

# Pacific Journal of Mathematics

**SOME PROPERTIES OF THE NASH BLOWING-UP**

AUGUSTO NOBILE

## SOME PROPERTIES OF THE NASH BLOWING-UP

A. NOBILE

**Intuitively, in the Nash blowing-up process each singular point of an algebraic (or analytic) variety is replaced by the limiting positions of tangent spaces (at non-singular points). The following properties of this process are shown: 1) It is, locally, a monoidal transform; 2) in characteristic zero, the process is trivial if and only if the variety is non-singular. Examples show that this is not true in characteristic  $p > 0$ ; that, in general, the transform of a hypersurface is not locally a hypersurface; and that this process does not give, in general, minimal resolutions.**

**Introduction.** In this paper, the term algebraic variety (over a field  $k$ ) means reduced, separated algebraic scheme over  $k$ ; the term analytic variety means reduced, separated analytic space over  $\mathbb{C}$ , the complex numbers. Let  $k$  be an algebraically closed field (resp.  $k = \mathbb{C}$ ),  $X$  a reduced closed subscheme of a Zariski open  $U \subset \mathbb{A}^n$  (resp. a reduced closed complex subspace of an open  $U \subset \mathbb{C}^n$ ) of pure dimension  $r$ , defined by  $\{f_1, \dots, f_m\} \subset \Gamma(U, \mathcal{O}_U)$ . By the Nash blowing-up of  $X$  we mean the pair  $(X^*, p)$  obtained by the following process. Let  $S(X)$  be the set of singular points of  $X$ ,  $X_0$  its complement in  $X$ ,  $\eta: X_0 \rightarrow X \times G_r^n$  ( $G_r^n$  is the grassmanian of  $r$ -planes in  $n$ -space) the morphism determined by  $\eta(x) = (x, T_{x,x})$  for each closed point  $x \in X_0$  (here  $T_{x,x}$  is the tangent space of  $X$  at  $x$ , which can be identified with an  $r$ -plane in  $n$ -space),  $X^*$  the closure of  $\eta(X_0)$  in  $X \times G_r^n$  (resp. the closure in the metric topology),  $p: X^* \rightarrow X$  induced by the first projection. In the complex case it is not obvious that  $X^*$  is an analytic variety; see [7], Theorem 16.4 for a proof (or see Theorem 1 of this note).

It is possible to prove that  $(X^*, p)$  is (up to unique  $X$ -isomorphism) independent of the immersion (as a locally closed subset) of  $X$  in an affine space, hence the process globalizes.

*Sketch of proof.* Working (to simplify) in the algebraic case with closed points only, and calling  $G_r(T) = \{r\text{-linear planes in } T\}$  for any vector space  $T$ , one verifies that  $Z = \bigcup_{x \in X} x \times G_r(T_{x,x})$  is a subvariety of  $X \times G_r^n$ , and  $X^*$  is contained in  $Z$ . If  $X'$  is a locally closed in  $\mathbb{A}^n$ , we have (using notations as above, but with primes):  $X'^* \subset X' \times G_r^{m'}$ . Assume  $q: X \rightarrow X'$  is an isomorphism. Then,

$$(x, L) \rightarrow (q(x), dq(L)),$$

for  $(x, L) \in Z$ , defines an isomorphism  $Z \rightarrow Z'$ . This clearly induces an isomorphism  $X^* \rightarrow X'^*$ , commuting with the projections.

A natural question, which apparently has not been seriously studied, is to determine the desingularization properties of this process. In this note we present some very basic results in this direction: (a) in characteristic zero,  $p: X^* \rightarrow X$  is an isomorphism if and only if  $X$  is nonsingular, (b) in positive characteristic, (a) is false. We also verify (which allows to show (b) in a very clear way) that, locally, a Nash blowing-up is a monoidal transform, with center a suitable ideal. The proof of (a) presented here is analytic, and uses results of J. Stutz on branched coverings (cf. [4] and [5]). It would be interesting to have an algebraic proof which probably would throw more light on the main question: if, in characteristic zero, this process desingularizes (cf. Remark 3).

We finish with some examples, which indicate other features of the process (see §3).

**1. Monoidal transforms.** In this section  $k$  is, in the algebraic case, an algebraically closed field; in the analytic case  $k = \mathbb{C}$ . Our arguments hold in either case.

