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ENUMERATION OF WEIGHTED p -LINE ARRAYS

MARGARET JONES HODEL

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Let $F_p(n, k; q_1, q_2, \dots, q_p) = F_p(n, k)$ be defined by

$$F_p(n, k) = \sum \prod_{i=1}^p q_i^{\sum_{j=1}^n a_{ij}},$$

where the summation is over all p -line arrays of positive integers

a_{11}	a_{12}	\cdots	a_{1n}
a_{21}	a_{22}	\cdots	a_{2n}
\vdots	\vdots		\vdots
a_{p1}	a_{p2}	\cdots	a_{pn}

subject to the following conditions:

$$\max\{a_{ij} : 1 \leq i \leq j\} \leq \min\{a_{i,j+1} : 1 \leq i \leq p\}, \quad 1 \leq j \leq n-1,$$

$$\max\{a_{ij} : 1 \leq i \leq p\} \leq j, \quad 1 \leq j \leq n,$$

and

$$a_{in} = k, \quad 1 \leq i \leq p.$$

Assuming $\prod_{i=1}^p q_i = 1$, formulas for $F_p(n, k)$ and two other enumerants, which are closely related to $F_p(n, k)$, are obtained in this paper. These three functions generalize enumerants which Carlitz has determined.

1. Introduction. We consider the enumeration of p -line arrays of positive integers

$$(1.1) \quad \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{array}$$

satisfying certain conditions. We first require that

$$(1.2) \quad \max\{a_{ij} : 1 \leq i \leq p\} \leq \min\{a_{i,j+1} : 1 \leq i \leq p\}, \quad 1 \leq j \leq n-1,$$

and

$$(1.3) \quad \max\{a_{ij} : 1 \leq i \leq p\} \leq j, \quad 1 \leq j \leq n.$$

We indicate further requirements by defining the enumerants we seek. Let $f_p(n, k; s_1, s_2, \dots, s_p)$ represent the number of arrays (1.1) subject to the restrictions (1.2), (1.3),

$$(1.4) \quad \sum_{j=1}^n a_{ij} = s_i, \quad 1 \leq i \leq p,$$

and

$$(1.5) \quad a_{in} = k, \quad 1 \leq i \leq p,$$

and let

$$F_p(n, k; q_1, q_2, \dots, q_p) = F_p(n, k) = F_p \\ = \sum^* f_p(n, k; s_1, s_2, \dots, s_p) q_1^{s_1} q_2^{s_2} \dots q_p^{s_p},$$

where \sum^* is the sum over the p -tuples (s_1, s_2, \dots, s_p) . (We may view (s_1, s_2, \dots, s_p) as the weight of the array (1.1).) Let $g_p(n, k; s_1, s_2, \dots, s_p)$ denote the number of arrays (1.1) satisfying (1.2), (1.3), (1.4) and

$$(1.6) \quad \max\{a_{in} : 1 \leq i \leq p\} = k,$$

and let

$$G_p(n, k; q_1, q_2, \dots, q_p) = G_p(n, k) = G_p \\ = \sum^* g_p(n, k; s_1, s_2, \dots, s_p) q_1^{s_1} q_2^{s_2} \dots q_p^{s_p}.$$

Finally, we use $h_p(n, k; s_1, s_2, \dots, s_p)$ to represent the number of arrays (1.1) subject to conditions (1.2), (1.3), (1.4), (1.6) and

$$(1.7) \quad a_{i+1,j} \leq a_{ij}, \quad 1 \leq i \leq p-1, \quad 1 \leq j \leq n,$$

and we let

$$H_p(n, k; q_1, q_2, \dots, q_p) = H_p(n, k) = H_p \\ = \sum^* h_p(n, k; s_1, s_2, \dots, s_p) q_1^{s_1} q_2^{s_2} \dots q_p^{s_p}.$$

The functions F_1 , G_1 and H_1 coincide, and if $q_1 = 1$, they enumerate what MacMahon [7, p. 167] called two-element lattice

permutations. Carlitz and Riordan [6] have studied these functions and a q -generalization. A related q -generalization, in fact $F_1(n, k; q_1)$, has also been investigated by Carlitz [5]. If $p = 2$ and $q_1 = q_2 = 1$, F_2, G_2 and H_2 are the enumerants f, g and h which Carlitz [4] has explicitly determined.

In this paper we first generalize some identities which Carlitz stated for f, g and h . Then, by assuming

$$(1.8) \quad \prod_{i=1}^p q_i = 1,$$

we are able to use these results and Carlitz's technique for finding f, g and h to obtain formulas for F_p, G_p and H_p . In general these formulas are in terms of functions $t(n, k)$ which are defined by

$$\Phi^n(x) = \sum_{k=0}^{\infty} t(n, k) x^k,$$

where $\Phi(x)$ is a known function. In some special cases the enumerants can be expressed in terms of binomial or q -binomial coefficients. For example, if $q_1 = q_2 = q$ and $q^2 = 1$,

$$F_2(n+1, k+1) = \frac{1}{n} \sum_{j=0}^k (n-j) b_2(n, j; q)$$

and

$$G_2(n, k+1) = \frac{n-k}{n} b_2(n, k; q)$$

where

$$\begin{aligned} b_2(n, k; q) &= \sum_{m=0}^k \binom{n}{k-m} \binom{2n+m-1}{m} q^m \\ &= \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{n}{k-2j} \binom{2n+2j-1}{2j} + \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \binom{n}{k-2j-1} \binom{2n+2j}{2j+1} q. \end{aligned}$$

We also find that

$$H_2(n, k+1) = \frac{n-k}{n} \sum_{m=0}^k \binom{n+k-m-1}{k-m} \binom{n+m-1}{m} q^m$$

if $q_1 q_2 = 1$, and

$$H_p(n, k+1) = \frac{n-k}{n} \left[\begin{matrix} np+k-1 \\ k \end{matrix} \right]_q$$

if $q_1 = q_2 = \cdots = q_p = q$ and $q^p = 1$. In §6 we interpret the formulas for F_p , G_p and H_p as partition theorems. It would be of interest to determine these enumerants without the restriction (1.8).

In a subsequent paper we shall consider

$$I_p(n, k; q_1, q_2, \cdots, q_p) = \sum^* i_p(n, k; s_1, s_2, \cdots, s_p) q_1^{s_1} q_2^{s_2} \cdots q_p^{s_p},$$

where $i_p(n, k; s_1, s_2, \cdots, s_p)$ represents the number of arrays (1.1) satisfying (1.2), (1.3), (1.4) and (1.7), and having k columns in which

$$a_{1j} = a_{2j} = \cdots = a_{pj}.$$

Carlitz [3] called such columns coincidences and has proved that

$$I_p(n, k; 1, 1, \cdots, 1) = \frac{1}{k} \binom{n-1}{k-1} \sum_{t=0}^{n-k} (-1)^{n-k-t} \binom{n-k}{t} \binom{2n+(p-1)t}{n-1}$$

for $q_1 = q_2 = \cdots = q_p = 1$.

