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RICHARD THOMAS BUMBY AND DAVID EARL DOBBS

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Computations of Amitsur cohomology (in the units functor U) for extensions of rings of algebraic integers have been achieved in two ways: via Mayer-Vietoris sequences (by Morris and Mandelberg) and via cohomology in the functor UK/U (by the second-named author). One of the goals of these computations has been to shed light on the Chase-Rosenberg homomorphism from Amitsur cohomology to the split Brauer group. In this paper we obtain, for quadratic ring extensions, formulas for cohomology in U and in UK/U , which have wider application than the corresponding work of Morris and Mandelberg. Our formulas lead to examples showing that the Chase-Rosenberg homomorphism, arising from a quadratic extension of rings of algebraic integers, need not be injective or surjective.

Our methods are direct and, in particular, avoid explicit use of Mayer-Vietoris sequences. Section 2 studies the embedding of certain Amitsur cochains in Cartesian products. Section 3 contains the cohomology computations which, together with [4, Corollary 1.5] and the Hasse norm principle of class field theory, lead to the desired examples in §4.

We employ the standard notation concerning Amitsur cohomology (cf. [1, p. 29]) and assume familiarity with [4, §1].

2. Cochain and coboundary computations. The *standing hypotheses* for §§2 and 3 are that R is an integral domain with quotient field K , that S is a flat R -subalgebra of a quadratic (two-dimensional separable) field extension L of K , and that the Galois group G of L/K fixes S as a set.

Note that R -flatness of S allows us to view $S^i = \bigotimes_R^i S$ as an R -subalgebra of $L^i = \bigotimes_K^i L$. Since G maps S into itself, the explicit K -algebra isomorphism $L^i \rightarrow \Pi_{G^{i-1}} L$ given in [1, Lemma 5.1] may be used to identify S^i with a subring of $\Pi_{G^{i-1}} S$. Provided that S is taken to act on S^i by multiplication with the first tensor factor, this identification clearly holds as S -algebras.

Denote the action of the nonidentity element σ of G by $a \rightarrow a'$. It was shown in the proof of [4, Proposition 1.8] that the Amitsur coboundary $d^1: U(L^2) \rightarrow U(L^3)$, viewed as a homomorphism from $\Pi_G U(L)$ to $\Pi_{G^2} U(L)$, sends (a, b) to $(a, a, a', a^{-1}bb')$. (Observe that the

indices of the above, and following, Cartesian products are subjected to lexicographic order, with $1 \leq \sigma$.) A similar routine computation reveals that $d^2: U(L^3) \rightarrow U(L^4)$ sends (a, b, c, d) to

$$(1, ba^{-1}, 1, ab^{-1}, a'c^{-1}, b'c^{-1}, c'a^{-1}, d'd^{-1}b^{-1}c).$$

Inasmuch as these formulas also describe the coboundaries in the Amitsur complex $C(S/R, U)$, it becomes imperative to know which tuples in $\Pi_{\sigma^{i-1}} U(S)$ arise from elements of $U(S^i)$, for $i = 2, 3$. The next two propositions settle this issue.

First, we give a key definition. Let I be the ideal of S generated by $\{a - a' : a \in S\}$.

- PROPOSITION. (i) $S^2 = \{(a, b) \in S \times S : a \equiv b(I)\}$.
(ii) $U(S^2) = [U(S) \times U(S)] \cap S^2$.

Proof. (i): Let $(a, b) \in S^2$; in other words, suppose that there exists $\xi = \sum \alpha_i \otimes \beta_i \in S^2$ such that $a = \sum \alpha_i \beta_i$ and $b = \sum \alpha_i \beta_i'$. If $m: L^2 \rightarrow L$ is the multiplication map, applying m and $m(1 \otimes \sigma)$ to ξ shows that S contains both a and b . It is clear that $a - b \in I$; i.e., $a \equiv b(I)$.