Recall that given a reduced subscheme  $X$  of  $\mathbb{A}_k^n$  (resp. a reduced subspace  $X$  of an open  $U$  in  $\mathbb{C}^n$ ) and  $\{g_0, \dots, g_s\} \subset \Gamma(X, \mathcal{O}_X)$ , the monoidal transform of  $X$  with center  $I = (g_0, \dots, g_s)$  can be constructed by taking the closure (in  $X \times \mathbb{P}^s$ ) of  $\varphi(Y)$ , where  $Y = X \setminus V(I)$  ( $V(I) = \text{locus of } I$ ) and  $\varphi: Y \rightarrow X \times \mathbb{P}^s$  is defined by  $\varphi(x) = (x, (g_0(x), \dots, g_s(x))) \in X \times \mathbb{P}^s$ , for any closed point  $x$  (see [1], Remark 2).

REMARK 1. We shall use the following notations:

(1) We have two closed embeddings of  $G_r^n$  in  $\mathbb{P}^N$ ,  $N = \binom{n}{r} - 1$ :

(i) the map  $\Lambda$ , which sends the point corresponding to the  $r$ -plane  $L$ , of parametric equations  $x_i = \sum_{d=1}^n b_d^i t_d$ ,  $i = 1, \dots, n$ , to the point of  $\mathbb{P}^N$  of homogeneous coordinates  $(\Delta_{i_1 \dots i_r})$ ,  $1 \leq i_1 < \dots < i_r \leq n$ , where  $\Delta_{i_1 \dots i_r}$  is the  $r \times r$  subdeterminant of  $\|b_d^i\|$  formed by the columns  $i_1, \dots, i_r$ .

(ii) the map  $\psi$ , which sends the point corresponding to the  $r$ -plane  $L$ , defined by the equations  $\sum_{j=1}^n a_j^i x_j = 0$ ,  $i = 1, \dots, n - r$ , to the point of  $\mathbb{P}^N$  of homogeneous coordinates  $(\Delta_{j_1 \dots j_{n-r}})$ , where  $\Delta_{j_1 \dots j_{n-r}}$  is the  $(n - r) \times (n - r)$  subdeterminant of  $\|a_j^i\|$  defined by the columns  $j_1, \dots, j_{n-r}$ . In the terminology of [2], Ch. VII,  $\Lambda$  corresponds to the use of Grassman coordinates and  $\psi$  to the dual Grassman coordinates; cf. Theorem I, p. 294 of [2] for their relations.

(2) Let  $X, \eta$  be as in the introduction. We shall write

$$\psi_0 = (\text{id} \times \psi)\eta: X_0 \rightarrow X \times \mathbf{P}^N$$

$$\Lambda_0 = (\text{id} \times \Lambda)\eta: X_0 \rightarrow X \times \mathbf{P}^N.$$

Clearly, there are natural isomorphisms:

$$X^* \approx \text{cl}(\psi_0(X_0)) \approx \text{cl}(\Lambda_0(X_0)),$$

where  $\text{cl}$  denotes closure in the corresponding ambient space.

(3) Given integers  $n \geq r > 0$ ,  $m \geq n - r$ , and an  $(m \times n)$ -matrix  $\|a_{ij}\| = A$ , let  $S$  (resp.  $S'$ ) denote the set of increasing sequences of  $(n - r)$ -positive integers less than  $m + 1$  (resp.  $n + 1$ ); if  $\alpha = (i_1, \dots, i_{n-r}) \in S$ ,  $\beta = (j_1, \dots, j_{n-r}) \in S'$ , then  $\Delta_{\alpha\beta}$  is the subdeterminant of  $A$  obtained by considering the rows  $i_1, \dots, i_{n-r}$  and the columns  $j_1, \dots, j_{n-r}$ .

**THEOREM 1.** *A Nash blowing-up is locally a monoidal transform (with center a suitable ideal).*

*Proof.* We may assume  $X$  is affine (resp. an analytic set in  $U \subset \mathbf{C}^n$ ); write  $X$  as union of its irreducible components,  $X = X_1 \cup \dots \cup X_d$  (in the analytic case, shrink  $U$  if necessary). Using the notations of the introduction, let  $M = \|\partial f_i / \partial x_j\|$ ,  $i = 1, \dots, m$ ,  $j = 1, \dots, n$ . Clearly, for each  $i = 1, \dots, d$ , there is  $(\alpha_i, \beta_i) \in S \times S'$  such that  $\Delta_{\alpha_i\beta_i}$  does not vanish on  $X_i$ ; hence  $W'_i = X_i \setminus V(\Delta_{\alpha_i\beta_i})$  is a nonempty open of  $X_i$ . For each  $i = 1, \dots, d$ , fix  $h_i \in \Gamma(X, \mathcal{O}_X)$  such that  $h_i = 0$  on  $\bigcup_{j \neq i} X_j$ ,  $h_i \neq 0$  on  $X_i$ . Consider the ideal  $I$  generated by  $\{g_\beta\}$ ,  $\beta \in S'$ , where  $g_\beta = \sum_{i=1}^d h_i \Delta_{\alpha_i\beta}$ . We claim that the monoidal transform with center  $I$  agrees with the Nash blowing-up of  $X$ .

