is obvious since the points of $P(A)$ are exactly the one-dimensional subspaces of A .

THEOREM 2. *Let A be a nontrivial homogeneous algebra over a field K . Then*

$$\text{tr } L_x = \text{tr } R_x = 0 \quad \forall x \in A$$

Proof. Let $\dim A = n$. It is well known that $\text{tr}: \text{End}(A) \rightarrow K$ is a linear functional and that $\dim \ker(\text{tr}) = n^2 - 1$. But then since $\dim L = \dim A = n > 1$ it follows that $L \cap \ker(\text{tr}) \neq 0$ and so there must exist at least one nonzero map $L_a \in L$ such that $\text{tr } L_a = 0$. But now the second result of the previous theorem implies that $\text{tr } L_x = 0$ for all $x \in A$. Similarly $\text{tr } R_x = 0$ for all $x \in A$.

THEOREM 3. *Let A be a homogeneous algebra over a field K and let $a \in A \setminus \{0\}$. If $\langle a \rangle$ denotes the subalgebra of A generated by a then $\langle a \rangle$ is also a homogeneous algebra enjoying the property that it is generated by each of its nonzero elements. Also $A = \bigcup A_i$ where each $A_i = \langle a_i \rangle$ for some $a_i \in A \setminus \{0\}$ and $A_i \cap A_j = \{0\}$ for $i \neq j$.*

Proof. Let $b \in \langle a \rangle$. Clearly $\langle b \rangle \subseteq \langle a \rangle$. But there must exist $\alpha \in \text{Aut}(A)$ such that $\alpha(a) = \lambda b$ for some nonzero $\lambda \in K$ and this implies that $\langle a \rangle \subseteq \langle b \rangle$ and so $\langle a \rangle = \langle b \rangle$. That is $\langle a \rangle$ is generated by each of its nonzero elements. Now let c and d be any nonzero elements in $\langle a \rangle$. Again there must exist $\beta \in \text{Aut}(A)$ such that $\beta(c) = \lambda d$ for some nonzero $\lambda \in K$. But the fact that both c and d generate $\langle a \rangle$ implies that $\langle a \rangle$ is invariant under β and so the restriction of β to $\langle a \rangle$ is in $\text{Aut}(\langle a \rangle)$. That is, $\langle a \rangle$ is also a homogeneous algebra. The final statement again follows directly from the fact that $\langle a \rangle$ is generated by each of its nonzero elements.

The above theorem implies that in some situations it is sufficient to consider the case where a homogeneous algebra A is generated by each of its nonzero elements.

DEFINITION. Let V be a vector space over a field K and suppose H is a subgroup of $GL(V)$ where $GL(V)$ is the general linear group. Then $C(H)$ is defined as

$$C(H) = \{u \in \text{End}(A) \mid uv = vu \text{ for all } V \in H\}.$$

DEFINITION. Let A be an algebra over a field K and suppose $S, T \in C(\text{Aut}(A))$. Then $A(S, T)$ indicates a new algebra which coincides with A when considered as a vector space over K but possesses a new multiplication defined by

$$a \circ b = S(a)b + T(b)a \text{ for all } a, b \in A$$

Note that the fact that S and T are linear maps on A ensure that $\circ: A \times A \rightarrow A$ is a bilinear map. Also the algebras $A(1, 1)$, $A(1, -1)$ and $A(0, 1)$ are well known and are usually denoted as A^+ , A^- and A^{opp} respectively.