2. Preliminary results. Generalizing (2.1)–(2.4), (2.7) and (2.9) of [4], we have

$$(2.1) \quad F_p(n+1, k) = \left[\prod_{i=1}^p q_i \right]^k \sum_{m=1}^k \left[\prod_{i=1}^p [k-m+1]_{q_i} - \prod_{i=1}^p q_i [k-m]_{q_i} \right] F_p(n, m), \quad k \leq n+1,$$

where

$$[k]_{q_i} = \sum_{j=0}^{k-1} q_i^j,$$

$$(2.2) \quad F_p(n+1, k) = \left[\prod_{i=1}^p q_i \right]^k \sum_{m=1}^k G_p(n, m), \quad k \leq n+1,$$

$$(2.3) \quad G_p(n+1, k) = \sum_{m=1}^k \left[\prod_{i=1}^p q_i \right]^m \left[\prod_{i=1}^p [k-m+1]_{q_i} - \prod_{i=1}^p [k-m]_{q_i} \right] G_p(n, m), \quad k \leq n+1,$$

$$(2.4) \quad G_p(n, k) = \sum_{m=1}^k \left[\prod_{i=1}^p [k-m+1]_{q_i} - \left(1 + \prod_{i=1}^p q_i \right) \prod_{i=1}^p [k-m]_{q_i} \right. \\ \left. + \prod_{i=1}^p q_i \prod_{i=1}^p [k-m-1]_{q_i} \right] F_p(n, m), \quad k \leq n,$$

$$(2.5) \quad G_p(n+k, k) = \sum_{m=1}^k \left[\prod_{i=1}^p q_i \right]^{(n+k-m)(m-1)} \\ G_p(m, m) G_p(n+k-m, k-m+1), \quad n \geq 1,$$

and

$$(2.6) \quad F_p(n+k, k) = \sum_{j=1}^k \left[\prod_{i=1}^p q_i \right]^{(n+k-m)(m-1)} \\ G_p(m, m) F_p(n+k-m, k-m+1), \quad n \geq 1.$$

Let

$$(2.7) \quad \theta_p(k; q_2, q_3, \dots, q_p) = \sum \prod_{i=2}^p q_i^{a_{in}}, \quad p \geq 2,$$

where the summation is over all $(p-1)$ -tuples $(a_{2n}, a_{3n}, \dots, a_{pn})$ with the a_{in} satisfying $0 \leq a_{pn} \leq \dots \leq a_{3n} \leq a_{2n} \leq k$. Then corresponding to (6.3) and (6.8) of [4] we have

$$(2.8) \quad H_p(n, k) = \sum_{m=1}^k \left[\prod_{i=1}^p q_i \right]^m q_1^{k-m} \theta_p(k-m; q_2, q_3, \dots, q_p) H_p(n-1, m), \\ 1 \leq k \leq n,$$

where it is understood that $H_p(n-1, n) = 0$, and

$$(2.9) \quad H_p(n+k, k) = \sum_{j=1}^k \left[\prod_{i=1}^p q_i \right]^{(n+k-m)(m-1)} \\ H_p(m, m) H_p(n+k-m, k-m+1), \quad n \geq 1.$$

The proofs of (2.1)–(2.6), (2.8) and (2.9) are simply generalizations of the proofs of their analogues in [4]. To prove (2.1) it suffices to assume $k \leq n$ since $F_p(n+1, n+1) = F_p(n+1, n)$. For $k \leq n$ we consider the array

a_{11}	a_{12}	\cdots	a_{1n}	k
a_{21}	a_{22}	\cdots	a_{2n}	k
\vdots	\vdots		\vdots	\vdots
a_{p1}	a_{p2}	\cdots	a_{pn}	k

satisfying (1.2), (1.3) and

$$(2.10) \quad \max\{a_{in} : 1 \leq i \leq p\} \leq k.$$

Let

$$(2.11) \quad \min\{a_{in} : 1 \leq i \leq p\} = m.$$

Using (2.10), (2.11) and the definition of $F_p(n, k)$, we have

$$(2.12) \quad F_p(n+1, k) = \left[\prod_{i=1}^p q_i \right]^k \sum_{m=1}^k \sum \prod_{i=1}^p q_i^{\sum_{j=1}^n a_{ij}},$$

where the inner summation is over all arrays (1.1) satisfying (1.2), (1.3), (2.10) and (2.11). From (2.12) we get

$$(2.13) \quad F_p(n+1, k) = \left[\prod_{i=1}^p q_i \right]^k \sum_{m=1}^k \sum' \prod_{i=1}^p q_i^{a_{in}-m} F_p(n, m),$$

where Σ' is the sum over all p -tuples $(a_{1n}, a_{2n}, \dots, a_{pn})$ subject to conditions (2.10) and (2.11). Since

$$\begin{aligned} \sum' \prod_{i=1}^p q_i^{a_{in}-m} &= \sum_{\substack{m \leq a_{in} \leq k \\ 1 \leq i \leq p}} \prod_{i=1}^p q_i^{a_{in}-m} - \sum_{\substack{m+1 \leq a_{in} \leq k \\ 1 \leq i \leq p}} \prod_{i=1}^p q_i^{a_{in}-m} \\ &= \sum_{\substack{0 \leq a_{in} \leq k-m \\ 1 \leq i \leq p}} \prod_{i=1}^p q_i^{a_{in}} - \sum_{\substack{0 \leq a_{in} \leq k-m-1 \\ 1 \leq i \leq p}} \prod_{i=1}^p q_i^{a_{in}+1} \\ &= \prod_{i=1}^p [k-m+1]_{q_i} - \prod_{i=1}^p q_i [k-m]_{q_i}, \end{aligned}$$

(2.1) follows from (2.13).

Equation (2.2) follows immediately from the definitions. The proof of (2.3) is similar to the proof of (2.1), but to obtain (2.3) we consider the array

a_{11}	a_{12}	\cdots	a_{1n}	$a_{1,n+1}$
a_{21}	a_{22}	\cdots	a_{2n}	$a_{2,n+1}$
\vdots	\vdots		\vdots	\vdots
a_{p1}	a_{p2}	\cdots	a_{pn}	$a_{p,n+1}$

where conditions (1.2) and (1.3) (with n replaced by $n + 1$) are satisfied and where

$$(2.14) \quad \max\{a_{in} : 1 \leq i \leq p\} = m$$

and

$$(2.15) \quad \max\{a_{i,n+1} : 1 \leq i \leq p\} = k.$$

As (2.14) and (1.2) imply that

$$(2.16) \quad \min\{a_{i,n+1} : 1 \leq i \leq p\} \geq m,$$

we find that

$$(2.17) \quad G_p(n+1, k) = \sum_{m=1}^k \sum \prod_{i=1}^p q_i^{a_{i,n+1}} G_p(n, m),$$

where the inner sum is over all p -tuples $(a_{1,n+1}; a_{2,n+1}; \cdots; a_{p,n+1})$ satisfying (2.15) and (2.16). From (2.17) we get (2.3).

To prove (2.4) consider the array (1.1) subject to conditions (1.2), (1.3), (1.6) and (2.11). Corresponding to (2.13) and (2.17) in the previous proofs, we have

$$(2.18) \quad G_p(n, k) = \sum_{m=1}^k \sum \prod_{i=1}^p q_i^{a_{in}-m} F_p(n, m),$$

where the inner summation is over the p -tuples $(a_{1n}, a_{2n}, \cdots, a_{pn})$ satisfying (1.6) and (2.11). From (2.18) we get

$$G_p(n, k) = \sum_{m=1}^k \left[\sum_{\substack{m \leq a_{in} \leq k \\ 1 \leq i \leq p}} \prod_{i=1}^p q_i^{a_{in}-m} - \sum_{\substack{m \leq a_{in} \leq k-1 \\ 1 \leq i \leq p}} \prod_{i=1}^p q_i^{a_{in}-m} \right. \\ \left. - \sum_{\substack{m+1 \leq a_{in} \leq k \\ 1 \leq i \leq p}} \prod_{i=1}^p q_i + \sum_{\substack{m+1 \leq a_{in} \leq k-1 \\ 1 \leq i \leq p}} \prod_{i=1}^p q_i \right] F_p(n, m),$$

and (2.4) follows.