In fact, first note that  $V(I) \supset S(X)$ . Call  $W = X \setminus V(I)$ , then (since all points in  $X_0 \setminus W$  are non-singular)  $X^* \approx$  closure of  $\psi_0(W)$  in  $\mathbf{P}^N$ . Hence, to show our contention we must check that the maps  $W \rightarrow \mathbf{P}^N$  given by  $p_2\psi_0 = \psi_1$  ( $p_2$  is the second projection) and by  $\{g_\beta\}$  agree. It is enough to check this at an arbitrary  $z \in W_i = W'_i \setminus V(h_i)$  ( $i = 1, \dots, d$ ). But for  $z \in W_i$ , as points of  $\mathbf{P}^N$ ,

$$(g_\beta(z)) = \left( \sum_{j=1}^d h_j(z) \Delta_{\alpha_j\beta}(z) \right) = (h_i(z) \Delta_{\alpha_i\beta}(z)) = (\Delta_{\alpha_i\beta}(z)) = \psi(L'),$$

where  $L'$  is the point of  $G_r^n$  corresponding to the  $r$ -plane  $L$ , defined by  $\{\sum (\partial f_{k_i}(z) / \partial x_j) x_j = 0\}$ ,  $(k_i) = \alpha_i$ , which is the tangent space to  $X$  at  $z$  (since  $\Delta_{\alpha_i\beta_i} \neq 0$ ). As clearly  $\psi(L') = \psi_i(z)$ , the assertion is proved.

REMARK 2. If  $X$  (of dimension  $r$ ) is defined by  $n - r$  equations, then the proof is simpler. In fact, we may take  $I$  to be the Jacobian ideal, formed by the  $(n - r) \times (n - r)$  minors of  $\|\partial f_i / \partial x_j\|$ ,  $i = 1, \dots, n - r$ ,  $j = 1, \dots, n$ . In general this is not true, as the example of two planes in  $A^4$ , meeting at one point, shows.

In this example also it can be seen that, in general, the support of the ideal of Theorem 1 is not the singular locus of  $X$ .

EXAMPLE 1. Let  $\text{ch}(k) = 2$ , consider the plane curve  $y^2 + x^3 = 0$ . By Remark 1, its Nash blowing-up is the monoidal transform with center  $I = (2y, 3x^2) = (x^2)$ . This is a principal ideal, hence  $p: X^* \rightarrow X$  is an isomorphism.

EXAMPLE 2. If  $\text{ch}(k) = q > 2$ , the Nash blowing-up of the plane curve  $y^2 - x^q = 0$  is trivial. The verification is as in Example 1.

## 2. Proof of the main theorem.

THEOREM 2. *Let  $k$  be an algebraically closed field of characteristic zero (resp.  $k = \mathbb{C}$ ),  $X$  a pure  $r$ -dimensional algebraic (resp. analytic) variety over  $k$ ,  $(X^*, p)$  the Nash blowing-up of  $X$ . Then,  $p$  is an isomorphism if and only if  $X$  is nonsingular.*

*Proof.* By descent theory we may assume, in the algebraic case, that  $k = \mathbb{C}$ . Moreover, it is clear (e.g., from Theorem 1) that  $(X^h)^* = (X^*)^h$  where  $X^h$  denotes the analytic variety associated to an algebraic variety  $X$ . Hence, it suffices to prove the theorem in the analytic case.

One implication is obvious. Let us show that if  $X$  has singularities the morphism  $p$  is not an isomorphism. Let  $S = S(X)$  be the singular set of  $X$ . We distinguish two cases.

Case a.  $S$  has a component of codimension 1.

Let  $W$  be a component of codimension 1. We claim that there is a point  $x_0 \in W$ , such that  $X$  can be embedded, locally about  $x_0$ , in a polydisk  $U \subset \mathbb{C}^n$ , in such a way that (with  $x_1, \dots, x_n$  coordinates in  $\mathbb{C}^n$ , and writing, to simplify,  $X \subset U$ ,  $W \subset U$ , etc.):

- (i)  $x_0$  corresponds to the origin
- (ii)  $W$  is defined by  $x_r = \dots = x_n = 0$ .
- (iii) Let  $X_1, \dots, X_m$  be the irreducible components of  $X$ . Then, there are analytic functions  $f_{ij}(x_1, \dots, x_r)$ , defined on

$$D = \{x \in \mathbb{C}^r / (x_1, \dots, x_r, 0, \dots, 0) \in U\}$$

such that

$$x \rightarrow (x_1, \dots, x_{r-1}, x_r^{s_j}, f_{r+1,j}(x), \dots, f_{n,j}(x))$$

defines a homeomorphism  $D \rightarrow X_j$ ,  $j = 1, \dots, m$ .

