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**A CONSTRUCTION OF THE IDEMPOTENT-SEPARATING
CONGRUENCES ON A BISIMPLE ORTHODOX SEMIGROUP**

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A CONSTRUCTION OF THE IDEMPOTENT-SEPARATING CONGRUENCES ON A BISIMPLE ORTHODOX SEMIGROUP

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For any bisimple orthodox semigroup S we show how to construct all idempotent-separating congruences on S , and give an explicit construction for the quotient semigroup of S modulo such a congruence.

Introduction. For an arbitrary bisimple inverse semigroup S , Reilly and Clifford [8] have shown (1) how to construct all idempotent-separating congruences on S , and (2) have obtained an explicit construction for the quotient semigroup of S modulo such a congruence. Their work is based on the construction of all bisimple inverse semigroups given by Reilly in [7].

The purpose of this article is to extend the above results to the case where the semigroup S is a bisimple orthodox semigroup, by making use of the elegant construction theorem for all such semigroups due to Clifford [1; 2]. The construction of the idempotent-separating congruences on S yields an immediate one-to-one correspondence between these congruences and certain pairs (V, V') of normal subgroups of some of the components used in Clifford's construction of S . When this correspondence is applied to an abstract bisimple orthodox semigroup, it reduces to the one given by Munn [5] for bisimple regular semigroups.

1. Preliminaries. We shall adopt the notation and terminology of [3]. Clifford's construction of any bisimple orthodox semigroup is given in Theorem A of [1; 2]. As this construction is basic for our study, we begin by reviewing it and certain associated concepts.

By a *right Reilly groupoid* we mean a partial groupoid R satisfying the following four axioms.

- (R1) If a, b, c are elements of R such that bc and $a(bc)$ are defined, then ab and $(ab)c$ are defined, and $(ab)c = a(bc)$.
- (R2) If a is an element of R such that ab is defined for some b in R , then ax is defined for all x in R .
- (R3) If a, b, c are elements of R such that $ac = bc$, then $ax = bx$ for all x in R .
- (R4) R contains at least one left identity element.

By the *core* \hat{R} of a right Reilly groupoid R we mean the set of all a in R such that ab is defined for some, hence for all, b in R . Now \hat{R} is a subsemigroup of R containing all left identities of R . For any left identity e of R let H_e denote the group of units of the semigroup $\hat{R}e$. Define $a\mathcal{L}b$ (a, b in R) to mean that $a = xb$ and $b = ya$ for some x, y in \hat{R} . By Proposition 1.2 of [2], $a\mathcal{L}b$ if and only if $a = ub$ for some $u \in H_e$.

A *left Reilly groupoid* L and its core \hat{L} are defined dually. If e' is a right identity of a left Reilly groupoid L , let $H_{e'}$ denote the group of units of the semigroup $e'\hat{L}$. The relation \mathcal{R} on L is defined dually.

Let $R[L]$ be a right [left] Reilly groupoid. Elements of L will be denoted by primed letters. By an *anti-correlation* between L and R we mean a subset K of $L \times R$ satisfying the following conditions.

- (AC1) The projection of K into $L[R]$ is onto $L[R]$.
- (AC2) $(a', a) \in K, (b', b) \in K, (b', a) \in K$ imply $(a', b) \in K$.
- (AC3) If $(a', a) \in K$ then $a' \in \hat{L}$ if and only if $a \in \hat{R}$.
- (AC4) Let $\hat{K} = K \cap (\hat{L} \times \hat{R})$. Then $(a', a) \in \hat{K}, (b', b) \in K$ imply $(b'a', ab) \in K$.

The following axioms are needed.

- (AI) $R[L]$ is a right [left] Reilly groupoid, and K is an anti-correlation between them; $e[e']$ is an arbitrary but fixed left [right] identity of $R[L]$.

We write $\kappa = K^{-1} \circ K$ and $\lambda = K \circ K^{-1}$, and let $H_e[H_{e'}]$ be the group of units of $\hat{R}e[\hat{L}e']$.

- (AII) $ackbc$ ($a, b \in \hat{R}, c \in R$) imply $aekbe$; and $c'a'\lambda c'b'(a', b' \in \hat{L}, c' \in L)$ imply $e'a'\lambda e'b'$.
- (AIII) For $a' \in L, (a', e) \in K$ if and only if a' is a right identity of L ; for $a \in R, (e', a) \in K$ if and only if a is a left identity of R .

Using (AI–III), one can show that K induces an anti-isomorphism $u \rightarrow u'$ from H_e onto $H_{e'}$. Here, for $u \in H_e$, u' denotes the unique element of $H_{e'}$ such that $(u', u) \in K$.

Define an equivalence relation τ on $L \times R$ by

- (1.1) $(a', b)\tau(c', d)$ if and only if $c' = a'u'$ and $d = ub$ for some $u \in H_e$.

Let $(a', b)_\tau$ be the τ -class containing (a', b) . Proposition 3.2 of [2] shows that the subsets $\hat{L} \times \hat{R}$, $e'\hat{L} \times \hat{R}e$, and K of $L \times R$ are unions of τ -classes, and so we write

$$\begin{aligned} T &= (L \times R)/\tau, & \hat{T} &= (\hat{L} \times \hat{R})/\tau, & T^0 &= (e'\hat{L} \times \hat{R}e)/\tau, \\ K_\tau &= K/\tau, & \hat{K}_\tau &= \hat{T} \cap K_\tau, & K_\tau^0 &= T^0 \cap K_\tau, \end{aligned}$$

and $\hat{K} = K \cap (\hat{L} \times \hat{R})$. For arbitrary $(x', y)_\tau$ in \hat{T} and (a', b) in $L \times R$, Proposition 3.3 of [2] allows us to define

$$(1.2) \quad (x', y)_\tau (a', b) = (a'x', yb)_\tau.$$

We now postulate a binary operation (\circ) on K_τ such that the following axioms hold.

(AIV) $K_\tau(\circ)$ is a band; and, for all $(a', a) \in \hat{K}$,

$$(e', e)_\tau \circ (a', a)_\tau = (e' a', a)_\tau,$$

$$(a', a)_\tau \circ (e', e)_\tau = (a', ae)_\tau.$$

(AVI) If $(a', a)_\tau$ and $(b', b)_\tau$ belong to K_τ^0 and $(c', c) \in K$, then $[(a', a)_\tau \circ (b', b)_\tau](c', c) = (a', a)_\tau(c', c) \circ (b', b)_\tau(c', c)$.