Conversely, let $(a, b) \in S \times S$ with $a \equiv b(I)$. Since $(a, b) = (a, a) + (0, b - a)$, it suffices to prove that S^2 contains both (a, a) and $\{0\} \times I$. For the former, observe that $(a, a) = a \otimes 1$. For the latter, our earlier remarks establish that S^2 is an S -submodule of $S \times S$, so that we need only to prove $(0, c - c') \in S^2$ for each $c \in S$. This, however, is immediate: $(0, c - c') = c \otimes 1 - 1 \otimes c$.

(ii): As the injection $S^2 \rightarrow S \times S$ is a ring homomorphism, it is clear that $U(S^2) \subset [U(S) \times U(S)] \cap S^2$. For the reverse inclusion, (i) reduces us to showing that, if a and b in $U(S)$ satisfy $a \equiv b(I)$, then $a^{-1} \equiv b^{-1}(I)$. As $a^{-1} - b^{-1} = a^{-1}b^{-1}(b - a)$, the proof is complete.

We pause to observe that the proof of part (i) of the preceding proposition was obtained by rendering basis-free Morris' proof of [7, Lemma 4.0]. The computational method used to establish (ii) replaces the Mayer-Vietoris argument of [7, Theorem 4.1].

- PROPOSITION. (i) $S^3 = \{(a, b, c, d) \in S \times S \times S \times S : a \equiv b \equiv c(I), a + c \equiv b + d(I^2)\}$.
(ii) $U(S^3) = [U(S) \times U(S) \times U(S) \times U(S)] \cap S^3$.

Proof. (i): Let $(a, b, c, d) \in S^3$; i.e., suppose that $\xi = \sum \alpha_i \otimes \beta_i \otimes \gamma_i \in S^3$ satisfies $a = \sum \alpha_i \beta_i \gamma_i$, $b = \sum \alpha_i \beta_i \gamma_i'$, $c = \sum \alpha_i \beta_i' \gamma_i'$ and $d = \sum \alpha_i \beta_i' \gamma_i$. It is clear that $a \equiv b \equiv c(I)$. Moreover $a + c \equiv b + d(I^2)$ since $a - b + c - d = \sum \alpha_i (\beta_i - \beta_i') (\gamma_i - \gamma_i')$.

Conversely, if a, b, c, d in S satisfy $a \equiv b \equiv c(I)$ and $a + c \equiv b + d(I^2)$, note that

$$\begin{aligned} (a, b, c, d) &= (a, a, a, a) + (0, 0, c - a, c - a) \\ &\quad + (0, b - a, 0, a - b) \\ &\quad + (0, 0, 0, -a + b - c + d). \end{aligned}$$

Since $(a, a, a, a) = a \otimes 1 \otimes 1$ and S^3 is an S -submodule of $S \times S \times S \times S$, it suffices to prove that S^3 contains $(0, 0, e - e', e - e')$, $(0, e - e', 0, e' - e)$ and $(0, 0, 0, (e - e')(f - f'))$ for each e and f in S . To this end, we need only to consider $(e \otimes 1 - 1 \otimes e) \otimes 1, 1 \otimes (e \otimes 1 - 1 \otimes e)$ and

$$(e \otimes 1 - 1 \otimes e) \otimes f - (f'e \otimes 1 - f' \otimes e) \otimes 1,$$

respectively.

(ii): By reasoning as in the preceding proposition, it suffices to show that, if $a, b, c, d \in U(S)$ satisfy $a \equiv b \equiv c(I)$ and $a + c \equiv b + d(I^2)$, then $a^{-1} + c^{-1} \equiv b^{-1} + d^{-1}(I^2)$. Taking congruences modulo I^2 , we have

$$\begin{aligned} a^{-1} - b^{-1} + c^{-1} - d^{-1} &\equiv a^{-1}b^{-1}c^{-1}d^{-1}[bc(a - b + c) \\ &\quad - ac(a - b + c) + ab(a - b + c) - abc] \\ &= a^{-1}b^{-1}c^{-1}d^{-1}[-c(a - b)^2 - a(b - c)^2 \\ &\quad + b(c - a)^2] \\ &\equiv 0, \end{aligned}$$

to complete the proof.