(iv) The integer  $s_j$  of (iii) is the multiplicity of  $X_j$  at any  $x \in W$ , and

$$f_{ij} = \sum_{k=s_j}^{\infty} a_{ij}^{(k)}(x_1, \dots, x_{r-1})x_r^k.$$

This is a consequence of the following results. Let  $W_0 = \{x \in W : S \text{ is nonsingular at } x, \dim C_4(X, x) = r, \dim C_5(X, x) = r + 1\}$ . Hence,  $C_i(X, x)$ ,  $i = 4, 5$ , are the indicated Whitney tangent cones to  $X$  at  $x$  (see [6], §3 for the definitions). In [4], Proposition 3.6, it is proved that  $W_0$  is a dense open set in  $W$ . Propositions 2.5 of [5] and 4.2, 4.6 of [4] imply that for any  $x_0 \in W_0$ , there is a local embedding of the type described above.

From now on, we shall assume that  $X$  is contained in such an open  $U \subset \mathbb{C}^n$ . (The result that we are proving is clearly local on  $X$ .)

Note that now, keeping the notations of Remark 1, the map  $\Lambda_0: X_0 \rightarrow X \times \mathbb{P}^N$  can be described as follows: calling  $\varphi_{ij} = x_i$ ,  $1 \leq i \leq r-1$ ,  $\varphi_{rj} = x_r^{s_j}$ ,  $\varphi_{r+k,j} = f_{r+k,j}$ ,  $k = 1, \dots, n-r$ , then, for  $j = 1, \dots, m$ :

$$\Lambda_0: (\varphi_{ij}(x))_{i=1, \dots, n} \rightarrow ((\varphi_{ij}(x))_{1 \leq i \leq n}, (\Delta_{i_1}^{(j)}, \dots, i_r)) \in X \times \mathbb{P}^N,$$

where  $0 < i_1 < i_2 < \dots < i_r \leq n$ , and  $\Delta_{i_1}^{(j)}, \dots, i_r$  is the subdeterminant of  $\|\partial \varphi_{ij} / \partial x_k\|$ ,  $i = 1, \dots, n$ ,  $k = 1, \dots, r$  formed of the rows  $i_1, \dots, i_r$ . Note that  $\Delta_{1,2, \dots, r}^{(j)} = s_j x_r^{s_j-1}$ .

We may assume  $p: X^* \rightarrow X$  to be bijective (otherwise, the theorem is trivial). Then, if  $A = \{x \in \mathbb{P}^N / z_{1, \dots, r} \neq 0\}$ , by using condition (iv) of the parametrization we see that  $\text{cl}(\Lambda_0(X_0)) \subset U \times A$ . Let us identify, by using  $\text{id} \times \Lambda$ , the varieties  $X^*$  and  $\text{cl}(\Lambda_0(X_0))$ . Then, the irreducible components  $X_j^*$  of  $X^* \subset U \times A$  are parametrized by:

$$(*) \quad (\varphi_{ij}(x), \dots, \varphi_{nj}(x), (s_j^{-1} x_r^{-s_j+1} \Delta_{i_1}^{(j)}, \dots, i_r)),$$

$1 \leq i_1 < \dots < i_r \leq n$  (except  $(1, 2, \dots, r)$ ),  $j = 1, \dots, m$ . By condition (iv),  $(s_j x_r^{s_j-1})^{-1} \Delta_{i_1}^{(j)}, \dots, i_r$  are analytic functions.

Now there are two possibilities: (1) A component of  $X$  is singular at  $x_0$ ; (2) All components of  $X$  are nonsingular at  $x_0$ .

To show that  $p$  is not an isomorphism, it clearly suffices to show that if  $X^{(q)}$  is the  $q$ th iterated blowing-up of  $X$  (i.e.,  $X^{(0)} = X$ ,  $X^{(1)} = X^*$ ,  $\dots$ ,  $X^{(q)} = (X^{(q-1)})^*$ ), then the induced canonical morphism

$p_q: X^{(q)} \rightarrow X$  is not an isomorphism. We shall see that in either situation (1) or (2) above, this is the case.