(AVII) For each element (c', b) of $L \times R$, there exists an element $(x', y)_\tau$ of \hat{T} such that $(b', b)_\tau \circ (c', c)_\tau = (x', y)_\tau(b', c) = (b'x', yc)_\tau$, for any $b' \in L$ and $c \in R$ such that (b', b) and (c', c) belong to K .

REMARK. In [1; 2] an axiom (AV) is also postulated. However, we have shown in [4] that this axiom is a consequence of axioms (AI–IV, VII).

Using these axioms, Proposition 3.5 of [2] shows that an element $(x', y)_\tau$ satisfying (AVII) exists in T^0 , and this element is uniquely determined by b and c' ; denoting it by $b \star c'$ the equation in (AVII) becomes

$$(1.3) \quad (b', b)_\tau \circ (c', c)_\tau = (b \star c')(b', c).$$

By a *box frame* we mean a system $(L, e'; R, e; K)$ satisfying axioms (AI–III). By a *banded box frame* $(L, e'; R, e; K_\tau(\circ))$ we mean a box frame $(L, e'; R, e; K)$ together with a binary operation (\circ) on K_τ satisfying (AIV–VII).

THEOREM A [1]. *Let $(L, e'; R, e; K_\tau(\circ))$ be a banded box frame, and let $T = (L \times R)/\tau$. Define a binary operation (\circ) on T by*

$$(1.4) \quad (a', b)_\tau \circ (c', d)_\tau = (b \star c')(a', d),$$

where $b \star c'$ is the element of T^0 defined by (1.3). Then the following hold:

- (i) $T(\circ)$ is a bisimple orthodox semigroup.
- (ii) The band of idempotents of $T(\circ)$ is precisely $K_\tau(\circ)$.
- (iii) The mapping $b \rightarrow (e', b)_\tau$ is an isomorphism of R onto the \mathcal{R} -class R_e of $T(\circ)$ containing the idempotent $\epsilon = (e', e)_\tau$, and $a' \rightarrow (a', e)_\tau$ is an isomorphism of L onto L_e .
- (iv) For arbitrary $a' \in L$ and $b \in R$, $(a', b) \in K$ if and only if $(a', e)_\tau$ and $(e', b)_\tau$ are inverse to each other in $T(\circ)$.

Conversely, if S is a bisimple orthodox semigroup, if e is an idempotent of S , and K_e is the set of all mutually inverse pairs (a', a) in $L_e \times R_e$, then K_e is an anti-correlation between L_e and R_e and $(L_e, e; R_e, e; K_e)$ is a box frame. Define τ on $L_e \times R_e$ by $(a', b)\tau(c', d)$ if and only if $c' = a'u^{-1}$ and $d = ub$ for some u in H_e , and let $T_e = (L_e \times R_e)/\tau$, $K_\tau = K_e/\tau$. The mapping $\theta: T_e \rightarrow S$ defined by $(a', b)_\tau \theta = a'b$ is a bijection, and maps K_τ onto E_S . The latter enables us to define a binary operation (\circ) on K_τ by

$$(a', a)_\tau \circ (b', b)_\tau = (c', c)_\tau \text{ if and only if } (a'a)(b'b) = c'c.$$

Then $(L_e, e; R_e, e; K_\tau(\circ))$ is a banded box frame. Defining $\star: R \times L \rightarrow T^0$ by (1.3), and then (\circ) on T_e by (1.4), the above mapping θ is an isomorphism of (T_e, \circ) onto S .

REMARKS. (1) From the proof of Theorem A in [2], it is clear that the operation (\circ) on T extends the band operation (\circ) on $K_\tau(\circ)$.

(2) For purposes in the next section, we reformulate the operation (\circ) on T as follows. According to equations (1.4), (1.3), (1.2), and axiom (AVII), for any $(a', b)_\tau$ and $(c', d)_\tau$ in T , we have $(a', b)_\tau \circ (c', d)_\tau = (a'x', yd)_\tau$, where $(x', y)_\tau$ is the unique element in T^0 such that

$$(b', b)_\tau \circ (c', c)_\tau = (b'x', yc)_\tau,$$

for any $b' \in L$ and $c \in R$ such that (b', b) , (c', c) are in K .

The following lemma is immediate from (R3); we shall use it and its left-right dual frequently without explicit mention.

LEMMA 1.1. Let e be a left identity for a right Reilly groupoid R . If $ac = bc$ with a, b in $\hat{R}e$ and c in R , then $a = b$. In particular, $\hat{R}e$ is right cancellative.

LEMMA 1.2. Let $T(\circ)$ be a bisimple orthodox semigroup constructed from a banded box frame $(L, e'; R, e; K_\tau(\circ))$ as in Theorem A. Then in $T(\circ)$

- (i) $(a', b)_{\tau} \mathcal{R}(c', d)_{\tau}$ if and only if $a' = c'u'$ for some $u \in H_e$;
- (ii) $(a', b)_{\tau} \mathcal{L}(c', d)_{\tau}$ if and only if $b = vd$ for some $v \in H_e$;
- (iii) $(a', b)_{\tau} \mathcal{H}(c', d)_{\tau}$ if and only if $a' = c'u'$ and $b = vd$ for some u, v in H_e .

Proof. Suppose $(a', b)_{\tau} \mathcal{R}(c', d)_{\tau}$ in $T(\circ)$; then there are elements $(x', y)_{\tau}$ and $(w', z)_{\tau}$ in T such that $(a', b)_{\tau} = (c', d)_{\tau} \circ (x', y)_{\tau}$ and $(c', d)_{\tau} = (a', b)_{\tau} \circ (w', z)_{\tau}$. Thus $(a', b)_{\tau} = (c'p', qy)_{\tau}$ for some $(p', q)_{\tau} \in T^0 = (e'\hat{L} \times \hat{R}e)/\tau$ and $(c', d)_{\tau} = (a'r', sz)_{\tau}$ for some $(r', s)_{\tau} \in T^0$. Then $a' = (c'p')u'$ and $b = u(qy)$, some $u \in H_e$, and $c' = (a'r')v'$ and $d = v(sz)$, some $v \in H_e$, so

$$a'e' = a' = c'(p'u') = (a'r'v')(p'u') = a'(r'v'p'u').$$

Since e' and $r'v'p'u'$ belong to $e'\hat{L}$, the dual of Lemma 1.1 gives $e' = r'v'p'u'$, whence $p'u'$ is a unit in H_e . Therefore $a' = c'(p'u')$ with $p'u' \in H_e$, as desired.