Since the proofs of (2.5), (2.6) and (2.9) are similar, we shall only establish (2.5). To this end we observe that

$$\max\{a_{i1} : 1 \leq i \leq p\} = 1$$

implies that there exists a greatest m such that

$$\max\{a_{im} : 1 \leq i \leq p\} = m.$$

Therefore

$$\max\{a_{i,m+1} : 1 \leq i \leq p\} = m,$$

$$a_{i,m+1} = m, \quad 1 \leq i \leq p,$$

and we can divide our original array into two sub-arrays as follows:

$$(2.19) \quad \begin{array}{cccccc} 1 & \cdots & a_{1m} & m & \cdots & a_{1,n+k} \\ 1 & \cdots & a_{2m} & m & \cdots & a_{2,n+k} \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cdots & a_{pm} & m & \cdots & a_{p,n+k} \end{array}$$

By subtracting $m - 1$ from each entry in the right sub-array of (2.19), we get

$$\begin{aligned} & g_p(n+k, k; s_1, s_2, \dots, s_p) \\ &= \sum_{m=1}^k g_p(m, m; u_1, u_2, \dots, u_p) g_p(n+k-m, k-m+1; v_1, v_2, \dots, v_p), \end{aligned}$$

where $u_i + v_i = s_i - (n+k-m)(m-1)$, $1 \leq i \leq p$. Now (2.5) follows immediately.

We obtain (2.8) by considering the array

$$\begin{array}{cccccc} a_{11} & a_{12} & \cdots & m & k \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{p,n-1} & a_{pn} \end{array}$$

where (1.2)–(1.4), (1.6), (1.7) and the condition

$$a_{1,n-1} = m$$

are satisfied. Clearly

$$\begin{aligned} h_p(n, k; s_1, s_2, \dots, s_p) \\ = \sum_{m=1}^k \sum'' h_p(n-1, m; s_1-k, s_2-a_{2n}, \dots, s_p-a_{pn}), \end{aligned}$$

where Σ'' is the sum over all $(p-1)$ -tuples such that

$$m \leq a_{pn} \leq a_{p-1,n} \leq \dots \leq a_{2n} \leq k.$$

Thus

$$H_p(n, k) = \sum_{m=1}^k q_1^k \sum'' \prod_{i=2}^p q_i^{a_{in}} H_p(n-1, m)$$

and we have (2.8).

3. Techniques for determining the enumerants. To obtain the following results we use Carlitz's method [4] for finding f , g and h in a somewhat more general setting. Assuming (1.8), Theorem 1 provides formulas for G_p and H_p while Corollary 1 yields an expression for F_p .

Before stating the theorem it is convenient to define some functions. Using \mathbf{N} to denote the nonnegative integers and \mathbf{N}^* to represent the positive integers, let $r(n, k)$ be a function from $\mathbf{N}^* \times \mathbf{N}^*$ into a field F and $\phi(n)$ be a function from \mathbf{N} into F . Let

$$R_n(x) = \sum_{k=1}^{\infty} r(n+k-1, k)x^{n+k-1}, \quad n \geq 1,$$

and

$$\Phi(x) = \sum_{n=0}^{\infty} \phi(n)x^n.$$

Furthermore we define $t(n, k)$, a function from $\mathbf{N} \times \mathbf{N}$ into F , by

$$\Phi^n(x) = \sum_{k=0}^{\infty} t(n, k)x^k.$$

THEOREM 1. *If $r(n, k)$ and $\phi(n)$ satisfy*

$$(3.1) \quad r(1, 1) = 1,$$

$$(3.2) \quad r(n, k) = 0, \quad n < k,$$

$$(3.3) \quad \phi(0) = 1,$$

$$(3.4) \quad r(n+1, k) = \sum_{m=1}^k \phi(k-m)r(n, m), \quad 1 \leq k \leq n+1,$$

and

$$(3.5) \quad r(n+k, k) = \sum_{m=1}^k r(m, m)r(n+k-m, k-m+1), \quad n \geq 1,$$

and if $\Phi(z)$ is analytic about $z = 0$, then

$$(3.6) \quad r(n, k+1) = \frac{n-k}{n} t(n, k).$$

Proof. By (3.5)

$$\begin{aligned} R_{n+1}(x) &= \sum_{k=1}^{\infty} x^{n+k} \sum_{m=1}^k r(m, m)r(n+k-m, k-m+1) \\ &= \sum_{m=1}^{\infty} r(m, m)x^m \sum_{k=1}^{\infty} r(n+k-1, k)x^{n+k-1} \\ &= R_1(x) R_n(x). \end{aligned}$$

Thus

$$R_n(x) = R_1^n(x) \quad n \geq 1.$$

Using (3.1)–(3.4), we find that

$$\begin{aligned} \Phi(R_1(x)) &= 1 + \sum_{n=1}^{\infty} \phi(n) R_n(x) \\ &= 1 + \sum_{n=1}^{\infty} \phi(n) \sum_{k=1}^{\infty} r(n+k-1, k)x^{n+k-1} \\ &= 1 + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \phi(n-k) r(n-1, k)x^{n-1} \\ &= 1 + \sum_{n=2}^{\infty} r(n, n)x^{n-1} \\ &= x^{-1} R_1(x). \end{aligned}$$

It follows that

$$x = \frac{z}{\Phi(z)},$$

where $z = R_1(x)$.

By the Lagrange expansion formula [9, p. 125] the equation

$$x = \frac{z}{\Phi(z)} \quad (\Phi(0) = 1),$$

where $\Phi(z)$ is analytic in a neighborhood of $z = 0$, implies

$$(3.7) \quad f(z) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} f'(z) \Phi^n(z) \right]_{z=0}.$$

Since

$$R_m(x) = R_1^m(x) = z^m,$$

we can determine $R_m(x)$ by letting $f(z) = z^m$ in (3.7). Thus we have

$$\begin{aligned} R_m(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} (mz^{m-1}) \sum_{k=0}^{\infty} t(n, k) z^k \right]_{z=0} \\ &= \sum_{n=1}^{\infty} x^n \frac{m}{n} t(n, n-m), \end{aligned}$$

and

$$(3.8) \quad \sum_{n=m-1}^{\infty} r(n, n-m+1) x^n = \sum_{n=1}^{\infty} x^n \frac{m}{n} t(n, n-m).$$

We obtain (3.6) by equating coefficients of x^n in (3.8).