Consider (1) first; let  $X_j$  ( $j = 1, \dots, M \leq m$ ) be the components of  $X$  which are singular at  $x_0$ . After changing coordinates (if necessary), we may assume that we have a parametrization of the components of  $X$  satisfying (i) to (iv), and also:

$$f_{r+1,1} = \sum_{k=d}^{\infty} a_{r+1,1}^{(k)}(x_1, \dots, x_{r-1})x_r^k$$

where  $a_{r+1,1}^{(d)} \neq 0$ , and  $d$  is not a multiple of  $s = s_1$ . Write  $(i) = (i_1, \dots, i_r)$ , and, for  $j = 1, \dots, m$ ,

$$(sx_r^{s-1})^{-1} \Delta_{(i)}^{(j)} = \psi'_{(i)}^{(j)} = \sum b_{(i),k}^{(j)}(x_1 \cdots x_{r-1})x_r^k;$$

let  $\psi_{(i)}^{(j)} = \psi'_{(i)}^{(j)} - b_{(i),0}^{(j)}$ . Consider  $X^* \subset U \times A$  (we maintain the identification of  $X^*$  with  $\text{cl } \Lambda_0(X_0)$ ). After an obvious change of coordinates we may assume that  $p^{-1}(W)$  is defined, in  $U \times A$ , by  $z_i = 0$ ,  $i \geq r$ , and the parametrization of  $X_j^* = p^{-1}(X_j)$  induced by  $(*)$  is:

$$(x_1, \dots, x_{r-1}, x_r^s, (\psi_{(i)}^{(j)})), \quad j = 1, \dots, m.$$

If for some  $j = 1, \dots, M$ , there is an  $(i)$  such that  $b_{(i),k}^{(j)} \neq 0$  with  $k < s_j$ , then the multiplicity of  $X_j^*$ , at some point near  $x_0$ , is less than  $s$ , and hence  $p$  is not an isomorphism. If not, then  $(*)$  induces a parametrization of  $X^* \subset (U \times A)$ , satisfying (i) to (iv). We can repeat the process. We claim that after at most  $\mu$  blowing-ups, with  $\mu = [d/s]$ , either  $p_\mu$  is not bijective, or the multiplicity  $s$  of  $p_\mu^{-1}(X_1)$ , at some point near  $p_\mu^{-1}(x_0)$ , drops. In fact, were  $p_0 = p, p_1, \dots, p_\mu$  bijective, then one of the entries of the induced parametrization of  $p_\mu^{-1}(X_1)$  is of the form

$$(**) \quad \sum_{k=d}^{\infty} \gamma_{k,\mu} a_{r+1,1}^{(k)}(x_1, \dots, x_{r-1})x_r^{k-\mu s},$$

where  $\gamma_{k,\mu} = s^{-\mu} \prod_{v=1}^{\mu} k - (v-1)s$ . Since  $a_{q_1}^{(m)} \neq 0$ , and  $(d, s) < n$ , then  $0 < d - \mu s < s$  for  $\mu = [d/s]$ , and hence the multiplicity of such  $X_1^{(\mu)}$ , at some point near  $p_\mu^{-1}(x_0)$ , is less than  $s$ .

Consider the case (2). Since we assume that  $p$  is bijective, then for all  $x \in W$ ,  $T_{x_i,x} = T_{X_i,x}$ ,  $i = 1, \dots, m$ . We also may assume, after a further change of coordinates, that, aside from (i) to (iv), we have:  $X_m$  is the  $r$ -plane  $x_{r+1} = \dots = x_r = 0$  (i.e.,  $f_{k,m}(x) = 0$ ,  $k = r+1, \dots, n$ ). As before, we see that if the iterated blowing-ups

$$p_u: X^{(u)} \rightarrow X^{(u-1)}, \quad 1 \leq u \leq q$$

were bijective, then we can obtain a parametrization of the components of  $X^{(q)}$ , such that  $X_m^{(q)} = p_q^{-1}(X_m)$  and  $X_1^{(q)} = p_q^{-1}(X_1)$  are given, respectively, by  $(x_1, \dots, x_r, 0, \dots, 0)$  and  $(x_1, \dots, x_r, \psi_1, \dots, \psi_L)$  (for some integer  $L$ ) where, for some  $i_0$ ,

$$\psi_{i_0} = \sum_{k=d}^{\infty} \frac{k!}{(k-q)!} a_{r+1,1}^{(k)}(x_1, \dots, x_{r-1}) x_r^{k-q}.$$

If  $a_{r+1,1}^{(k)} = 0$  for  $k < d$ , and nonzero for  $k = d$ , then for  $q = d - 1$ ,  $\psi_{i_0}$  has the form

$$\psi_{i_0} = a(x_1, \dots, x_{r-1}) x_r + \dots, \quad a \neq 0.$$

Clearly, if  $a(z_1, \dots, z_n) \neq 0$ , then  $T_{X_1^{(q)}, z} \neq T_{X_m^{(q)}, z}$ , and the next Nash blowing-up has at least two points lying over  $z$ . Hence,  $p$  cannot be an isomorphism.