Conversely, suppose $(a', b)_{\tau}, (c', d)_{\tau}$ belong to $T(\circ)$ with $a' = c'u'$ for some $u \in H_e$ (hence $u' \in H_e$). By (AC1), let $b' \in L$ such that $(b', b) \in K$. Then

$$\begin{aligned} (a', b)_{\tau} \circ (b', ud)_{\tau} &= (b \star b')(a', ud) \\ &= (e', e)_{\tau}(a', ud) \text{ by Proposition 4.5 of [2]} \\ &= (a'e', eud)_{\tau} \\ &= (a', ud)_{\tau} \\ &= (a'(u')^{-1}, d)_{\tau} \\ &= (c', d)_{\tau}. \end{aligned}$$

By similarity we conclude that $(a', b)_{\tau} \mathcal{R}(c', d)_{\tau}$ in $T(\circ)$. Assertion (ii) is dual to (i), and (iii) is immediate from (i) and (ii).

2. Idempotent-separating congruences. Let e be a left identity of a right Reilly groupoid R , and let us call a subgroup V of H_e a *left normal divisor* of $\hat{R}e$ provided that $aV \subseteq Va$ for every a in $\hat{R}e$. Clearly, such a subgroup is a normal subgroup of H_e . Dually, if e' is a right identity for a left Reilly groupoid L , we call a subgroup V' of H_e a *right normal divisor* of $e'\hat{L}$ if $V'b' \subseteq b'V'$ for every $b' \in e'\hat{L}$. These concepts are due to Rees [6].

Let $T(\circ)$ be a bisimple orthodox semigroup constructed from a banded box frame $(L, e'; R, e; K_{\tau}(\circ))$ as in Theorem A. If $V[V']$ is a left [right] normal divisor of $\hat{R}e[e'\hat{L}]$ such that the anti-isomorphism from H_e to H_e maps V onto V' , we call the pair (V, V') a *linked pair* of

left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$. The following theorem is the analogue of Theorem 2.4 of [8].

THEOREM 2.1. *Let $T(\circ)$ be a bisimple orthodox semigroup constructed from a banded box frame $(L, e'; R, e; K, (\circ))$ as in Theorem A. Let (V, V') be a linked pair of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$. Define a relation $\rho_{(V, V')}$ on $T(\circ)$ as follows:*

$(a', b)_{\tau} \rho_{(V, V')} (c', d)_{\tau}$ if and only if there are elements u, v of H_e such that $a' = c'u'$, $b = vd$, and $u^{-1}v \in V$. Then $\rho_{(V, V')}$ is an idempotent-separating congruence on $T(\circ)$. Moreover, if (V_1, V'_1) and (V_2, V'_2) are two pairs of linked left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$, then $\rho_{(V_1, V'_1)} \subseteq \rho_{(V_2, V'_2)}$ if and only if $V_1 \subseteq V_2$.

Conversely, if ρ is any idempotent-separating congruence on $T(\circ)$ then there is a linked pair (V, V') of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$ such that $\rho = \rho_{(V, V')}$, namely, $V = \{v \in H_e : (e', v)_{\tau} \rho (e', e)_{\tau}\}$ and $V' = \{v' \in H_e : (v', e)_{\tau} \rho (e', e)_{\tau}\}$.

Proof. It is clear from Lemma 1.2(iii) that the relation $\rho_{(V, V')}$ is contained in \mathcal{H} , so that $\rho_{(V, V')}$ separates idempotents. It is also clear that $\rho_{(V, V')}$ is reflexive.

For symmetry, suppose $(a', b)_{\tau} \rho_{(V, V')} (c', d)_{\tau}$, say $a' = c'u'$ and $b = vd$ for some $u, v \in H_e$ with $u^{-1}v \in V$. Then $c' = a'(u')^{-1} = a'(u^{-1})'$, $d = v^{-1}b$, and since V is a normal subgroup of H_e , $u^{-1}v \in V$ implies $uv^{-1} \in V$.

For transitivity, let $(a', b)_{\tau} \rho_{(V, V')} (c', d)_{\tau}$ and $(c', d)_{\tau} \rho_{(V, V')} (g', f)_{\tau}$, say $a' = c'u'$, $b = vd$, $c' = g'w'$ and $d = xf$ with $u, v, w, x \in H_e$ and $u^{-1}v, w^{-1}x \in V$. Then

$$a' = c'u' = (g'w')u' = g'(w'u') = g'(uw)' \quad \text{and}$$

$$b = vd = v(xf) = (vx)f \quad \text{with}$$

$(uw)^{-1}(vx) = w^{-1}(u^{-1}v)w w^{-1}x \in V$ since $u^{-1}v, w^{-1}x \in V$. Thus $\rho_{(V, V')}$ is an equivalence on $T(\circ)$.

To see that $\rho_{(V, V')}$ is a congruence on $T(\circ)$, let $(a', b)_{\tau} \rho_{(V, V')} (c', d)_{\tau}$, say $a' = c'u'$ and $b = vd$ with $u, v \in H_e$, $u^{-1}v \in V$. Let $(x', y)_{\tau}$ be arbitrary in $T(\circ)$; for right compatibility we must show $(a', b)_{\tau} \circ (x', y)_{\tau} \rho_{(V, V')} (c', d)_{\tau} \circ (x', y)_{\tau}$. Now $(a', b)_{\tau} \circ (x', y)_{\tau} = (a'p', qy)_{\tau}$, where $(p', q)_{\tau}$ is the unique element of $T^0 = (e'\hat{L} \times \hat{R}e)/\tau$ such that

$$(2.1) \quad (b', b)_{\tau} \circ (x', x)_{\tau} = (b'p', qx)_{\tau}$$

for any $b' \in L$ and $x \in R$ such that (b', b) and (x', x) are in K .

Likewise, $(c', d)_{\tau} \circ (x', y)_{\tau} = (c'r', sy)_{\tau}$ where $(r', s)_{\tau}$ is the unique element of T^0 such that

$$(2.2) \quad (d', d)_\tau \circ (x', x)_\tau = (d'r', sx)_\tau$$

for any $d' \in L$ and $x \in R$ such that (d', d) and (x', x) are in K .