To state the corollary we must introduce two more functions. Let $s(n, k)$ be a function from $\mathbf{N}^* \times \mathbf{N}^*$ into F and define $S_n(x)$ by

$$S_n(x) = \sum_{k=1}^{\infty} s(n+k-1, k) x^{n+k-1}, \quad n \geq 1.$$

COROLLARY 1. If $r(n, k)$, $s(n, k)$ and $\phi(n)$ satisfy (3.1)–(3.5),

$$(3.9) \quad s(1, 1) = 1,$$

$$(3.10) \quad s(n, n) = s(n, n-1), \quad n \geq 2,$$

and

$$(3.11) \quad s(n+k, k) = \sum_{m=1}^k r(m, m) s(n+k-m, k-m+1), \quad n \geq 1,$$

then

$$(3.12) \quad s(n+1, k+1) = \frac{1}{n} \sum_{j=0}^k (n-j) t(n, j).$$

Proof. Using (3.9)–(3.11), we find that

$$\begin{aligned} S_1(x) &= x + \sum_{k=1}^{\infty} x^{k+1} \sum_{m=1}^k r(m, m) s(k-m+1, k-m+1) \\ &= x + R_1(x) S_1(x), \end{aligned}$$

and from (3.11) we get

$$\begin{aligned} S_{n+1}(x) &= \sum_{k=1}^{\infty} x^{n+k} \sum_{m=1}^k r(m, m) s(n+k-m, k-m+1) \\ &= S_n(x) R_1(x), \end{aligned} \quad n \geq 1.$$

Thus

$$\begin{aligned} S_{n+1}(x) &= S_1(x) R_1^n(x) \\ &= \frac{x R_1^n(x)}{1 - R_1(x)} \end{aligned}$$

or

$$\frac{S_{m+1}(x)}{x} = \frac{z^m}{1-z},$$

where $z = R_1(x)$. Again making use of (3.7), this time with

$$f(z) = \frac{z^m}{1-z},$$

we see that

$$x^{-1} S_{m+1}(x) = \delta_{m,0} + \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} \left[\frac{mz^{m-1}}{1-z} + \frac{z^m}{(1-z)^2} \right] \sum_{k=0}^{\infty} t(n, k) z^k \right]_{z=0}.$$

Hence we have

$$\sum_{n=m-1}^{\infty} s(n+1, n-m+1)x^n = \delta_{m,0} + \sum_{n=1}^{\infty} x^n \frac{1}{n} \sum_{j=0}^{n-m} (m+j) t(n, n-m-j)$$

and (3.12) follows.

4. The functions F_p and G_p . Throughout the rest of this paper we assume condition (1.8) holds. With this assumption we can use the results of the two previous sections to determine F_p , G_p and H_p . In this section we consider F_p and G_p and in §5 we find H_p .

THEOREM 2. *If $\prod_{i=1}^p q_i = 1$, then*

$$(4.1) \quad G_p(n, k+1) = \frac{n-k}{n} b_p(n, k)$$

and

$$(4.2) \quad F_p(n+1, k+1) = \frac{1}{n} \sum_{j=0}^k (n-j) b_p(n, j),$$

where $b_p(n, k; q_1, \dots, q_p) = b_p(n, k)$ is defined by

$$\begin{aligned} & \left[\sum_{k=0}^{\infty} \left[\prod_{i=1}^p [k+1]_{q_i} - \prod_{i=1}^p [k]_{q_i} \right] x^k \right]^n \\ &= \sum_{k=0}^{\infty} b_p(n, k) x^k. \end{aligned}$$

Proof. This theorem is an immediate consequence of Theorem 1 and Corollary 1, for (1.8), (2.3), (2.5), (2.6) and the definitions of F_p and G_p imply that the hypotheses of these results are satisfied if $r(n, k) = G_p(n, k)$, $s(n, k) = F_p(n, k)$ and

$$\phi(k) = \prod_{i=1}^p [k+1]_{q_i} - \prod_{i=1}^p [k]_{q_i}.$$

Instead of Corollary 1 we can use (2.2) to obtain (4.2). We remark that by virtue of (1.8) it is possible to reduce

$$\prod_{i=1}^p [k+1]_{q_i} - \prod_{i=1}^p [k]_{q_i},$$

and thus $b_p(n, k)$, to a function of $p-1$ q_i 's.

Now

$$\sum_{k=0}^{\infty} \left[\prod_{i=1}^p [k+1]_{q_i} - \prod_{i=1}^p [k]_{q_i} \right] x^k = x^{-1} H_p(x^{-1} | q_1, q_2, \dots, q_p),$$

where $H_p(x | q_1, q_2, \dots, q_p) = H_p(x)$ is the generalized Eulerian function defined and studied by Roselle [12]. Because $H_p(x)$ is quite complicated for $p \geq 3$, in general it is not feasible to find a simple formula for $b_p(n, k)$ in terms of more familiar coefficients. However, we can find $b_2(n, k)$ without difficulty.

COROLLARY 2. Let $p = 2$. If $q_1 q_2 = 1$, then

$$G_2(n, k+1) = \frac{n-k}{n} b_2(n, k)$$

and

$$F_2(n+1, k+1) = \frac{1}{n} \sum_{j=0}^k (n-j) b_2(n, j),$$

where

$$(4.3) \quad b_2(n, k) = \sum_{m=0}^k \binom{n}{k-m} \sum_{j=0}^m \binom{n+j-1}{j} \binom{n+m-j-1}{n-1} q_1^{m-2j}.$$

Proof. Since

$$H_2(x) = \frac{x + q_1 q_2}{(x - q_1)(x - q_2)}$$

and $q_1 q_2 = 1$, we find that

$$x^{-1} H_2(x^{-1}) = \frac{1+x}{(1-q_1 x)(1-q_1^{-1} x)}.$$

Thus

$$\begin{aligned} (x^{-1} H_2(x^{-1}))^n &= \sum_{m=0}^n \binom{n}{m} x^m \sum_{k=0}^{\infty} \binom{n+k-1}{k} q_1^k x^k \sum_{j=0}^{\infty} \binom{n+j-1}{j} q_1^{-j} x^j \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{n}{k-m} \sum_{j=0}^m \binom{n+j-1}{j} \binom{n+m-j-1}{m-j} q_1^{m-2j} x^k \end{aligned}$$

and (4.3) follows.

Another special case of interest is that in which $q_1 = q_2 = \cdots = q_p = q$. In this case condition (1.8) becomes $q^p = 1$ and the possible values for q are the p th roots of unity. Let $H_p(x \mid q_1, q_2, \cdots, q_p) = H_p(x \mid q)$ and $b_p(n, k; q_1, q_2, \cdots, q_p) = b_p(n, k; q)$ when $q_1 = \cdots = q_p = q$. We also require the notation

$$[x] = [x]_q = \frac{q^x - 1}{q - 1}$$

and

$$\begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} x \\ m \end{bmatrix}_q = \prod_{j=1}^m \frac{q^{x-j+1} - 1}{q^j - 1}.$$

Carlitz [2] has proved that

$$(4.4) \quad H_p(x \mid q) = \sum_{j=1}^p A_{p,j}(q) x^{j-1} / \prod_{n=1}^p (x - q^n),$$

where $A_{p,j}(q)$ are the q -Eulerian numbers defined by

$$[x]^m = \sum_{s=1}^m A_{m,s}(q) \begin{bmatrix} x + s - 1 \\ m \end{bmatrix}, \quad m \geq 1.$$

Now (4.4) implies

$$x^{-1} H_p(x^{-1} \mid q) = \sum_{j=1}^p A_{p,j}(q) x^{j-1} / \prod_{n=1}^p (1 - q^n x).$$