*Case b.*  $S$  has codimension  $> 1$  at each of its points.

The only nontrivial case is the following: assume that for all  $x \in S$ , for all  $\{x_i\} \rightarrow x$ ,  $i = 1, 2, \dots, x$ , nonsingular, such that  $\{T_{X, x_i}\}$  converges (in  $G_r^n$ ),  $\lim T_{X, x_i}$  is a fixed space  $T_x$  (otherwise,  $p^{-1}(x)$  has more than one point). Assume this is the case. Pick any  $x_0 \in S$ , and embed locally  $X$  in a polydisk  $U$  in  $\mathbb{C}^n$  (as before, we just write  $X \subset U$ ,  $S \subset U$ , etc.). Then,  $C_4(X, x_0)$  has dimension  $r$ . In fact, this cone is the set of limit positions of lines, tangent to nonsingular points of  $X$ . By [7] (Part I, Preliminaries), the function  $d: G_r^n \times \mathbf{P}^1 \rightarrow \mathbf{R}$  where, for an  $r$ -plane  $L \in G_r^n$  and a line  $\ell \in \mathbf{P}^1$ ,  $d(L, \ell) = \text{distance between } L \text{ and } \ell$  (intuitively, the sine of the angle between  $L$  and  $\ell$ ) is continuous. From this, it follows that  $C_4(X, x_0) \subset T_{x_0} = T$ , hence  $\dim C_4(X, x_0) \leq r$ . Since the inequality  $\dim C_4(X, x_0) \geq r$  always holds, we get  $\dim C_4(X, x_0) = r$ . By Proposition 2.6 of [4], this equality implies that, after shrinking  $U$  if necessary, the projection  $\pi$  on  $T$ , along a  $s(n-r)$ -plane transversal to  $T$  satisfies:  $B(\pi) = \{x \in X/\pi \text{ ramifies at } x\} = S$ . Hence,  $\dim B(\pi) < n - 1$ . This inequality implies, by the statement 1.8 of [4], that (after further shrinking of  $U$ , if necessary) all the irreducible components  $X_i$  of  $X$  are nonsingular, and  $S(X) = B(\pi) = \bigcup_{i \neq j} (X_i \cap X_j)$ . Thus, there is more than a component at  $x_0$  (since  $x_0$  is a singular point of  $X$ ), and  $T_{X, x_0} = T$  for all  $i$ . By changing coordinates (if necessary), we may assume  $X_1 = T$ , and by the implicit function theorem we may parametrize simultaneously the components  $X_j$ :

$$(x_1, \dots, x_r, \varphi_{r+1,j}(x'), \dots, \varphi_{n,j}(x')), \quad x' = (x_1, \dots, x_r).$$



Exactly as in part (2) of case (a), we see that after a finite number of Nash blowing-ups, the components of  $X$  get separated, and hence  $p$  cannot be an isomorphism.

The proof of the theorem is complete.

We have the following corollary, which seems to be well known:

**COROLLARY 1.** *If  $C$  is an algebraic curve, defined over an algebraically closed field of characteristic zero (resp. an analytic curve), a finite sequence of Nash blowing-ups desingularizes the curve.*

*Proof.* In general, the Nash blowing-up  $p: C^* \rightarrow C$  is a finite birational (resp. bimeromorphic) morphism. Then it is clear that after a finite number of Nash blowing-ups, we reach the normalization. Since this is nonsingular, the result follows.

As we saw, this is false in positive characteristic.

**REMARK 3.** In [7], J. Lipman proves, in a purely algebraic way, that for an algebraic variety  $X$ , the monoidal transform with center the sheaf of Jacobian ideals is trivial if and only if  $X$  is smooth. Since, by Remark 2, for complete intersections this transform agrees with the Nash blowing-up, it gives an algebraic proof of Theorem 2 in this case.

**3. Some remarks and examples.** In general, the Nash blowing-up of a hypersurface is not locally a hypersurface, as the following example shows.