3. Formulas for cohomology. It will be convenient to let N denote the field norm $N_{L/K}: U(L) \rightarrow U(K)$ and to view $H^1(S/R, UK/U)$ as a subgroup of $H^2(S/R, U)$ by means of the (injective) connecting homomorphism (cf. [4, p. 240], [5]). In conjunction with the standing hypotheses announced earlier, we now *assume* that S is not contained in K . This readily implies that the multiplication map $S \otimes_R K \rightarrow L$ is an isomorphism since $[L: K] = 2$.

THEOREM. Let $A = \{x \in U(S): x \equiv 1(I^2)\}$ and $B = \{x \in U(S): x \equiv 1(I)\}$. Then:

- (i) $H^1(S/R, UK/U) \cong [N(U(L)) \cap A]/N(B)$.
- (ii) $H^2(S/R, U) \cong [K \cap A]/N(B)$.
- (iii) $H^2(S/R, U)/H^1(S/R, UK/U) \cong [K \cap A]/[N(U(L)) \cap A]$.

Proof. (i): As usual, the R -flatness of S and the isomorphism $S \otimes_R K \rightarrow L$ yield $C^n(S/R, UK/U) \cong U(L^{n+1})/U(S^{n+1})$. If $D =$

$\{\xi \in U(L^2): d^1(\xi) \in U(S^3)\}$, then the first cocycle group of $C(S/R, UK/U)$ is $\{\xi \cdot U(S^2): \xi \in D\}$, so that a standard isomorphism theorem implies $H^1(S/R, UK/U) \cong D/[d^0(U(L)) \cdot U(S^2)]$. Since N is given by $N(a) = aa'$, the material in §2 permits us to identify D with

$$\begin{aligned} E &= \{(a, b) \in U(L) \times U(L): a \in U(S), N(b) \in U(S), \\ &\quad N(a) \equiv N(b) \times (I^2)\} \\ &= \{a(1, c) \in U(L) \times U(L): a \in U(S), c \in U(L), N(c) \in A\}. \end{aligned}$$

As d^0 is given by $d^0(v) = v^{-1} \otimes v$, Hilbert's Theorem 90 shows that $d^0(U(L))$ is regarded as $\{1\} \times \ker(N)$; the preceding identification of D with E then causes $d^0(U(L)) \cdot U(S^2)$ to be identified with

$$F = [\{1\} \times \ker(N)] \cdot \{a(1, c) \in U(S) \times U(S): c \in B\}.$$

Thus, $H^1(S/R, UK/U) \cong E/F$. Observe that the homomorphism $h: U(L) \times U(L) \rightarrow U(L) \times U(K)$, given by $h(x, y) = (x, N(yx^{-1}))$, carries E onto $U(S) \times [N(U(L)) \cap A]$ and F onto $U(S) \times N(B)$. Since $\ker(h) \subset F$, standard isomorphism theorems apply, and establish (i).

(ii): The material in section 2 allows us to describe the second cocycle and coboundary groups of $C(S/R, U)$, so that $H^2(S/R, U) \cong J/M$, where

$$\begin{aligned} J &= \{(a, a, a', d) \in U(S) \times U(S) \times U(S) \times U(S): \\ &\quad a' \equiv d(I^2), d'd^{-1} = a(a')^{-1}\} \end{aligned}$$

and

$$\begin{aligned} M &= \{(a, a, a', a^{-1}bb') \in U(S) \times U(S) \times U(S) \times U(S): \\ &\quad b \in U(S), a \equiv b(I)\}. \end{aligned}$$

Projection onto the last two coordinates is an isomorphism that identifies J with

$$\begin{aligned} P &= \{(a, d) \in U(S) \times U(S): a \equiv d(I^2), d'd^{-1} = a'a^{-1}\} \\ &= \{a(1, c) \in U(S) \times U(S): c' = c, c \equiv 1(I^2)\} \end{aligned}$$

and identifies M with

$$\begin{aligned} Q &= \{(a, (a')^{-1}bb') \in U(S) \times U(S): b \in U(S), a' \equiv b(I)\} \\ &= \{a(1, N(c)) \in U(S) \times U(S): c \in B\}. \end{aligned}$$

Thus, $H^2(S/R, U) \cong P/Q$. Since K is the fixed field of G , the isomorphism given by $(x, y) \rightarrow (x, yx^{-1})$ carries P onto $U(S) \times$

$(K \cap A)$. As Q is sent onto $U(S) \times N(B)$, isomorphism theorems apply again, and establish (ii).