If we define $a_p(n, k; q)$ by

$$(4.5) \quad \left[\sum_{j=1}^p A_{p,j}(q) x^{j-1} \right]^n = \sum_{k=0}^{\infty} a_p(n, k; q) x^k,$$

then

$$\begin{aligned} (x^{-1} H_p(x^{-1} \mid q))^n &= \sum_{k=0}^{\infty} a_p(n, k; q) x^k \sum_{m=0}^{\infty} \begin{bmatrix} np + m - 1 \\ m \end{bmatrix} (qx)^m \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^k a_p(n, k - m; q) \begin{bmatrix} np + m - 1 \\ m \end{bmatrix} q^m x^k. \end{aligned}$$

Thus we have proved

COROLLARY 3. *If $q_1 = q_2 = \cdots = q_p = q$ and $q^p = 1$, then*

$$G_p(n, k+1) = \frac{n-k}{n} b_p(n, k; q)$$

and

$$F_p(n+1, k+1) = \frac{1}{n} \sum_{j=0}^k (n-j) b_p(n, k; q),$$

where

$$(4.6) \quad b_p(n, k; q) = \sum_{m=0}^k a_p(n, k-m; q) \left[\begin{matrix} np+m-1 \\ m \end{matrix} \right] q^m$$

and $a_p(n, k; q)$ is defined by (4.5).

The following result follows immediately from Corollaries 2 and 3.

COROLLARY 4. *Let $p = 2$. If $q_1 = q_2 = q$ and $q^2 = 1$, then*

$$G_2(n, k+1) = \frac{n-k}{n} b_2(n, k; q)$$

and

$$F_2(n+1, k+1) = \frac{1}{n} \sum_{j=0}^k (n-j) b_2(n, j; q),$$

where

$$(4.7) \quad b_2(n, k; q) = \sum_{m=0}^k \binom{n}{k-m} \binom{2n+m-1}{m} q^m$$

or

$$(4.8) \quad b_2(n, k; q) = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{n}{k-2j} \binom{2n+2j-1}{2j} \\ + \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \binom{n}{k-2j-1} \binom{2n+2j}{2j+1} q$$

or

$$(4.9) \quad b_2(n, k; q) = \sum_{m=0}^k \binom{n}{k-m} \left[\begin{matrix} 2n+m-1 \\ m \end{matrix} \right] q^{n-k}.$$

Proof. We deduce (4.7) from Corollary 2 by using the hypothesis $q^2 = 1$ and a binomial identity found in Riordan [10, p. 9]. Because $q^2 = 1$, (4.8) follows from (4.7). We get (4.9) from Corollary 3 by observing that

$$a_2(n, k; q) = \binom{n}{k} q^{n-k}.$$

If $q = 1$ in Corollary 4 we have

COROLLARY 5. Let $p = 2$. If $q_1 = q_2 = 1$, then

$$(4.10) \quad G_2(n, k+1) = \frac{n-k}{n} \sum_{m=0}^k \binom{n}{k-m} \left(\begin{matrix} 2n+m-1 \\ m \end{matrix} \right)$$

and

$$(4.11) \quad F_2(n+1, k+1) = \frac{1}{n} \sum_{j=0}^k (n-j) \sum_{m=0}^j \binom{n}{j-m} \left(\begin{matrix} 2n+m-1 \\ m \end{matrix} \right).$$

We note that (4.10) is precisely Carlitz's formula for $g(n, k+1)$, while (4.11) is equivalent to his formula for $f(n+1, k+1)$.

If $p = 1$, condition (1.8) implies $q_1 = 1$. In this case we get

COROLLARY 6. Let $p = 1$. If $q_1 = 1$, then

$$(4.12) \quad F_1(n, k+1) = G_1(n, k+1) = \frac{n-k}{n} \binom{n+k-1}{k}.$$

Proof. Letting $p = 1$ and $q = 1$ in Corollary 3 we obtain

$$(4.13) \quad G_1(n, k+1) = \frac{n-k}{n} \sum_{m=0}^k a_1(n, k-m; 1) \binom{n+m-1}{m}.$$

Since

$$a_1(n, k; 1) = \delta_{k,0},$$

(4.13) reduces to

$$G_1(n, k+1) = \frac{n-k}{n} \binom{n+k-1}{k}.$$

Then (4.12) follows because $F_1(n, k+1) = G_1(n, k+1)$ by definition. Formula (4.12) is the result given by Bertrand [1] as the number of two-element lattice permutations.

5. The function H_p . By virtue of (1.8), (2.8) and (2.9), the hypotheses of Theorem 1 are satisfied if $r(n, k) = H_p(n, k)$ and

$$\phi(n) = q_1^n \theta_p(n; q_2, q_3, \dots, q_p),$$

where $\theta_p(n; q_2, q_3, \dots, q_p)$, or more briefly $\theta_p(n)$, is defined by (2.7) for $p \geq 2$ and by $\theta_p(n) = 1$ for $p = 1$. Thus we can express $H_p(n, k)$ in terms of the coefficients $c_p(n, k) = c_p(n, k; q_1, q_2, \dots, q_p)$ defined by

$$(5.1) \quad \Theta_p^n(x) = \sum_{k=0}^{\infty} c_p(n, k) x^k,$$

where

$$\Theta_p(x) = \Theta_p(x \mid q_1, q_2, \dots, q_p) = \sum_{k=0}^{\infty} \theta_p(k) (q_1 x)^k.$$

In fact we have

THEOREM 3. *If $\prod_{i=1}^p q_i = 1$, then*

$$(5.2) \quad H_p(n, k+1) = \frac{n-k}{n} c_p(n, k),$$

where $c_p(n, k)$ is defined by (5.1).

(In view of condition (1.8) it is possible to express $\Theta_p(x)$ and $c_p(n, k)$ as a function of $p-1$ q_i 's.)

Since the coefficients $c_p(n, k)$ are so closely related to $\theta_p(k)$ and $\Theta_p(x)$, we reduce $\theta_p(k)$ from a $(p-1)$ -tuple summation to a single sum. Using this simplification of $\theta_p(k)$, we can write $\Theta_p(x)$ as a single, finite sum.

THEOREM 4. *If $p \geq 2$, then*

$$(5.3) \quad \theta_p(k) = \frac{1}{\gamma_1(q_2, \dots, q_p)} + \sum_{j=2}^p \frac{(-1)^{j+1} q_2 q_3^2 \cdots q_j^{j-1} (q_2 q_3 \cdots q_j)^k}{\gamma_j(q_2, \dots, q_p)},$$

where

$$\gamma_1(q_2, \dots, q_p) = (1 - q_2)(1 - q_2 q_3) \cdots (1 - q_2 \cdots q_p),$$

and

$$(5.4) \quad \gamma_j(q_2, \dots, q_p) = (1 - q_2 \cdots q_j)(1 - q_3 \cdots q_j) \cdots (1 - q_j)(1 - q_{j+1}) \cdots (1 - q_{j+1} q_{j+2}) \cdots (1 - q_{j+1} \cdots q_p), \quad j \geq 2,$$

with the understanding that

$$(5.5) \quad (1 - q_i \cdots q_j) = 1 \quad \text{if} \quad i > j.$$

Proof. We use induction on p . From definition (2.7) we get

$$\theta_2(k) = \sum_{m=0}^k q_2^m.$$

Thus

$$\theta_2(k) = \frac{1 - q_2^{k+1}}{1 - q_2}$$

and we have verified (5.3) for $p = 2$. Using first the definition of $\theta_{p+1}(k)$ and then (5.3) (as the induction hypothesis), we find that