**EXAMPLE 3.** Let  $\text{char}(k) = 0$ . Consider the plane curve  $X$  of parametric equations:

$$x = t^4, \quad y = \varphi(t) = t^{11} + t^{13}.$$

Let  $p: X^* \rightarrow X$  be the Nash blowing-up,  $x_0 \in X$  the origin. Then  $p^{-1}(x_0)$  has only one point  $x_1$ , and a neighborhood  $X_1$  of  $x_1$  in  $X^*$  is naturally contained in  $\mathbf{A}^2 \times \mathbf{A}^1 \subset \mathbf{A}^2 \times G_1^2$ , and has a parametrization

$$x = t^4, \quad y = t^{11} + t^{13}, \quad u = \frac{11}{4} t^7 + \frac{13}{4} t^9$$

(cf. proof of Theorem 2). We claim that the embedding dimension of  $X_1$  at  $x_1$  is 3. In fact, if  $\text{emb. dim}_{x_1} X_1 = 2$ , it would follow

$$y = t^{11} + t^{13} = g\left(t^4, \frac{11}{4} t^7 + \frac{13}{4} t^9\right)$$

for some  $g(x, u) \in k[[x, u]]$ . An elementary computation shows that this is impossible.

Even if it were true that the Nash blowing-up process desingularizes, in general one would not get "minimal" resolutions. Consider this example:

EXAMPLE 4. Let  $\text{ch}(k) \neq 2$ . Consider the surface  $X$  (in  $A^3$ ) defined by  $y^2 - x^2z = 0$ . It is well known that the normalization  $X'$  of this surface is nonsingular. Moreover, this normalization can be obtained by applying the monoidal transform with center the  $z$ -axis. Thus, one can get a desingularization  $\pi: X' \rightarrow X$ , with  $\pi$  finite. However, the Nash blowing-up  $p: X^* \rightarrow X$  is not finite, in fact  $p^{-1}(0)$  is a projective line. But  $X^*$  is nonsingular; in fact, using Remark 1, and the fact that (for any ideal  $I$ ), the monoidal transforms with center the ideal  $I$  and  $I^2$  coincide, it is easy to see that the Nash blowing-up of  $X$  can be obtained by composing  $\pi: X' \rightarrow X$  and the quadratic transform of  $X'$  with center the point  $\pi^{-1}(0)$ .

## REFERENCES

1. H. Hironaka, and M. Rossi, *On the equivalence of imbeddings of exceptional complex spaces*, Math. Annalen, **156** (1964), 313–333.
2. W. Hodge, and B. Pedoe, *Methods of Algebraic Geometry*, Vol. 1, Cambridge U. Press, 1947.
3. J. Lipman, *On the Jacobian ideal of the module of differentials*, Proc. Amer. Math. Soc., **21**, (1969), 422–426.
4. J. Stutz, *Analytic sets as branched coverings*, Trans. Amer. Math. Soc., **166** (1972), 241–259.
5. ———, *Equisingularity and equisaturation in codimension 1*, Amer. J. Math., **XCIV** (1972), 1245–1268.
6. H. Whitney, *Local properties of analytic varieties*, Differential and Combinatorial Topology, (Edited by S. Cairns), Princeton U. Press, 1965.
7. ———, *Tangents to an analytic variety*, Ann. Math., **81** (1965), 496–549.

Received June 12, 1974 and in revised form November 1, 1974.