(iii): It suffices to prove that the isomorphism in (i) is the restriction to $H^1(S/R, UK/U)$ of the isomorphism in (ii). Let $\xi = \sum \alpha_i \otimes \beta_i \in D$; set $a = \sum \alpha_i \beta_i$ and $b = \sum \alpha_i \beta_i'$. It is routine to check that the connecting homomorphism sends the $H^1(S/R, UK/U)$ -cohomology class of ξ to the coset in J/M represented by $(a, a, a', a^{-1}bb')$. The map in (ii) then sends this coset (cohomology class) to the $N(B)$ -coset represented by $N(ba^{-1})$. This is precisely the effect of the isomorphism in (i) on the cohomology class of ξ , and so the proof is complete.

REMARK. Suppose $U(S) \cap K \subset R$. If $W = \{x \in U(R) : x \equiv 1(I^2)\}$, then the formulas in the preceding theorem may be restated as $H^1(S/R, UK/U) \cong [N(U(L)) \cap W]/N(B)$, $H^2(S/R, U) \cong W/N(B)$, and $H^2(S/R, U)/H^1(S/R, UK/U) \cong W/[N(U(L)) \cap W]$. This formula for $H^2(S/R, U)$ was obtained by Mandelberg [6, Theorem 4.24] for the special case in which R is integrally closed, S is integral over R , $\text{char}(K) \neq 2$, and there exists $a \in S$ such that S is R -free with basis $\{1, a\}$. As our work does not place restrictions on characteristic or bases, it applies to examples such as:

- (i) $R = \mathbf{F}_2[t]$, $L =$ splitting field of $x^2 + x + 1$ over K ;
- (ii) $R = \mathbf{Z}[(-30)^{1/2}]$, $L = K(6^{1/2})$ for which [6, Theorem 4.24] cannot be used.

4. Number-theoretic examples. We fix notation and assumptions for the remarks and examples given below: L is a biquadratic field extension of \mathbf{Q} , R is the ring of algebraic integers of a quadratic subfield K of L , and S is a ring properly containing R and contained in the ring of algebraic integers of L . The standing hypotheses of §§2 and 3 hold in this context. We also define I as in §2, and let N, A and B be as in the theorem of §3. Note that I^2 may be interpreted as the discriminant ideal of S/R . Because of the explicit description of the algebraic integers in biquadratic fields given by Williams [10], we will customarily leave to the reader, without further comment, verification of the values and basic properties of the ideals I^2 occurring in our examples. One such result, which occurs frequently in our examples, states: whenever $K = \mathbf{Q}(d^{1/2})$ and $L = K((d_1)^{1/2})$ and we write $d_1 d_2 = dk^2$ with discriminants d, d_1 , and d_2 ; then $I^2 = (k)$.

Since L/K is a quadratic extension of algebraic number fields, the expressions studied in §3 may be reinterpreted. The Hasse norm theorem [8, page 185] implies that an element $x \in K \cap A$ belongs to $N(U(L))$ precisely in case x is a local norm at all places. At a place of K which splits in L , all elements are norms. Moreover, x is a local

norm at any place arising from an inertial prime, since norms are characterized as being of even order and $x \in A \subset U(S)$ has order zero. If p is a ramified prime of R , then $p \mid I^2$, and so $x \equiv 1(p)$; in case p does not lie over (2) in \mathbf{Z} , this congruence suffices to make x a local square, and thus a local norm. The ramified primes lying over (2) require further analysis. While $x \equiv 1(I^2)$ does suffice to show that x is a local norm in general, a discussion of the relevant local class field theory would lead us far afield. For application to our examples, however, we need only consider such primes in biquadratic extensions of \mathbf{Q} . This reduces the problem to computing the properties of a finite number of extensions of \mathbf{Q}_2 . Hence we will not distinguish the primes dividing (2) from other primes. Finally, if an archimedean place does not split, we obtain an embedding of K in \mathbf{R} with $\mathbf{R} \otimes_K L \cong \mathbf{C}$; at such a place, x is a local norm if and only if x is positive in \mathbf{R} . Thus, $x \in K \cap A$ belongs to $N(U(L))$ precisely when x is positive at each real place of K which does not split in L .