THEOREM 4. *Let A be a homogeneous algebra over a field K and suppose $S, T \in C(\text{Aut}(A))$. Then $A(S, T)$ is also a homogeneous algebra.*

Proof. Let $\sigma \in \text{Aut}(A)$. Then

$$\begin{aligned} \sigma(a \circ b) &= \sigma(S(a)b + T(b)a) \\ &= \sigma(S(a)b) + \sigma(T(b)a) \\ &= (\sigma S(a))\sigma(b) + (\sigma T(b))\sigma(a) \\ &= (S\sigma(a))\sigma(b) + (T\sigma(b))\sigma(a) \\ &= \sigma(a) \circ \sigma(b) \end{aligned}$$

and so the result is true since $\text{Aut}(A) \subset \text{Aut}(A(S, T))$

DEFINITION. Let A be an algebra over a field K . Then A is left (right) simple if A possesses no nonzero proper left (right) ideals. Also A is simple if A possesses no nonzero, proper, two-sided ideals and $A^2 \neq 0$.

THEOREM 5. *If A is a nonzero homogeneous algebra then A is left simple and right simple.*

Proof. Assume that A has proper nonzero left ideals. When

B runs through minimal left ideals then the sets $B \setminus \{0\}$ form a partition of $A \setminus \{0\}$. Suppose $a \in A \setminus \{0\}$ and let $I(a)$ denote the minimal left ideal which contains a . Now R_a map $A \rightarrow I(a)$ and since $I(a) \neq A$ it follows that R_a has a nonzero kernel. That is, there exists $b \in A \setminus \{0\}$ such that $ba = 0$. Let c be any point in $A \setminus I(a)$. Then $I(c) \cap I(a) = \{0\}$ which implies that $I(c) \cap I(c + a) = \{0\}$. But $b(c + a) = bc$ and so $bc \in I(c) \cap I(c + a)$ which implies that $bc = 0$. Now fix some nonzero $c \in A \setminus I(a)$ and let d be any point in $I(a) \setminus \{0\}$. Then $c + d \in A \setminus I(a)$ and so $b(c + d) = bd = 0$. Hence $bA = 0$ which is impossible since A is a nonzero homogeneous algebra. Hence A has no proper nonzero left ideals and similarly A has no proper nonzero right ideals.

II. Alternative homogeneous algebras. The following definition is well known.

DEFINITION. An algebra A over a field K is said to be alternative if

$$\begin{aligned} a^2b &= a(ab) \\ ab^2 &= (ab)b \end{aligned}$$

for all $a, b \in A$.

THEOREM 6. *There are no nontrivial alternative homogeneous algebras.*

Proof. Let A be a nontrivial alternative homogeneous algebra. Then the previous theorem implies that A is simple. But it is known that a simple alternative algebra has an identity element 1 (see Corollary 3.11 of Schafer's book [5]). But then A is certainly not homogeneous since $\alpha(1) = 1$ for all $\alpha \in \text{Aut}(A)$.

Note that the above theorem of course implies that there are no nontrivial associative homogeneous algebras.

III. Power-associative homogeneous algebras.

THEOREM 7. *Let A be a power-associative nontrivial homogeneous algebra over a field K . Then either $a^2 = 0$ for all $a \in A$ or $a^2 = a$ for all a in A and in the latter case A is a Jordan algebra and $K = GF(2)$.*

Proof. Let a be some fixed element in $A \setminus \{0\}$. Then Theorem 3 implies that $\langle a \rangle$ is an associative homogeneous algebra and so the previous theorem implies that $\langle a \rangle$ is a trivial homogeneous algebra.

It follows that either $a^2 = 0$ or $a^2 = \lambda a$ for some nonzero $\lambda \in K$. In the former case the homogeneity condition implies that $x^2 = 0$ for all $x \in A$ and so we may assume the latter case. The homogeneity condition implies that $x^2 = \lambda(x)x$ where $\lambda(x)$ is a nonzero scalar in K possibly depending on x . Since $\dim A > 1$ we may choose two independent vectors in A , say e_1 and e_2 . Since $a^2 = \lambda a$ implies that $(a/\lambda)^2 = a/\lambda$ we may assume without loss of generality that both e_1 and e_2 are idempotents. It is now necessary to perform several simple calculations. First

$$\begin{aligned}(e_1 + e_2)^2 &= e_1 + e_2 + e_1e_2 + e_2e_1 = \lambda(e_1 + e_2)(e_1 + e_2) \\ (e_1 - e_2)^2 &= e_1 + e_2 - e_1e_2 - e_2e_1 = \lambda(e_1 - e_2)(e_1 - e_2)\end{aligned}$$