$$\begin{aligned} \theta_{p+1}(k; q_2, \dots, q_{p+1}) &= \sum_{a_2=0}^k q_2^{a_2} \sum_{a_3=0}^{a_2} q_3^{a_3} \cdots \sum_{a_{p+1}=0}^{a_p} q_{p+1}^{a_{p+1}} \\ &= \sum_{a_2=0}^k q_2^{a_2} \theta_p(a_2; q_3, \dots, q_{p+1}) \\ &= \sum_{m=0}^k q_2^m \left[\frac{1}{\gamma_1(q_3, \dots, q_{p+1})} + \sum_{j=2}^p \frac{(-1)^{j+1} q_3 q_4^2 \cdots q_{j+1}^{j-1} (q_3 \cdots q_{j+1})^m}{\gamma_{j+1}(q_3, \dots, q_{p+1})} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1 - q_2^{k+1}}{(1 - q_2)\gamma_1(q_3, \dots, q_{p+1})} + \sum_{j=3}^{p+1} \frac{(-1)^j q_3 q_4^2 \cdots q_j^{j-2} (1 - (q_2 \cdots q_j)^{k+1})}{(1 - q_2 \cdots q_j)\gamma_j(q_3, \dots, q_{p+1})} \\
&= \sum_{j=2}^{p+1} \frac{(-1)^{j+1} q_2 q_3^2 \cdots q_j^{j-1} (q_2 \cdots q_j)^k}{\gamma_j(q_2, \dots, q_{p+1})} + \sum_{j=2}^{p+1} \frac{(-1)^j q_3 q_4^2 \cdots q_j^{j-2}}{\gamma_j(q_2, \dots, q_{p+1})}.
\end{aligned}$$

To complete the induction it suffices to prove

LEMMA 1. *We have*

$$\begin{aligned}
&\frac{1}{(1 - q_1)(1 - q_2)(1 - q_2 q_3) \cdots (1 - q_2 \cdots q_p)} + \sum_{j=2}^p \frac{(-1)^{j+1} q_2 q_3^2 \cdots q_j^{j-1}}{\gamma_j(q_1, \dots, q_p)} \\
(5.6) \quad &= \frac{1}{(1 - q_1)(1 - q_1 q_2) \cdots (1 - q_1 \cdots q_p)},
\end{aligned}$$

where γ_j is defined by (5.4) and (5.5).

Proof. This proof was suggested by Carlitz. In the expression

$$\begin{aligned}
(5.7) \quad &\frac{(1 - q_1 q_2) \cdots (1 - q_1 \cdots q_p)}{(1 - q_2)(1 - q_2 q_3) \cdots (1 - q_2 \cdots q_p)} \\
&+ \sum_{j=2}^p \frac{(-1)^{j+1} q_2 q_3^2 \cdots q_j^{j-1} (1 - q_1) \cdots (1 - q_1 \cdots q_p)}{\gamma_j(q_1, \dots, q_p)}
\end{aligned}$$

fix q_2, \dots, q_p . Then (5.7) is a polynomial in q_1 of degree $p - 1$. Since this polynomial is 1 for p values of q_1 , namely

$$q_1 = 1, \frac{1}{q_2}, \dots, \frac{1}{q_2 q_3 \cdots q_p},$$

it is identically 1 and (5.6) follows. Using the definition of $\Theta_p(x)$ and (5.3), we find that

$$\Theta_p(x) = \frac{1}{(1 - q_1 x)\gamma_1(q_2, \dots, q_p)} + \sum_{j=2}^p \frac{(-1)^{j+1} q_2 q_3^2 \cdots q_j^{j-1}}{(1 - q_1 \cdots q_j x)\gamma_j(q_2, \dots, q_p)}.$$

In general $\theta_p(x)$ is quite complicated and it is not feasible to determine the coefficients $c_p(n, k)$ explicitly. However, for $p = 2$ and $q_1 q_2 = 1$, we have

$$\begin{aligned}\Theta_2(x) &= \frac{1}{(1-q_1x)(1-q_2)} - \frac{q_2}{(1-x)(1-q_2)} \\ &= \frac{1}{(1-q_1x)(1-x)}.\end{aligned}$$

It follows that

$$\theta_2^n(x) = \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{n+k-m-1}{k-m} \binom{n+m-1}{m} q_1^m x^k$$

and

$$c_2(n, k) = \sum_{m=0}^k \binom{n+k-m-1}{k-m} \binom{n+m-1}{m} q_1^m.$$

Hence, as a corollary of Theorem 3, we have

COROLLARY 7. *Let $p = 2$. If $q_1 q_2 = 1$, then*

$$(5.8) \quad H_2(n, k+1) = \frac{n-k}{n} \sum_{m=0}^k \binom{n+k-m-1}{k-m} \binom{n+m-1}{m} q_1^m.$$

If, in addition to (1.8), we assume $q_1 = q_2 = \cdots = q_p = q$, $\Theta_p(x)$ is considerably simpler. In this case let $\Theta_p(x) = \Theta_p(x | q)$, $\theta_p(k) = \theta_p(k; q)$ and $c_p(n, k) = c_p(n, k; q)$. From the definition of $\theta_p(k; q)$ it follows that

$$\theta_p(k; q) = \sum q^m,$$

where the summation is over all $(p-1)$ -tuples (a_2, \cdots, a_p) such that $1 \leq a_p \leq \cdots \leq a_2 \leq k+1$, and

$$m + p - 1 = \sum_{i=2}^p a_i.$$

Hence it is evident that $\theta_p(k; q)$ generates the number of partitions of $m + p - 1$ into $p - 1$ parts with each part at most $k + 1$. It is well-known (see, for example, [8, p. 5]) that such a function is $\begin{bmatrix} k+p-1 \\ k \end{bmatrix}$. Thus

$$\begin{aligned}\Theta_p(x | q) &= \sum_{k=0}^{\infty} \left[\begin{matrix} k+p-1 \\ k \end{matrix} \right] (qx)^k \\ &= \prod_{k=1}^p (1 - q^k x)^{-1}.\end{aligned}$$

Since $q^p = 1$,

$$(5.9) \quad \Theta_p(x | q) = \prod_{k=0}^{p-1} (1 - q^k x)^{-1},$$

and

$$\begin{aligned}\Theta_p^n(x | q) &= \prod_{k=0}^{p-1} (1 - q^k x)^{-n} \\ &= \prod_{k=0}^{np-1} (1 - q^k x)^{-1} \\ &= \sum_{k=0}^{\infty} \left[\begin{matrix} np+k-1 \\ k \end{matrix} \right] x^k.\end{aligned}$$

Therefore

$$c_p(n, k; q) = \left[\begin{matrix} np+k-1 \\ k \end{matrix} \right],$$

and we have proved

COROLLARY 8. *If $q_1 = q_2 = \cdots = q_p = q$ and $q^p = 1$, then*

$$(5.10) \quad H_p(n, k+1) = \frac{n-k}{n} \left[\begin{matrix} np+k-1 \\ k \end{matrix} \right].$$

We observe that if q is a primitive p th root of unity, say ξ , (5.9) reduces to

$$\Theta_p(x | \xi) = (1 - x^p)^{-1}.$$

Then

$$(5.11) \quad \Theta_p^n(x | \xi) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^{pk}.$$

From (5.11) it follows that

$$(5.12) \quad c_p(n, k; \xi) = \begin{cases} \binom{n + k/p - 1}{n - 1} & \text{if } p/k \\ 0 & \text{otherwise} \end{cases}$$

As an immediate consequence of Corollary 8 we have

COROLLARY 9. *If $q_i = 1$, $1 \leq i \leq p$, then*

$$(5.13) \quad H_p(n, k + 1) = \frac{n - k}{n} \binom{np + k - 1}{k}.$$

We can also obtain (5.13) by viewing $\theta_p(k; 1)$ as the number of $(p - 1)$ -combinations with repetition of $k + 1$ distinct objects. Then we know that (see Riordan [11, p. 7])

$$(5.14) \quad \theta_p(k; 1) = \binom{k + p - 1}{p - 1}.$$

From (5.14) we can deduce (5.13).