By (AC1), choose $d' \in L$ and $x \in R$ such that $(d', d), (x', x)$ are in K . Since $(d', d) \in K$ and $(v', v) \in \hat{K}$, (AC4) implies $(d'v', vd) \in K$, i.e., $(d'v', b) \in K$. Thus from (2.1) we have

$$(d'v'; b)_\tau \circ (x', x)_\tau = (d'v'p', qx)_\tau, \quad \text{i.e.,}$$

$$(d'v', vd)_\tau \circ (x', x)_\tau = (d'v'p', qx)_\tau, \quad \text{so that}$$

$$(2.3) \quad (d', d)_\tau \circ (x', x)_\tau = (d'v'p', qx)_\tau. \quad \text{Comparing}$$

(2.3) with (2.2), the uniqueness of $(r', s)_\tau$ in T^0 forces $(r', s)_\tau = (v'p', q)_\tau$ [note that $v' \in V' \subseteq H_e \subseteq e'\hat{L}$, and $p' \in e'\hat{L}$, implies $v'p' \in e'\hat{L}$; and $q \in \hat{R}e$, so that $(v'p', q)_\tau \in (e'\hat{L} \times \hat{R}e)/\tau = T^0$].

From $(r', s)_\tau = (v'p', q)_\tau$, there exists $\alpha \in H_e$ such that $q = \alpha s$ and $v'p' = r'\alpha'$. Then $p' = (v')^{-1}r'\alpha' = (v^{-1})'r'\alpha'$. Now

$$\begin{aligned} a'p'(\alpha')^{-1} &= a'[(v^{-1})'r'\alpha'](\alpha')^{-1} \\ &= a'(v^{-1})'r' \\ &= c'u'(v^{-1})'r' \\ &= c'(v^{-1}u)'r', \end{aligned}$$

and $u^{-1}v \in V$ implies $v^{-1}u \in V$, whence $(v^{-1}u)' \in V'$. Since $r' \in e'\hat{L}$ and V' is a right normal divisor of $e'\hat{L}$, $V'r' \subseteq r'V'$. Thus $(v^{-1}u)'r' = r'z'$, some $z' \in V'$ (so $z \in V$). Consequently, $a'p'(\alpha')^{-1} = c'(v^{-1}u)'r' = c'r'z'$ and so $a'p' = (c'r'z')\alpha' = (c'r')(\alpha z)'$ with $\alpha z \in H_e$ (since $\alpha \in H_e$ and $z \in V \subseteq H_e$).

Then $a'p' = (c'r')(\alpha z)'$ with $\alpha z \in H_e$, and $qy = (\alpha s)y = \alpha(sy)$ with $\alpha \in H_e$, and $(\alpha z)^{-1}\alpha = z^{-1}\alpha^{-1}\alpha = z^{-1} \in V$. Therefore $(a'p', qy)_\tau$, $\rho_{(V, V')}(c'r', sy)_\tau$ and $\rho_{(V, V')}$ is right compatible; left compatibility is dual.

If (V_1, V'_1) and (V_2, V'_2) are two pairs of linked left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$, the definitions of $\rho_{(V_i, V'_i)}$ make it clear that if $V_1 \subseteq V_2$ (and hence $V'_1 \subseteq V'_2$) then $\rho_{(V_1, V'_1)} \subseteq \rho_{(V_2, V'_2)}$. The converse is evident since, for any linked pair (V, V') , we have

$$V = \{v \in H_e : (e', v)_\tau \rho_{(V, V')}(e', e)_\tau\}, \quad \text{and}$$

$$V' = \{v' \in H_e : (v', e)_\tau \rho_{(V, V')}(e', e)_\tau\}.$$

Turning to the converse assertion of the theorem, let ρ be any idempotent-separating congruence on $T(\circ)$, and let N be the ρ -class containing $\epsilon = (e', e)_\tau$. Since $\rho \subseteq \mathcal{H}$ in $T(\circ)$, $N \subseteq H_e = R_e \cap L_e$, and by Theorem A(iii),

$$R_\epsilon = \{(e', b)_\tau : b \in R\} \quad \text{and} \\ L_\epsilon = \{(a', e)_\tau : a' \in L\}.$$

For any $x \in N$, $x \in R_\epsilon$ implies $x = (e', b)_\tau$ for some $b \in R$; but $x \in L_\epsilon$ implies $(e', b)_\tau \mathcal{L}(e', e)_\tau$, whence Lemma 1.2(ii) gives $b = ue = u$, some $u \in H_\epsilon$. Thus any $x \in N$ can be written as

$$x = (e', u)_\tau = ((u^{-1})', e)_\tau, \quad \text{some } u \in H_\epsilon.$$

Let
$$V = \{v \in H_\epsilon : (e', v)_\tau \text{ is in } N\} \quad \text{and} \\ V' = \{v' \in H_\epsilon : (v', e)_\tau \text{ is in } N\}.$$

Then $V[V']$ is a subgroup of $H_\epsilon[H_\epsilon]$, and we shall show that (V, V') is a linked pair of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$ such that $\rho = \rho_{(V, V')}$.

To see that the anti-isomorphism $u \rightarrow u'$ from H_ϵ to $H_{\epsilon'}$ maps V onto V' , let $v \in V$. Then $(e', v)_\tau \in N$, and

$$(e', v)_\tau = [e'(v^{-1})', v^{-1}v]_\tau = [(v^{-1})', e]_\tau$$

implies $(v^{-1})' \in V'$. Since $(v^{-1})' = (v')^{-1}$ and V' is a group, we have $v' \in V'$. Thus $u \rightarrow u'$ maps V into V' . On the other hand, if $v' \in V'$ then $(v', e)_\tau \in N$, so $(v', e)_\tau = [v'(v^{-1})', v^{-1}e]_\tau = (e', v^{-1})_\tau$ shows v^{-1} , and hence v , is in V . Thus V maps onto V' .