Now adding and comparing coefficients gives

$$\begin{aligned}2 &= \lambda(e_1 + e_2) + \lambda(e_1 - e_2) \\ 2 &= \lambda(e_1 + e_2) - \lambda(e_1 - e_2)\end{aligned}$$

or

$$2\lambda(e_1 - e_2) = 0$$

which implies that $\text{char } K = 2$.

For convenience let $\mu = \lambda(e_1 + e_2)$. Then from above

$$e_1e_2 + e_2e_1 = (\mu + 1)(e_1 + e_2).$$

Now consider

$$\begin{aligned}(e_1 + e_2)^2 &= e_1 + \mu^2e_2 + \mu(e_1e_2 + e_2e_1) \\ &= (\mu^2 + \mu + 1)e_1 + \mu e_2\end{aligned}$$

from which it follows that $\mu^2 + \mu + 1 = 1$ which implies with $\text{char } K = 2$ that $\mu = 1$ and so

$$e_1e_2 + e_2e_1 = 0.$$

Now let δ be any nonzero scalar in K . Then

$$(e_1 + \delta e_2)^2 = e_1 + \delta^2e_2 + \delta(e_1e_2 + e_2e_1) = e_1 + \delta^2e_2$$

which implies that $\delta^2 = \delta$ and so $\delta = 1$ and indeed $K = GF(2)$. Hence $x^2 = x$ for all $x \in A$. But then

$$(x + y)^2 = x + y + xy + yx = x + y$$

and so $xy = yx$ for all $x, y \in A$ and thus A is a commutative algebra. The second identity for a Jordan algebra is trivially satisfied and so A is a Jordan algebra over $GF(2)$.

It is interesting to note that Dyokovic has shown that all nontrivial real homogeneous algebras are of the first type [2] and Gross has shown that some, but not all, of the known homogeneous algebras over $GF(2)$ are of the second type [3].

IV. Homogeneous quasi-division algebras.

DEFINITION. An algebra A over a field K is said to be a quasi-division algebra if the nonzero elements of A form a quasi-group under multiplication.

One of the reasons for devoting a separate section to homogeneous quasi-division algebras is that Shult [6] and Gross [3] have shown that all nontrivial finite homogeneous algebras are in fact quasi-division algebras.

THEOREM 8. *Let A be a nontrivial homogeneous quasi-division algebra with the property that A is generated by each of its nonzero elements. Then*

(i) $\text{Aut}(A)$ is sharply transitive on the one-dimensional subspaces of A

(ii) If a is any element in $A \setminus \{0\}$ then L_a has precisely one eigenvalue denoted by $\lambda_a \in K$ and the corresponding eigenspace is one-dimensional

(iii) Finally $\lambda_a = \lambda_b$ if and only if there exists some $\alpha \in \text{Aut}(A)$ such that $\alpha(a) = b$.

Proof. (1) It is sufficient to show that no automorphism of A , except the identity Id , can have an eigenvalue in K . Let $\alpha \in \text{Aut}(A)$ and suppose that α has eigenvalue $\lambda \in K$. Then there exists $a \in A \setminus \{0\}$ such that

$$\alpha(a) = \lambda a.$$

Since A is not associative by Theorem 6 we define inductively

$$a^n = L_a^{n-1}(a) \qquad n = 2, 3, 4, \dots$$

But now

$$\alpha(a^n) = \lambda^n a^n \qquad n = 1, 2, 3, \dots$$

and so there must exist positive integers m, n with $m > n$ such that $\lambda^m = \lambda^n$ since α can only have a finite number of eigenvalues. Letting $k = m - n$ we have

$$\alpha(a^k) = \lambda^k a^k = a^k \neq 0$$

and so $\alpha = \text{Id}$ since from the hypothesis we are assuming that a^k generates A .