If $p = 2$, $q_1 = q_2 = q$ and $q^2 = 1$, the formulas for $H_2(n, k + 1)$, given in the following corollary, are particularly simple.

COROLLARY 10. *Let $p = 2$. If $q_1 = q_2 = 1$, then*

$$(5.15) \quad H_2(n, k + 1) = \frac{n - k}{n} \binom{2n + k - 1}{k};$$

if $q_1 = q_2 = -1$,

$$(5.16) \quad H_2(n, k + 1) = \begin{cases} \frac{n - k}{n} \binom{n + k/2 - 1}{n - 1} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} . \end{cases}$$

Proof. From either (5.8) or (5.13) we get (5.15), while (5.16) follows from (5.12). Carlitz's result for $h(n, k + 1)$ coincides with (5.15).

Since (5.13) is valid for $p = 1$, we have the expected result

COROLLARY 11. *Let $p = 1$. If $q_1 = 1$, then*

$$H_1(n, k + 1) = \frac{n - k}{n} \binom{n + k - 1}{k}.$$

6. Partitions. From the definition of $g_p(n, k; s_1, s_2, \dots, s_p)$ it is clear that this enumerant is the number of partitions of the p -partite (s_1, s_2, \dots, s_p) of the form

$$\sum_{j=1}^n a_{ij} = s_i, \quad 1 \leq i \leq p,$$

where the a_{ij} are positive integers subject to conditions (1.2), (1.4) and (1.6). Thus $G_p(n, k)$ generates these partitions. If we replace (1.6) by (1.5), these statements are true for $f_p(n, k)$ and $F_p(n, k)$. Adding (1.7) to the conditions for $g_p(n, k)$, we have a partition interpretation for $h_p(n, k)$ and $H_p(n, k)$.

Since we have only obtained F_p, G_p and H_p under the assumption (1.8), we describe how this restriction affects the partitions. If $\prod_{i=1}^p q_i = 1$ and $p \geq 2$,

$$G_p(n, k) = \sum^* g_p(n, k; s_1, s_2, \dots, s_p) q_1^{s_1 - s_p} q_2^{s_2 - s_p} \cdots q_{p-1}^{s_{p-1} - s_p},$$

where Σ^* is as defined in §1. Thus we have

THEOREM 5. *If $\prod_{i=1}^p q_i = 1$, the function $G_p(n, k)$ generates the number of partitions of the $(p-1)$ -partite $(m_1, m_2, \dots, m_{p-1})$ of the form*

$$m_i = \sum_{j=1}^n a_{ij} - \sum_{j=1}^n a_{pj},$$

where the a_{ij} are positive integers satisfying (1.2), (1.4) and (1.6).

We have corresponding interpretations for F_p and H_p . For example, by Corollaries 2 and 7 we have

$$\begin{aligned} G_2(n, k+1) &= \frac{n-k}{n} \sum_{m=-k}^k \sum_{j=\max\{0, -m\}}^{\lfloor (k-m)/2 \rfloor} \binom{n}{k-m-2j} \binom{n+j-1}{j} \binom{n+m+j-1}{m+j} q_1^m, \\ F_2(n+1, k+1) &= \frac{1}{n} \sum_{m=-k}^k \sum_{j=\max\{m, -m\}}^k (n-j) \sum_{s=\max\{0, -m\}}^{\lfloor (j-m)/2 \rfloor} \binom{n}{j-m-2s} \binom{n+s-1}{s} \binom{n+m+s-1}{m+s} q_1^m, \end{aligned}$$

and

$$H_2(n, k+1) = \frac{n-k}{n} \sum_{m=0}^k \binom{n+k-m-1}{k-m} \binom{n+m-1}{m} q_1^m.$$

Thus

$$(6.1) \quad G_2(3, 2) = 2q_1^{-1} + 2 + 2q_1,$$

$$(6.2) \quad F_2(3, 2) = q_1^{-1} + 2 + q_1$$

and

$$(6.3) \quad H_2(3, 2) = 2 + 2q_1.$$

The following arrays are enumerated by $G_2(3, 2)$:

$$\begin{array}{lll} (1) \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 2 \\ \hline \end{array} & (2) \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 1 & 2 & 2 \\ \hline \end{array} & (3) \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 1 & 1 & 2 \\ \hline \end{array} \\ (4) \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 1 & 2 & 2 \\ \hline \end{array} & (5) \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 1 & 1 & 2 \\ \hline \end{array} & (6) \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \end{array}$$

Now arrays (1) and (2) satisfy

$$-1 = \sum_{j=1}^3 a_{1j} - \sum_{j=1}^3 a_{2j};$$

for (3) and (4) we have

$$0 = \sum_{j=1}^3 a_{1j} - \sum_{j=1}^3 a_{2j},$$

while (5) and (6) are subject to the condition

$$1 = \sum_{j=1}^3 a_{1j} - \sum_{j=1}^3 a_{2j}.$$

The arrays counted by $F_2(3, 2)$ are (2)–(5). Array (2) accounts for the coefficient of q_1^{-1} , (3) and (4) are enumerated by the constant term, and (5) is counted by the coefficient of q_1 . Finally, $H_2(3, 2)$ counts arrays (3)–(6). The first two of these are enumerated by the constant term; (5) and (6) account for the coefficient of q_1 .

If we assume $q_1 = \cdots = q_p = q$,

$$G_p(n, k) = \sum^* g_p(n, k; s_1, \cdots, s_p) q^{\sum_{i=1}^p s_i}.$$

Then condition (1.8) implies $q^p = 1$ and we have

$$G_p(n, k) = \sum_{m=0}^{p-1} \sum g_p(n, k; s_1, \cdots, s_p) q^m$$

where the inner sum is over all p -tuples satisfying

$$\sum_{i=1}^p s_i \equiv m \pmod{p}.$$

Therefore, we have

THEOREM 6. *If $q_1 = q_2 = \cdots = q_p = q$, $q^p = 1$, and $\Pi_{p,n,k}(r)$ is the number of partitions of the form*

$$r = \sum_{i=1}^p s_i = \sum_{i=1}^p \sum_{j=1}^n a_{ij},$$

where the a_{ij} are positive integers subject to conditions (1.2), (1.3), (1.4) and (1.6), then $G_p(n, k)$ generates

$$\sum_t \Pi_{p,n,k}(m + tp), \quad 0 \leq m \leq p-1.$$

For example, from Corollary 4 we see that

$$\sum_t \Pi_{2,n,k+1}(2t) = \frac{n-k}{n} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{n}{k-2j} \binom{2n+2j-1}{2j}$$

and

$$\sum_t \Pi_{2,n,k+1}(2t+1) = \frac{n-k}{n} \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \binom{n}{k-2j-1} \binom{2n+2j}{2j+1}.$$

We have corresponding results for F_p and H_p .