To see that V is a left normal divisor of $\hat{R}e$, first note that by Lemma 3 of [5], N is a left normal divisor of

$$P_\epsilon = \{x \text{ in } R_\epsilon : x \circ \epsilon = x\} \\ = \{(e', b)_\tau : b \in R \text{ and } (e', b)_\tau \circ (e', e)_\tau = (e', b)_\tau\}$$

and a right normal divisor of

$$Q_\epsilon = \{x \text{ in } L_\epsilon : \epsilon \circ x = x\} \\ = \{(a', e)_\tau : a' \in L \text{ and } (e', e)_\tau \circ (a', e)_\tau = (a', e)_\tau\}.$$

Since the isomorphism $a \rightarrow (e', a)_\tau$ of R onto R_ϵ carries $\hat{R}e$ onto P_ϵ and V onto N , it follows that V is a left normal divisor of $\hat{R}e$. Similarly, the isomorphism $a' \rightarrow (a', e)_\tau$ of L onto L_ϵ carries $e'\hat{L}$ onto Q_ϵ and V' onto N , so V' is a right normal divisor of $e'\hat{L}$.

Before proceeding to show that $\rho = \rho_{(V, V')}$, we note a lemma and corollary which will shorten our work.

LEMMA 2.2. *If $(b', b) \in K$ then $(a', b)_\tau \circ (b', c)_\tau = (a', c)_\tau$.*

Proof.

$$\begin{aligned} (a', b)_\tau \circ (b', c)_\tau &= (b \star b')(a', c)_\tau = (e', e)_\tau (a', c) \\ &= (a'e', ec)_\tau = (a', c)_\tau. \end{aligned}$$

COROLLARY 2.3. *If $u, v \in H_e$, and (c', c) and (d', d) are in K then*

$$(a', c)_\tau \circ (c'u', vd)_\tau \circ (d', b)_\tau = (a', u^{-1}vb)_\tau.$$

Proof.

$$\begin{aligned} (a', c)_\tau \circ (c'u', vd)_\tau \circ (d', b)_\tau &= (a', c)_\tau \circ (c', u^{-1}vd)_\tau \circ (d', b)_\tau \\ &= (a', u^{-1}vd)_\tau \circ (d', b)_\tau \\ &= (a'(v^{-1}u)', d)_\tau \circ (d', b)_\tau \\ &= (a'(v^{-1}u)', b)_\tau \\ &= (a', u^{-1}vb)_\tau. \end{aligned}$$

To see that $\rho = \rho_{(V, V)}$, let $(a', b)_\tau \rho_{(V, V)}(c', d)_\tau$, say $a' = c'u'$, $b = vd$, $u, v \in H_e$, and $u^{-1}v \in V$. Then $(e', u^{-1}v)_\tau \in N$, so $(e', u^{-1}v)_\tau \rho(e', e)_\tau$. Therefore

$$(e', u^{-1}v)_\tau \circ (e', d)_\tau \rho(e', e)_\tau \circ (e', d)_\tau,$$

that is, $(e', u^{-1}vd)_\tau \rho(e', d)_\tau$. But then

$$(c', e)_\tau \circ (e', u^{-1}vd)_\tau \rho(c', e)_\tau \circ (e', d)_\tau,$$

that is, $(c', u^{-1}vd)_\tau \rho(c', d)_\tau$. But $(c', u^{-1}vd)_\tau = (c'u', vd)_\tau = (a', b)_\tau$, whence $(a', b)_\tau \rho(c', d)_\tau$ and $\rho_{(V, V)} \subseteq \rho$.

Conversely, let $(a', b)_\tau \rho(c', d)_\tau$. Since $\rho \subseteq \mathcal{H}$, Lemma 1.2(iii) says $a' = c'u'$ and $b = vd$ for some $u, v \in H_e$. Then $(a', b)_\tau = (c'u', vd)_\tau$, so that $(c'u', vd)_\tau \rho(c', d)_\tau$. By (AC1) let $c \in R$ and $d' \in L$ such that (c', c) and (d', d) are in K . Then

$$(e'c)_\tau \circ (c'u', vd)_\tau \circ (d', e)_\tau \rho(e', c)_\tau \circ (c', d)_\tau \circ (d', e)_\tau.$$

Applying Corollary 2.3 to both sides of this last relation we obtain

$$(e', u^{-1}v)_\tau \rho(e', e)_\tau.$$

This puts $u^{-1}v \in V$, and so $\rho \subseteq \rho_{(V, V')}$. The proof is complete.

As an immediate corollary we obtain

COROLLARY 2.4. *Let $T(\circ)$ be a bisimple orthodox semigroup constructed from a banded box frame $(L, e'; R, e; K_r(\circ))$ as in Theorem A. There is a one-to-one, inclusion preserving correspondence between the idempotent-separating congruences on $T(\circ)$ and the linked pairs (V, V') of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$.*

COROLLARY 2.5. *Let $T(\circ)$ be as in Corollary 2.4. Then \mathcal{H} is a congruence on $T(\circ)$ if and only if $(H_e, H_{e'})$ is a linked pair of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$.*

REMARK. Let S be any bisimple orthodox semigroup, and e any idempotent in S . By Theorem A, $S \approx T_e$ where T_e is constructed from the box frame $(L_e, e; R_e, e; K_e)$ as in Theorem A. Now by definition, $\hat{R}_e = \{a \in R_e : ae \in R_e\}$ and clearly $\hat{R}_e e = \{x \in R_e : xe = x\} = R_e \cap eSe = P_e$. Similarly, $e\hat{L}_e = \{x \in L_e : ex = x\} = L_e \cap eSe = Q_e$. Since H_e is the group of units of eSe , H_e is then the group of units of $\hat{R}_e e = P_e$ and of $e\hat{L}_e = Q_e$. Thus the anti-isomorphism of Proposition 3.1 in [2] is the mapping $u \rightarrow u^{-1}$ on H_e .

Suppose we identify S with T_e . Then if (V, V') is a linked pair of left and right normal divisors of $\hat{R}_e e$ and $e\hat{L}_e$, we have $V' = V^{-1} = V$, so that V is a subgroup of H_e satisfying (1) $aV \subseteq Va$ for every a in $\hat{R}_e e = P_e$, and (2) $Vb \subseteq bV$ for every b in $e\hat{L}_e = Q_e$. Thus V is a subgroup of H_e which is a left normal divisor of P_e and a right normal divisor of Q_e .

So the one-to-one correspondence in Corollary 2.4 is just the one stated in Munn's theorem [5].