(2) Let a and b be any two nonzero elements of A . Since A is a quasi-division algebra the equation

$$xb = b$$

must have a solution, say c and the homogeneity condition implies that there exists $\alpha \in \text{Aut}(A)$ such that

$$\alpha(c) = \lambda a \quad \text{for some } \lambda \in K \setminus \{0\}.$$

But then

$$a\alpha(b) = 1/\lambda\alpha(b)$$

and so L_a has at least one eigenvalue.

Now suppose there exist nonzero elements $b, c \in A$ such that

$$ab = \lambda b$$

$$ac = \mu c$$

where $\lambda, \mu \in K$. If $\{b, c\}$ is an independent set then there must exist $\alpha \in \text{Aut}(A)$ such that

$$\alpha(c) = \delta b \quad \text{for some } \delta \in K.$$

But then

$$\alpha(a)b = \mu b$$

and thus

$$(\lambda\alpha(a) - \mu a)b = 0$$

which implies that $\alpha = \text{Id}$ by the previous part of this theorem. Thus L_a has precisely one eigenvector (up to a scalar multiple) which completes the proof of the second statement.

(3) Finally if $\alpha \in \text{Aut}(A)$ then $ax = \lambda_a x$ for some $\lambda \in A \setminus \{0\}$ implies that

$$\alpha(a)\alpha(x) = \lambda_a\alpha(x)$$

and so

$$\lambda_{\alpha(a)} = \lambda_a$$

also if $\lambda_a = \lambda_b$ then there exists $x, y \in A \setminus \{0\}$ such that

$$ax = \lambda_a x$$

$$by = \lambda_b y = \lambda_a y.$$

Now choose $\beta \in \text{Aut}(A)$ such that $\beta(x) = \mu y$ for some $\mu \in K \setminus \{0\}$ and applying β we obtain

$$\beta(a)y = \lambda_a y = by$$

and so it follows that $\beta(a) = b$ as required.

IV. On homogeneous algebras of dimension 2. We now investigate arbitrary homogeneous algebras of dimension 2.

THEOREM 9. *Let A be a nonzero, 2-dimensional, homogeneous algebra over a field K . Then $K = GF(2)$ and A has a basis $\{a, b\}$ so that A is isomorphic to one of the following algebras.*

	a	b
a	a	$a + b$
b	$a + b$	b

	a	b
a	b	a
b	a	$a + b$

Proof. Let $a \in A \setminus \{0\}$. Then there are exactly three possibilities which will be considered separately

(i) $a^2 = 0$

(ii) $a^2 = \lambda a$ for some nonzero $\lambda \in K$

(iii) $\{a, a^2\}$ is a basis of A

(1) If $a^2 = 0$ then the homogeneity condition implies that $x^2 = 0$ for all $x \in A$ and the linearized form of this identity implies that A is anticommutative. Extend a to a basis of A , say $\{a, b\}$. Using the fact that $\text{tr } L_a = 0$ and $L_a \neq 0$ it follows that $ab = \lambda a$ for some nonzero $\lambda \in K$. But now $ab = \lambda a$ and $b^2 = 0$ imply that $\text{tr } L_b = -\lambda \neq 0$ which is impossible. Hence this case does not occur.

(2) If $a^2 = \lambda a$ where $\lambda \neq 0$ then the homogeneity condition implies that A is power-associative and so Theorem 7 implies that $K = GF(2)$. Again extend a to a basis of A , say $\{a, b\}$. Using the fact that $\text{tr } L_a = \text{tr } L_b = 0$ and $L_a \neq 1$ and $L_b \neq 1$ it follows that A must be of the form

	a	b
a	a	$a + b$
b	$a + b$	b

By direct computations it can be shown that $\text{Aut}(A) = GL(2, 2)$ and so $\text{Aut}(A)$ is in fact triply transitive on $A \setminus \{0\}$.