Returning to the illustration used above, if $q_1 = q_2 = q$, we may write (6.1), (6.2) and (6.3) as

$$(6.4) \quad G_2(3, 2) = 2 + 4q,$$

$$(6.5) \quad F_2(3, 2) = 2 + 2q,$$

and

$$(6.6) \quad H_2(3, 2) = 2 + 2q.$$

Since the sums of arrays (3) and (4) are even, these arrays are counted by the constant term in (6.4)–(6.6). In each case the coefficient of q enumerates the arrays having an odd sum.

REFERENCES

1. J. Bertrand, *Solution d'un probleme*, Comptes Rendus Académie des Sciences. Paris, **105** (1887), 369.
2. L. Carlitz, q -Bernoulli and Eulerian numbers, Trans. Amer. Math. Soc., **76** (1954), 332–350.
3. ———, Enumeration of rectangular arrays by length and coincidences, Annali di Matematica pura ed applicata, **99** (1974), 155–182.
4. ———, Enumeration of two-line arrays, The Fibonacci Quarterly, **2** (1973), 113–130.
5. ———, Sequences, paths, ballot numbers, The Fibonacci Quarterly, **10** (1972), 531–550.
6. L. Carlitz and J. Riordan, Two element lattice permutation numbers and their q -generalization, Duke Math. J., **31** (1964), 371–388.
7. P. A. MacMahon, *Combinatorial Analysis*, Vol. 1, Cambridge, 1915.
8. ———, *Combinatorial Analysis*, vol. 2, Cambridge, 1916.
9. G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, Springer, Berlin, **1** (1925).
10. J. Riordan, *Combinatorial Identities*, Wiley, New York, 1968.
11. ———, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.
12. D. P. Roselle, Generalized Eulerian functions and a problem in partitions, Duke Math. J., **33** (1966), 293–304.

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DUKE UNIVERSITY

D. E. Bennett, <i>Strongly uncoherent continua</i>	1
Walter R. Bloom, <i>Sets of p-spectral synthesis</i>	7
R. T. Bumby and D. E. Dobbs, <i>Amitsur cohomology of quadratic extensions: Formulas and number-theoretic examples</i>	21
W. W. Comfort, <i>Compactness-like properties for generalized weak topological sums</i>	31
D. R. Dunninger and J. Locker, <i>Monotone operators and nonlinear biharmonic boundary value problems</i>	39
T. S. Erickson, W. S. Martindale, 3rd and J. M. Osborn, <i>Prime nonassociative algebras</i>	49
P. Fischer, <i>On the inequality $\sum_{i=1}^n p_i \frac{f(p_i)}{f(q_i)} \geq 1$</i>	65
G. Fox and P. Morales, <i>Compact subsets of a Tychonoff set</i>	75
R. Gilmer and J. F. Hoffmann, <i>A characterization of Prüfer domains in terms of polynomials</i>	81
L. C. Glaser, <i>On tame Cantor sets in spheres having the same projection in each direction</i>	87
Z. Goseki, <i>On semigroups in which $X = XYX = XZX$ if and only if $X = XYZX$</i>	103
E. Grosswald, <i>Rational valued series of exponentials and divisor functions</i>	111
D. Handelman, <i>Strongly semiprime rings</i>	115
J. N. Henry and D. C. Taylor, <i>The $\bar{\beta}$ topology for w^*-algebras</i>	123
M. J. Hodel, <i>Enumeration of weighted p-line arrays</i>	141
S. K. Jain and S. Singh, <i>Rings with quasiprojective left ideals</i>	169
S. Jeyaratnam, <i>The diophantine equation $Y(Y+m)(Y+2m) \times (Y+3m) = 2X(X+m)(X+2m)(X+3m)$</i>	183
R. Kane, <i>On loop spaces without p torsion</i>	189
Alvin J. Kay, <i>Nonlinear integral equations and product integrals</i>	203
A. S. Kechris, <i>Countable ordinals and the analytic hierarchy, I</i>	223
Ka-Sing Lau, <i>A representation theorem for isometries of $C(X, E)$</i>	229
I. Madsen, <i>On the action of the Dyer-Lashof algebra in $H_*(G)$</i>	235
R. C. Metzler, <i>Positive linear functions, integration, and Choquet's theorem</i>	277
A. Nobile, <i>Some properties of the Nash blowing-up</i>	297
G. E. Petersen and G. V. Welland, <i>Plessner's theorem for Riesz conjugates</i>	307

Donald Earl Bennett, <i>Strongly uncoherent continua</i>	1
Walter Russell Bloom, <i>Sets of p-spectral synthesis</i>	7
Richard Thomas Bumby and David Earl Dobbs, <i>Amitsur cohomology of quadratic extensions: formulas and number-theoretic examples</i>	21
W. Wistar (William) Comfort, <i>Compactness-like properties for generalized weak topological sums</i>	31
Dennis Robert Dunninger and John Stewart Locker, <i>Monotone operators and nonlinear biharmonic boundary value problems</i>	39
Theodore Erickson, Wallace Smith Martindale, III and J. Marshall Osborn, <i>Prime nonassociative algebras</i>	49
Pál Fischer, <i>On the inequality $\sum_{i=0}^n [f(p_i)/f(q_i)]p_i \geq i$</i>	65
Geoffrey Fox and Pedro Morales, <i>Compact subsets of a Tychonoff set</i>	75
Robert William Gilmer, Jr. and Joseph F. Hoffmann, <i>A characterization of Prüfer domains in terms of polynomials</i>	81
Leslie C. Glaser, <i>On tame Cantor sets in spheres having the same projection in each direction</i>	87
Zensiro Goseki, <i>On semigroups in which $x = xyx = xzx$ if and only if $x = xyzx$</i>	103
Emil Grosswald, <i>Rational valued series of exponentials and divisor functions</i>	111
David E. Handelman, <i>Strongly semiprime rings</i>	115
Jackson Neal Henry and Donald Curtis Taylor, <i>The $\tilde{\beta}$ topology for W^*-algebras</i>	123
Margaret Jones Hodel, <i>Enumeration of weighted p-line arrays</i>	141
Surender Kumar Jain and Surjeet Singh, <i>Rings with quasi-projective left ideals</i>	169
S. Jeyaratnam, <i>The Diophantine equation $Y(Y+m)(Y+2m)(Y+3m) = 2X(X+m)(X+2m)(X+3m)$</i>	183
Richard Michael Kane, <i>On loop spaces without p torsion</i>	189
Alvin John Kay, <i>Nonlinear integral equations and product integrals</i>	203
Alexander S. Kechris, <i>Countable ordinals and the analytical hierarchy. I</i>	223
Ka-Sing Lau, <i>A representation theorem for isometries of $C(X, E)$</i>	229
Ib Henning Madsen, <i>On the action of the Dyer-Lashof algebra in $H_*(G)$</i> ...	235
Richard C. Metzler, <i>Positive linear functions, integration, and Choquet's theorem</i>	277
Augusto Nobile, <i>Some properties of the Nash blowing-up</i>	297
Gerald E. Peterson and Grant Welland, <i>Plessner's theorem for Riesz conjugates</i>	307