LOUISIANA STATE UNIVERSITY, BATON ROUGE



D. E. Bennett, <i>Strongly unicoherent continua</i> .....	1
Walter R. Bloom, <i>Sets of <math>p</math>-spectral synthesis</i> .....	7
R. T. Bumby and D. E. Dobbs, <i>Amitsur cohomology of quadratic extensions: Formulas and number-theoretic examples</i> .....	21
W. W. Comfort, <i>Compactness-like properties for generalized weak topological sums</i> .....	31
D. R. Dunninger and J. Locker, <i>Monotone operators and nonlinear biharmonic boundary value problems</i> .....	39
T. S. Erickson, W. S. Martindale, 3rd and J. M. Osborn, <i>Prime nonassociative algebras</i> .....	49
P. Fischer, <i>On the inequality <math>\sum_{i=1}^n p_i \frac{f(p_i)}{f(q_i)} \geq 1</math></i> .....	65
G. Fox and P. Morales, <i>Compact subsets of a Tychonoff set</i> .....	75
R. Gilmer and J. F. Hoffmann, <i>A characterization of Prüfer domains in terms of polynomials</i> .....	81
L. C. Glaser, <i>On tame Cantor sets in spheres having the same projection in each direction</i> .....	87
Z. Goseki, <i>On semigroups in which <math>X = XYX = XZX</math> if and only if <math>X = XYZX</math></i> .....	103
E. Grosswald, <i>Rational valued series of exponentials and divisor functions</i> .....	111
D. Handelman, <i>Strongly semiprime rings</i> .....	115
J. N. Henry and D. C. Taylor, <i>The <math>\bar{\beta}</math> topology for <math>w^*</math>-algebras</i> .....	123
M. J. Hodel, <i>Enumeration of weighted <math>p</math>-line arrays</i> .....	141
S. K. Jain and S. Singh, <i>Rings with quasiprojective left ideals</i> .....	169
S. Jeyaratnam, <i>The diophantine equation <math>Y(Y+m)(Y+2m) \times (Y+3m) = 2X(X+m)(X+2m)(X+3m)</math></i> .....	183
R. Kane, <i>On loop spaces without <math>p</math> torsion</i> .....	189
Alvin J. Kay, <i>Nonlinear integral equations and product integrals</i> .....	203
A. S. Kechris, <i>Countable ordinals and the analytic hierarchy, I</i> .....	223
Ka-Sing Lau, <i>A representation theorem for isometries of <math>C(X, E)</math></i> .....	229
I. Madsen, <i>On the action of the Dyer-Lashof algebra in <math>H_*(G)</math></i> .....	235
R. C. Metzler, <i>Positive linear functions, integration, and Choquet's theorem</i> .....	277
A. Nobile, <i>Some properties of the Nash blowing-up</i> .....	297
G. E. Petersen and G. V. Welland, <i>Plessner's theorem for Riesz conjugates</i> .....	307

Donald Earl Bennett, <i>Strongly unicoherent continua</i> .....	1
Walter Russell Bloom, <i>Sets of <math>p</math>-spectral synthesis</i> .....	7
Richard Thomas Bumby and David Earl Dobbs, <i>Amitsur cohomology of quadratic extensions: formulas and number-theoretic examples</i> .....	21
W. Wistar (William) Comfort, <i>Compactness-like properties for generalized weak topological sums</i> .....	31
Dennis Robert Dunninger and John Stewart Locker, <i>Monotone operators and nonlinear biharmonic boundary value problems</i> .....	39
Theodore Erickson, Wallace Smith Martindale, III and J. Marshall Osborn, <i>Prime nonassociative algebras</i> .....	49
Pál Fischer, <i>On the inequality <math>\sum_{i=0}^n [f(p_i)/f(q_i)]p_i \geq i</math></i> .....	65
Geoffrey Fox and Pedro Morales, <i>Compact subsets of a Tychonoff set</i> .....	75
Robert William Gilmer, Jr. and Joseph F. Hoffmann, <i>A characterization of Prüfer domains in terms of polynomials</i> .....	81
Leslie C. Glaser, <i>On tame Cantor sets in spheres having the same projection in each direction</i> .....	87
Zensiro Goseki, <i>On semigroups in which <math>x = xyx = xzx</math> if and only if <math>x = xyzx</math></i> .....	103
Emil Grosswald, <i>Rational valued series of exponentials and divisor functions</i> .....	111
David E. Handelman, <i>Strongly semiprime rings</i> .....	115
Jackson Neal Henry and Donald Curtis Taylor, <i>The <math>\tilde{\beta}</math> topology for <math>W^*</math>-algebras</i> .....	123
Margaret Jones Hodel, <i>Enumeration of weighted <math>p</math>-line arrays</i> .....	141
Surender Kumar Jain and Surjeet Singh, <i>Rings with quasi-projective left ideals</i> .....	169
S. Jeyaratnam, <i>The Diophantine equation <math>Y(Y+m)(Y+2m)(Y+3m) = 2X(X+m)(X+2m)(X+3m)</math></i> .....	183
Richard Michael Kane, <i>On loop spaces without <math>p</math> torsion</i> .....	189
Alvin John Kay, <i>Nonlinear integral equations and product integrals</i> .....	203
Alexander S. Kechris, <i>Countable ordinals and the analytical hierarchy. I</i> .....	223
Ka-Sing Lau, <i>A representation theorem for isometries of <math>C(X, E)</math></i> .....	229
Ib Henning Madsen, <i>On the action of the Dyer-Lashof algebra in <math>H_*(G)</math></i> ...	235
Richard C. Metzler, <i>Positive linear functions, integration, and Choquet's theorem</i> .....	277
Augusto Nobile, <i>Some properties of the Nash blowing-up</i> .....	297
Gerald E. Peterson and Grant Welland, <i>Plessner's theorem for Riesz conjugates</i> .....	307