(3) Suppose that $\{a, a^2\}$ is a basis of A . First pass from A to A^- . By Theorem 4, A^- is also a homogeneous algebra and clearly A^- is of type (1) as defined above and so A^- must be a zero algebra

which implies that A is commutative. If $aa^2 = 0$ then $\text{tr } L_{a^2} = 0$ and $L_{a^2} \neq 0$ implies that L_a is nilpotent but L_{a+a^2} is invertible and so A is a quasi-division algebra generated by each of its nonzero elements and so we may apply Theorem 8. Assume $aa^2 = \mu a$.

Let b be any fixed nonzero element of A . The equation $xb = b$ must have a solution and without loss of generality we may assume that $x = a$. Hence the only eigenvalue of L_a is 1 and it follows that $\mu = 1$ and $\text{char } K = 2$. Also $a^2a^2 = va + a^2$ for some nonzero $v \in K$. Now since L_a and L_{a^2} both have eigenvalue 1 it follows from Theorem 8 that there must exist $\alpha \in \text{Aut}(A)$ such that $\alpha(a) = a^2$. But then

$$\begin{aligned}\alpha(a^3) &= \alpha(a)\alpha(a) = a^2a^2 = va + a^2 \\ \alpha(a^2a^2) &= \alpha(va + a^2) = va^2 + va + a^2 \\ &= \alpha(a^2)\alpha(a^2) = (va + a^2)(va + a^2) = v^2a^2 + va + a^2.\end{aligned}$$

It follows that $v = 1$ and so the multiplication table of A is of the form

	a	b
a	a^2	a
b	a	$a + a$

If $K = GF(2)$ it is easily shown that A is in fact a homogeneous algebra. If $K = GF(4)$ it can be shown that $\det(L_a + \lambda L_{a^2}) = 1 + \lambda + \lambda^2 = 0$ for some $\lambda \in GF(4)$ and so A is not homogeneous since it is not a quasi-division algebra. Now assume that $K \neq GF(2)$ and $K \neq GF(4)$. Then there must exist $\lambda_0 \in K$ such that λ_0 is not a root of the polynomial $x^2 + x + 1$ or of the polynomial $x^4 + x^3 + x^2 + 1$. Since A is homogeneous there must exist $\alpha \in \text{Aut}(A)$ such that

$$\alpha(a) = \lambda(a + \lambda_0 a^2) \quad \text{for some nonzero } \lambda \in K.$$

But then

$$\begin{aligned}\alpha(aa^2) &= \lambda^3(1 + \lambda_0 + \lambda_0^2)a + \lambda^3\lambda_0(1 + \lambda_0 + \lambda_0^2)a^2 \\ &= \alpha(a) = \lambda a + \lambda\lambda_0 a^2\end{aligned}$$

and so

$$(1) \quad \lambda^2 = \frac{1}{1 + \lambda_0 + \lambda_0^2}.$$

Also

$$\begin{aligned}\alpha(a^2a^2) &= \lambda^4(1 + \lambda_0^4)a + \lambda^4a^2 \\ &= \alpha(a) + \alpha(a^2) \\ &= (\lambda + \lambda^2\lambda_0^2)a + [\lambda\lambda_0 + \lambda^2(1 + \lambda_0^2)]a^2\end{aligned}$$

which implies using (1) that

$$(2) \quad \lambda^2 = \frac{1 + \lambda_0^2 + \lambda_0^4}{1 + \lambda_0^4 + \lambda_0^6}$$

and together (1) and (2) imply that

$$\lambda_0^4 + \lambda_0^3 + \lambda_0^2 + 1 = 0$$

which contradicts our choice of λ_0 . Hence A is a homogeneous algebra if and only if $K = GF(2)$.

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