3. A construction of the quotient semigroup. For any bisimple orthodox semigroup T and any idempotent-separating congruence ρ on T , T/ρ is also a bisimple orthodox semigroup, and so the converse half of Theorem A gives a construction for T/ρ in terms of a banded box-frame whose components are internal to T/ρ . In this section we show how Theorem A may be used to describe a construction of T/ρ in terms involving the original box-frame used to construct T .

Recall from [1] that by a congruence on a right Reilly groupoid R we mean an equivalence relation ρ on R satisfying the following conditions.

(CR1) \hat{R} is a union of ρ -classes.

(CR2) If $(a, b) \in \rho \cap (\hat{R} \times \hat{R})$ and $c \in R$, then $(ac, bc) \in \rho$.

(CR3) If $(a, b) \in \rho$ and $c \in \hat{R}$, then $(ca, cb) \in \rho$.

A congruence on a left Reilly groupoid is defined dually. The following lemma generalizes Lemma 2.2 of [8] to right Reilly groupoids.

LEMMA 3.1. *Let R be a right Reilly groupoid, e a fixed left identity for R , and V a left normal divisor of $\hat{R}e$. Then*

$$(3.1) \quad \sigma_v = \{(a, b) \in R \times R : a = ub, \text{ some } u \in V\}$$

is a congruence on R such that $\sigma_v \subseteq \mathcal{L}$ and (R3) holds for R/σ_v .

Conversely, if σ is a congruence on R such that $\sigma \subseteq \mathcal{L}$ and (R3) holds for R/σ , then $\sigma = \sigma_v$ where $V = e\sigma$, and V is a left normal divisor of $\hat{R}e$.

Proof. Clearly σ_v is an equivalence relation on R . For (CR1), suppose $a\sigma_v b$ with $a \in \hat{R}$. Then $b = ua$, some $u \in V$. Since $a, u \in \hat{R}$, so is $b = ua$, whence it follows that \hat{R} is a union of σ_v -classes.

To show (CR2), let $a\sigma_v b$ with $a, b \in \hat{R}$ and let $c \in R$. Then $a = ub$, some $u \in V$, so that $ac = (ub)c = u(bc)$ and $a\sigma_v bc$.

Turning to (CR3), let $a\sigma_v b$ and $c \in \hat{R}$, say $a = ub$, some $u \in V$. Now ce lies in $\hat{R}e$, so by left normality, $(ce)u = v(ce)$ for some $v \in V$. Then $ca = c(ea) = (ce)a = (ce)(ub) = vceb = v(cb)$, and so $ca\sigma_v cb$. Thus σ_v is a congruence on R , and $\sigma_v \subseteq \mathcal{L}$ is clear.

As noted on p. 15 of [2], R/σ_v becomes a partial groupoid satisfying (R1), (R2), and (R4) if we define $(a\sigma_v)(b\sigma_v) = (ab)\sigma_v$ if $a \in \hat{R}$, and $(a\sigma_v)(b\sigma_v)$ is undefined otherwise. To see that R/σ_v satisfies (R3), let $a\sigma_v, b\sigma_v$, and $c\sigma_v$ belong to R/σ_v such that $(a\sigma_v)(c\sigma_v) = (b\sigma_v)(c\sigma_v)$. Then $a, b \in \hat{R}$ and $(ac)\sigma_v(bc)$. Thus $ac = u(bc) = (ub)c$, some $u \in V$. Condition (R3) in R then implies $ax = (ub)x$ for any $x \in R$, so $ax = u(bx)$. Thus $ax\sigma_v bx$, or $(a\sigma_v)(x\sigma_v) = (b\sigma_v)(x\sigma_v)$ as desired.

Conversely, let σ be a congruence on R such that $\sigma \subseteq \mathcal{L}$ and (R3) holds in R/σ . Let $V = e\sigma$. If $a\sigma b$ then $a\mathcal{L}b$, whence $a = ub$ for some $u \in H_e$ by Proposition 1.2 of [2]. Then

$$(3.2) \quad (e\sigma)(b\sigma) = (eb)\sigma = b\sigma = a\sigma = (ub)\sigma = (u\sigma)(b\sigma).$$

Since (R3) holds for R/σ , $e\sigma = (e\sigma)(e\sigma) = (u\sigma)(e\sigma) = (ue)\sigma = u\sigma$ so $u \in V$. Thus $a\sigma_v b$ and $\sigma \subseteq \sigma_v$. On the other hand, if $a\sigma_v b$, say $a = ub$ with $u \in V = e\sigma$, then $a\sigma = (ub)\sigma = (u\sigma)(b\sigma) = (e\sigma)(b\sigma) = b\sigma$. Thus $\sigma = \sigma_v$.

Now $\sigma \subseteq \mathcal{L}$, so $x \in V$ implies $x = ue = u$ for some $u \in H_e$. Also, conditions (CR1)–(CR3) may be applied to show that V is a subgroup of H_e . If $a \in \hat{R}e$ and $v \in V$, then $(av)\sigma = (a\sigma)(v\sigma) = (a\sigma)(e\sigma) = (ae)\sigma = a\sigma$ so that $av\sigma_v a$. Thus $av = ua$ for some $u \in V$, and V is a left normal divisor of $\hat{R}e$. The proof is complete.

Now let $T(\circ)$ be a bisimple orthodox semigroup constructed from a banded box-frame $(L, e'; R, e; K, (\circ))$ as in Theorem A. Let $\rho = \rho_{(V, V')}$ be an idempotent-separating congruence on $T(\circ)$, where (V, V') is a linked pair of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$. Define $\sigma = \sigma_v$ as in Lemma 3.1 and $\sigma' = \sigma_{v'}$ dually; i.e.,

$$\sigma' = \sigma_{V'} = \{(a', b') \in L \times L : a' = b'u', \text{ some } u' \in V'\}.$$