The exact sequence

$$0 \rightarrow H^1(S/R, UK/U) \rightarrow H^2(S/R, U) \xrightarrow{\rho} B(S/R)$$

was developed in [4, Corollary 1.5] in order to study the map ρ appearing in [2, Theorem 7.6]. Examples in which ρ is an isomorphism abound ([2, Corollary 7.7], [3, Corollary 4.2]); we shall use this sequence to give some examples for which ρ fails to be an isomorphism. Our examples include the first for which $H^2(S/R, U) \neq 0$. In addition, they are simpler than one might imagine in light of recent results of Mandelberg [6, Corollary 4.25 and Remark 4.26], in which $H^2(S/R, U)$ is shown to vanish for a wide range of quadratic extensions of rings of algebraic integers in imaginary quadratic fields.

Before presenting our examples, we pause to note that previous calculations showing $H^2(T/\mathbf{Z}, U) = 0$ for an order T in a quadratic extension of \mathbf{Q} ([7, Theorems 3.0 and 3.2], [4, Proposition 1.9 and Remark 1.10(b)], [6, Theorem 4.27]) follow from the theorem in §3 and the observation that no discriminant divides 2. Indeed, no proper extension of \mathbf{Z} has discriminant 1 [9, Proposition 3-7-15 and Theorem 5-4-10]. Moreover, no extension of \mathbf{Z} has discriminant 2, because of the theorem of Stickelberger [9, Proposition 4-8-19], whose proof yields the statement: the discriminant of any finite extension of a principal ideal domain is congruent to a square modulo (4). We conjecture that the theorem in §3 generalizes to a class of higher-dimensional extensions, with “ I^2 ” replaced by “the discriminant” in the definition of A ; if so, the preceding argument implies $H^2(T/\mathbf{Z}, U) = 0$ for such extensions T/\mathbf{Z} .

EXAMPLE 1. This example treats various imaginary K . First, let K be either (i) $\mathbf{Q}((-30)^{1/2})$ or (ii) $\mathbf{Q}((-42)^{1/2})$; let $L = K((6)^{1/2})$. Then

$I^2 = (2)$ in (i) and $I^2 = (1)$ in (ii). For both cases, $K \cap A = \{\pm 1\}$ and $B = U(S) \subset \{\pm(5 + 2(6)^{1/2})^n : n \in \mathbf{Z}\}$. As $N(B) = \{1\}$, the formula in §3 implies that $H^2(S/R, U) \cong \mathbf{Z}/2\mathbf{Z}$. Since K has no real places, $H^1(S/R, UK/U) \cong \mathbf{Z}/2\mathbf{Z}$ also, and ρ is the zero map.

Next, let K be any imaginary quadratic algebraic number field other than $\mathbf{Q}((-3)^{1/2})$ and $\mathbf{Q}((-1)^{1/2})$. As $U(R) = \{\pm 1\}$, we find that $H^2(S/R, U)$ is nonzero (and, hence, isomorphic to $\mathbf{Z}/2\mathbf{Z}$) if and only if $N(B) = \{1\}$ and I^2 is either (1) or (2). One verifies from [10] that $I^2 = (1)$ when the discriminant d of K can be written as $d = d_1 d_2$, such that $L = K((d_1)^{1/2}) = \mathbf{Q}((d_1)^{1/2}, (d_2)^{1/2})$ and d_1, d_2 are each discriminants. Similarly, $I^2 = (2)$ arises from $4d = d_1 d_2$. In these cases, $U(S)$ is contained in the real quadratic subfield of L and, hence, is equal to B .