Since (V, V') is a linked pair, we may write

$$\sigma' = \sigma_{V'} = \{(a', b') \in L \times L : a' = b'u', \text{ some } u \in V\}$$

where, as usual, u' is the unique element of H_e such that $(u', u) \in K$. By Lemma 3.1 $R_1 = R/\sigma$ is a right Reilly groupoid with left identity $e\sigma$; $\hat{R}_1 = \hat{R}/\sigma$, and $\hat{R}_1(e\sigma) = (\hat{R}e)/\sigma$ since $\hat{R}e$, like \hat{R} , is a union of σ -classes. Moreover, the group of units $H_{e\sigma}$ of $\hat{R}_1(e\sigma)$ is precisely H_e/σ . To verify this last assertion, assume $a\sigma \in H_{e\sigma}$ say $(b\sigma)(a\sigma) = e\sigma$ for some $b\sigma$ in $\hat{R}_1(e\sigma) = (\hat{R}e)/\sigma$. Since $a\sigma, b\sigma$ are in $(\hat{R}e)/\sigma$ we have $a, b \in \hat{R}e$. From $(ba)\sigma e$ there exists $u \in V$ such that $e = u(ba) = (ub)a$. Now $u \in V \subseteq \hat{R}e \subseteq \hat{R}$ and $b \in \hat{R}e$ imply $ub \in \hat{R}\hat{R}e \subseteq \hat{R}e$. Then $e = (ub)a$ with ub, a in $\hat{R}e$ implies, by Proposition 1.3 of [2] that $a \in H_e$, whence $a\sigma \in H_e/\sigma$. The converse is immediate.

Dually, $L_1 = L/\sigma'$ is a left Reilly groupoid with right identity $e'\sigma'$ and $\hat{L}_1 = \hat{L}/\sigma'$, $(e'\sigma')\hat{L}_1 = (e'\hat{L})/\sigma'$, and the group of units $H_{e'\sigma'}$ of $(e'\sigma')\hat{L}_1$ is just $H_{e'}/\sigma'$.

We now define an anti-correlation K_1 between L_1 and R_1 as follows:

$$(3.3) \quad K_1 = \{(A', B) \in L_1 \times R_1 : \text{there exists } a' \in A', b \in B \text{ with } (a', b) \in K\}.$$

Instead of checking directly to see that K_1 is an anti-correlation by showing (AC1)–(AC4), we shall deduce the result from the converse half of Theorem A. This approach will likewise be taken throughout the remainder of this section.

Let $\epsilon = (e', e)_\tau$ in T . From the definition of $\rho = \rho_{(V, V')}$ we have

$$(e', b)_\tau, \rho(e', d)_\tau \Leftrightarrow b = vd \text{ with } v \in V \Leftrightarrow b\sigma d,$$

$$(a', e)_\tau, \rho(c', e)_\tau \Leftrightarrow a' = c'u' \text{ with } u \in V \Leftrightarrow a'\sigma'c'.$$

Thus it follows that the map $b\sigma \rightarrow (e', b)_\tau, \rho$ is a (partial) isomorphism of $R_1 = R/\sigma$ onto R_ϵ/ρ , and $a'\sigma' \rightarrow (a', e)_\tau, \rho$ is an isomorphism of $L_1 = L/\sigma'$ onto L_ϵ/ρ . Moreover, since ρ is idempotent separating, $R_\epsilon/\rho = R_{\epsilon\rho}$ and $L_\epsilon/\rho = L_{\epsilon\rho}$ in $T(\circ)/\rho$.

By the converse part of Theorem A, there is an anti-correlation, say K_1 , between $L_{\epsilon\rho}$ and $R_{\epsilon\rho}$ in $T(\circ)/\rho$, namely

$$K_1 = \{(\alpha', \alpha) \in L_{\epsilon\rho} \times R_{\epsilon\rho} : \alpha' \text{ and } \alpha \text{ are mutually inverse}\}.$$

Since $\alpha \in R_{\epsilon\rho}$ implies $\alpha = (e', b)_\tau, \rho$ for some $(e', b)_\tau$ in R_ϵ , and $\alpha' \in L_{\epsilon\rho}$ implies $\alpha' = (a', e)_\tau, \rho$ for some $(a', e)_\tau$ in L_ϵ , K_1 consists of all mutually

inverse pairs $((a', e)_{\tau\rho}, (e', b)_{\tau\rho})$ in $T(\circ)/\rho$. So identifying $a'\sigma'$ with $(a', e)_{\tau\rho}$, and $b\sigma$ with $(e', b)_{\tau\rho}$ by the above isomorphisms, we may regard K_1 as consisting of all mutually inverse pairs $(a'\sigma', b\sigma)$ in $L_1 \times R_1$. Therefore,

$$\begin{aligned} (a'\sigma', b\sigma) \in K_1 &\Leftrightarrow (a', e)_{\tau\rho} \text{ and } (e', b)_{\tau\rho} \text{ are inverse pairs in } T(\circ)/\rho \\ &\Leftrightarrow (e', b)_{\tau\rho} \cdot (a', e)_{\tau\rho} = (e', e)_{\tau\rho} \text{ by Lemma 2.12 of [3]} \\ &\Leftrightarrow (b \star a')_{\rho}(e', e)_{\tau}. \end{aligned}$$

Let $(b \star a') = (x', y)_{\tau}$. Now $(x', y)_{\tau\rho}(e', e)_{\tau}$ if and only if $x, y \in H_e$ and $x^{-1}y \in V$. Since $(x, y)_{\tau} = (e', x^{-1}y)_{\tau}$ it follows that

$$(a'\sigma', b\sigma) \in K_1 \Leftrightarrow b \star a' = (e', v)_{\tau}, \text{ some } v \in V.$$

Using Proposition 3.6 of [2] we have

$$(b \star a'v')(e', v) = b \star a' = (e', v)_{\tau} = (e', e)_{\tau}(e', v)$$

so Proposition 3.4 of [2] gives $b \star a'v' = (e', e)_{\tau}$. Conversely, $b \star a'v' = (e', e)_{\tau}$ implies $b \star a' = (e', v)_{\tau}$. By Proposition 4.5 of [2], $b \star a'v' = (e', e)_{\tau}$ is equivalent to $(a'v', b) \in K$. Since $(a'v')\sigma'a'$, we can choose a' (namely, replace it by $a'v'$) so that $(a', b) \in K$. Thus K_1 is the same as defined by (3.3), and is an anti-correlation between L_1 and R_1 . Moreover, under the isomorphisms we have $e'\sigma' \rightarrow (e', e)_{\tau\rho} = \epsilon\rho$ and $e\sigma \rightarrow (e', e)_{\tau\rho} = \epsilon\rho$, so Theorem A assures us that $(L_1, e'\sigma'; R_1, e\sigma; K_1)$ is a box-frame.