To fabricate examples, (including (i) and (ii) above), let d_1 be a positive square-free rational integer such that each unit of $\mathbf{Q}((d_1)^{1/2})$ has norm 1. (For instance, choose square-free positive d_1 divisible by a prime congruent to 3 modulo 4.) Then choose square-free negative $d_2 \in \mathbf{Z}$ such that $(d_1, d_2) = (1)$ and not both d_1, d_2 are congruent to 3 modulo 4. Set $K = \mathbf{Q}((d_1 d_2)^{1/2})$ and $L = K((d_1)^{1/2})$. The preceding work shows that $H^2(S/R, U) \cong \mathbf{Z}/2\mathbf{Z} \cong H^1(S/R, UK/U)$ and ρ is the zero map. Note that $I^2 = (2)$, so that L/K is ramified, if one of d_1, d_2 is even and the other is congruent to 3 modulo 4; in the remaining case, $I^2 = (1)$ and L/K is unramified.

If $K = \mathbf{Q}((-3)^{1/2})$, the general version of Stickelberger's theorem implies that $H^2(S/R, U) = 0$ for each quadratic extension S of R , since, by analogy with the case $R = \mathbf{Z}$, it shows that no difference of units could be divisible by a discriminant.

EXAMPLE 2. Let K be real and L complex. Then $K \cap A$ has the form $\{\pm \alpha^n : n \in \mathbf{Z}\}$ if $I^2 | (2)$; otherwise, $K \cap A = \{\alpha^n\}$. In either case, $N(U(L))$ does not contain -1 (since norms are totally positive) and $N(B)$ contains α^2 . The possible cases are tabulated below.

	$K \cap A$	$N(U(L)) \cap A$	$N(B)$	$H^2(S/R, U)$	$H^1(S/R, UK/U)$
(a)	$\{\pm \alpha^n\}$	$\{\alpha^n\}$	$\{\alpha^n\}$	$\mathbf{Z}/2\mathbf{Z}$	0
(b)	$\{\pm \alpha^n\}$	$\{\alpha^n\}$	$\{\alpha^{2n}\}$	$\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$
(c)	$\{\pm \alpha^n\}$	$\{\alpha^{2n}\}$	$\{\alpha^{2n}\}$	$\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$	0
(d)	$\{\alpha^n\}$	$\{\alpha^n\}$	$\{\alpha^n\}$	0	0
(e)	$\{\alpha^n\}$	$\{\alpha^n\}$	$\{\alpha^{2n}\}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$
(f)	$\{\alpha^n\}$	$\{\alpha^{2n}\}$	$\{\alpha^{2n}\}$	$\mathbf{Z}/2\mathbf{Z}$	0

As $H^1(S/R, UK/U) = \ker(\rho)$ and the split Brauer group $B(S/R)$ is known to be $\mathbf{Z}/2\mathbf{Z}$, case (c) cannot arise. A direct proof of this will now

be given. If case (c) holds, then $I^2 \mid (2)$; moreover, neither α nor $-\alpha$ is totally positive, whence $N_{K/\mathbb{Q}}(\alpha) = -1$. As every odd divisor of the discriminant d of K is congruent to 1 modulo 4, d cannot be expressed as the product of two negative discriminants. Thus $I^2 = (2)$, and $R = \{a + b\sqrt{D} : a, b \in \mathbb{Z}\}$ for some $D \equiv 2(4)$. If $\alpha = a + b\sqrt{D}$, then $N_{K/\mathbb{Q}}(\alpha) = -1$ implies that a and b are each odd; this contradicts $\alpha \equiv 1(I^2)$, thus proving (again) that case (c) cannot arise.

We now proceed to show that the other five cases do arise. First, examples giving (a) and (b) with $I^2 = (1)$ are (a) $K = \mathbb{Q}(6^{1/2})$, $L = K((-2)^{1/2})$ and (b) $K = \mathbb{Q}(3^{1/2})$, $L = K((-1)^{1/2})$. Whenever $I^2 = (1)$, we are in case (a) or (b); the difference is whether the extension L contains $(-\alpha)^{1/2}$ (case (a)) or not (case (b)). If $I^2 = (2)$, we must have $K = \mathbb{Q}(\sqrt{D})$ with $D > 0$ and $D \equiv 2 \pmod{4}$ and $L = K((D_1)^{1/2})$ with $D_1 \mid D$, $D_1 < 0$, and $D_1 \equiv 3 \pmod{4}$. As in our analysis of case (c), a unit, β , with $N_{K/\mathbb{Q}}(\beta) = -1$ cannot belong to $K \cap A$. On the other hand, $\beta \in B$. Thus, when R contains such β , any L with $I^2 = (2)$ gives case (a). If R contains no such β , every unit belongs to A , and the test is as in the case of $I^2 = (1)$. Thus $K = \mathbb{Q}(10^{1/2})$, $L = K((-5)^{1/2})$ or $K((-1)^{1/2})$ gives (a) since $N_{K/\mathbb{Q}}(3 + 10^{1/2}) = -1$. If $K = \mathbb{Q}(34^{1/2})$, the fundamental unit is $35 + 6(34)^{1/2} = (18^{1/2} + 17^{1/2})^2$; then $L = K((-17)^{1/2})$ gives (a), while $L = K((-1)^{1/2})$ gives (b).

In constructing examples of (d), (e) and (f), we expect that R and S will have the same units (actually, if $I^2 \nmid 2$, any new units must be roots of unity). Case (d) then requires that there be units congruent to 1 modulo I but not modulo I^2 , whose square is congruent to 1 modulo I^2 . Some examples of (d) can be constructed with $I^2 = (4)$ if $K = \mathbb{Q}(\sqrt{D})$ with $D > 0$, $D \equiv 1 \pmod{4}$, when the fundamental unit of R has norm -1 . In this case, there are units of norm -1 congruent to 1 modulo 2; these cannot be congruent to 1 modulo 4. To achieve $I^2 = (4)$, take $L = K((D_1)^{1/2})$ with $D_1 < 0$, $D_1 \mid D$, $D_1 \equiv 3 \pmod{4}$ (e.g., $D_1 = -1$). Thus we could take $K = \mathbb{Q}(5^{1/2})$, $L = K((-1)^{1/2})$. Here $B = \{(2 + (5)^{1/2})^n\}$ and $A = N(B) = \{(2 + (5)^{1/2})^{2n}\}$.

Another family of examples of (d) can be constructed as follows. Choose $D > 0$, $D \equiv 3 \pmod{4}$ with fundamental unit of $R = \mathbb{Z}[\sqrt{D}]$ denoted β where $\beta \equiv \sqrt{D} \pmod{2}$ and $\beta > 0$ (e.g. $D = 3$, $\beta = 2 + (3)^{1/2}$). Then $(\beta^{n+1} - \beta^{-n})/(\beta - 1)$ is an odd integer c_n such that $\beta^{2n+1} \equiv 1 \pmod{c_n}$. Take $L = K((-2c_n)^{1/2})$ which has $I^2 = (4c_n)$. Thus $B = \{\beta^{2(2n+1)k}\}$ and $A = N(B) = \{\beta^{4(2n+1)k}\}$. On the other hand, if $K = \mathbb{Q}((13 \cdot 17)^{1/2})$, case (d) cannot arise for any L (for any unit $\alpha \equiv 1 \pmod{2}$, $\alpha \equiv 1 \pmod{8}$).

It is easy to give examples over any R for which $A = B$: e.g. by choosing odd factors of I^2 one can force the elements of B to be congruent to 1 modulo 8 and hence to belong to A . The generator, α , of A must have $N_{K/\mathbb{Q}}(\alpha) = +1$. Indeed, $\alpha \equiv 1 \pmod{I^2}$ requires $\alpha' \equiv 1 \pmod{I^2}$

since I^2 is a rational ideal, and $\alpha' = -\alpha^{-1}$ would require $2 \in I^2$. Both conjugates of α have the same sign: if positive, then (e); if negative, then (f). If $K = \mathbf{Q}(d^{1/2})$ and $L = \mathbf{Q}((d_1)^{1/2}, (d_2)^{1/2})$ where d, d_1, d_2 are discriminants and $d, d_2 = dk^2$, then $I^2 = (k)$, so one can easily find extensions having any value of I^2 which satisfies: (i) odd primes dividing d do not occur, (ii) other odd primes occur to at most the first power, (iii) 2 occurs to a power depending on the power of 2 occurring in d, d_1 , and d_2 (at most the third power).

If the positive generator β of the units of R with norm 1 has odd order modulo any divisor of I^2 , then the generator of A cannot be negative. This makes examples of case (e) easy to construct over any R . For example, if $R = \mathbf{Z}[2^{1/2}]$, $\beta = 3 + 2(2)^{1/2}$ has order 3 modulo 7. Taking $L = K((-7)^{1/2})$, $I^2 = (7)$, and hence $A = \{\beta^{3n}\} = A \cap N(U(L))$. This procedure produces examples over any real quadratic R for which the map ρ is neither a monomorphism nor an epimorphism.

To produce examples of (f) requires more care since we must find a possible value of I^2 arising from an L/K of this type modulo which $(-\beta)$ has odd order. To do this, consider the factors of $t_{2k+1} = (\beta^{k+1} + \beta^{-k})/(\beta + 1)$ or $t_{2k} = \beta^k + \beta^{-k}$ for possible values of I^2 . If K is generated by the square root of a square-free positive *even* integer, 2 will occur to at most the first power in I^2 ; thus, there is no difficulty synthesizing examples of L from the t_n . If $R = \mathbf{Z}[2^{1/2}]$, $t_3 = 5$ and $t_2 = 6$. For $L = K((-3)^{1/2})$, $I^2 = (3)$ and $A = \{(-\beta)^{2n}\}$; for $L = K((-5)^{1/2})$, $I^2 = (10)$ and $\beta^3 = 99 + 70(2)^{1/2} \equiv -1(I^2)$, giving $A = \{(-\beta)^{3n}\}$. If $K = \mathbf{Q}(d^{1/2})$ with $d > 0$ and divisible by a prime of the form $4k - 1$, then every odd value of I^2 can be realized. Thus over $K = \mathbf{Q}(21^{1/2})$, the fundamental unit is $(5 + 21^{1/2})/2$, $t_2 = 5$, $t_3 = 4$, $t_4 = 23$, $t_5 = 19$. We can get $I^2 = (5)$, (23) , or (19) from $L = K((-15)^{1/2})$, $K((-23)^{1/2})$, or $K((-19)^{1/2})$, respectively.

In the remaining cases, synthesis of examples may be required to follow a different route. To illustrate, consider $K = \mathbf{Q}(17^{1/2})$ for which $\beta = (4 + 17^{1/2})^2 = 33 + 8(17)^{1/2}$. Here $t_{2n} \equiv 2 \pmod{8}$, $t_{2n+1} \equiv 1 \pmod{8}$; hence one would have difficulty identifying any t_n which could be divisible by an admissible I^2 . However, if q is any prime of the form $4k - 1$ which is also a quadratic non residue modulo 17, then: (i) q is an inertial prime of K ; (ii) the units of R modulo q form a cyclic group of order $q^2 - 1$; (iii) the subgroup of elements of norm 1 has order $q + 1$; (iv) an element of the subgroup which is the square of an element not in the subgroup has order divisible by the largest power of 2 dividing $q + 1$, and has a power which is congruent to -1 modulo q ; (v) thus q must divide some t_n . For this particular K , we may take $L = K((-q)^{1/2})$ where q is a prime congruent to 3, 7, 11, 23, 27, 31, 39, or 63 modulo 68. This procedure can be modified to cover those choices of R ,

whether or not they contain units of norm -1 , whose discriminant over \mathbf{Q} is divisible only by primes of the form $4k+1$. Over any real quadratic field K , one can give infinitely many choices of L which give (e) and infinitely many L which give (f).

The values of $H^2(S/R, U)$ given above exceed the bounds given by Mandelberg for special types of quadratic ring extensions [6, Corollary 4.25 and Remark 4.26]. On the other hand, they do sharpen the bound of $\prod_{i=1}^8 (\mathbf{Z}/2\mathbf{Z})$ which follows from the bound on the cochain group given by Dobbs [3, Proposition 2.1]. We hope that our examples will serve to clarify the role of the units of finite order in the computation of Amitsur cohomology.

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