Now $H_e = R_e \cap L_e = \{(e', u)_{\tau} : u \in H_e\}$. Therefore the isomorphism $b\sigma \rightarrow (e', b)_{\tau\rho}$ of R/σ onto R_e/ρ , when restricted to H_e/σ , is an isomorphism of H_e/σ onto H_e/ρ . Also, since ρ is idempotent-separating, we have $(L_e \cap R_e)/\rho = L_e/\rho \cap R_e/\rho$, whence

$$H_e/\rho = (L_e \cap R_e)/\rho = L_e/\rho \cap R_e/\rho = L_{\epsilon\rho} \cap R_{\epsilon\rho} = H_{\epsilon\rho}.$$

Similarly, the map $u'\sigma' \rightarrow (u', e)_{\tau\rho}$ is an isomorphism of H_e/σ' onto $H_e/\rho = H_{\epsilon\rho}$.

By the converse part of Theorem A, consider the relation τ_1 on $L_{\epsilon\rho} \times R_{\epsilon\rho}$ defined by

$$\begin{aligned} ((a', e)_{\tau\rho}, (e', b)_{\tau\rho})\tau_1((c', e)_{\tau\rho}, (e', d)_{\tau\rho}) &\Leftrightarrow (c', e)_{\tau\rho} \\ &= (a', e)_{\tau\rho} \cdot [(e', u)_{\tau\rho}]^{-1} \end{aligned}$$

and

$$(e', d)_{\tau\rho} = (e', u)_{\tau\rho} \cdot (e', b)_{\tau\rho}$$

for some $(e', u)_{\tau\rho}$ in $H_{e\rho}$. Note that $[(e', u)_{\tau\rho}]^{-1} = (e', u^{-1})_{\tau\rho} = (u', e)_{\tau\rho}$ in $H_{e\rho}$. Identifying under our isomorphisms, we may regard τ_1 as being defined on $L_1 \times R_1$ by

$$(3.4) \quad (a'\sigma', b\sigma)\tau_1(c'\sigma', d\sigma) \Leftrightarrow c'\sigma' = (a'\sigma')(u'\sigma') \text{ and } d\sigma = (u\sigma)(b\sigma)$$

for some $u\sigma \in H_{e\sigma} = H_e/\sigma$.

Let $T_1 = (L_1 \times R_1)/\tau_1$ and $K_{1\tau_1} = K_1/\tau_1$. By Theorem A, the mapping $\theta: T_1 \rightarrow T(\circ)/\rho$ is a bijection, where θ is defined by $((a', e)_{\tau\rho}, (e', b)_{\tau\rho})_{\tau_1}\theta = (a', e)_{\tau\rho} \cdot (e', b)_{\tau\rho}$, and maps $K_{1\tau_1}$ upon the band $E_{T/\rho}$ of idempotents of T/ρ . Thus θ is essentially given by

$$(3.5) \quad (a'\sigma', b\sigma)_{\tau_1}\theta = (a', b)_{\tau\rho}.$$

Again, Theorem A shows that $(L_1, e'\sigma'; R_1, e\sigma; K_{1\tau_1}(\circ_1))$ becomes a banded box-frame, where we use θ to transfer the operation on $E_{T/\rho}$ to an operation (\circ_1) on $K_{1\tau_1}$ in the obvious way. Under our identifications the definition of (\circ_1) reads as follows:

$$\begin{aligned} (a'\sigma', a\sigma)_{\tau_1}\circ_1(b'\sigma', b\sigma)_{\tau_1} &= (c'\sigma', c\sigma)_{\tau_1} \\ &\Leftrightarrow (a', a)_{\tau\rho} \cdot (b', b)_{\tau\rho} = (c', c)_{\tau\rho} \text{ in } T(\circ)/\rho, \text{ where } (a', a), \\ &\quad (b', b), \text{ and } (c', c) \text{ are in } K. \end{aligned}$$

Since $K_{\tau}(\circ)$ is the band of idempotents of $T(\circ)$, and ρ is idempotent-separating on $T(\circ)$, we have

$$(3.6) \quad \begin{aligned} (a'\sigma', a\sigma)_{\tau_1}\circ_1(b'\sigma', b\sigma)_{\tau_1} &= (c'\sigma', c\sigma)_{\tau_1} \\ &\Leftrightarrow (a', a)_{\tau} \circ (b', b)_{\tau} = (c', c)_{\tau} \text{ in } K_{\tau}(\circ). \end{aligned}$$

Because $(L_1, e'\sigma'; R_1, e\sigma; K_{1\tau_1}(\circ_1))$ is a banded box-frame, Proposition 3.5 of [2] holds. That is, for each element $(c'\sigma', b\sigma)$ of $L_1 \times R_1$ there exists a unique element $(X', Y)_{\tau_1}$ of $T_1^0 = ((e'\sigma')\hat{L}_1 \times \hat{R}_1(e\sigma))/\tau_1$ such that $(b'\sigma', b\sigma)_{\tau_1}\sigma_1(c'\sigma', c\sigma)_{\tau_1} = ((b'\sigma')X', Y(c\sigma))_{\tau_1}$ for any $b'\sigma'$ in L_1 and $c\sigma \in R_1$ such that $(b'\sigma', b\sigma)$ and $(c'\sigma', c\sigma)$ are in K_1 . If we denote $(X', Y)_{\tau_1}$ by $(b\sigma) \star_1(c'\sigma')$, and then extend the band operation (\circ_1) on $K_{1\tau_1}$ to an operation (\circ_1) on T_1 by

$$(3.7) \quad (a'\sigma', b\sigma)_{\tau_1}\circ_1(c'\sigma', d\sigma)_{\tau_1} = ((b\sigma) \star_1(c'\sigma'))(a'\sigma', d\sigma)$$

then θ becomes an isomorphism of $T_1(\circ_1)$ onto $T(\circ)/\rho$.

The following theorem summarizes all of the results in this section.

THEOREM 3.2. *Let $T(\circ)$ be a bisimple orthodox semigroup constructed from a banded box-frame $(L, e'; R, e; K_{\tau}(\circ))$ as in Theorem*

A. Let $\rho = \rho_{(V, V')}$ be an idempotent-separating congruence on $T(\circ)$ where (V, V') is a linked pair of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$. Define $\sigma = \sigma_V$ on R by (3.1) and $\sigma' = \sigma_{V'}$ on L dually. Then $\sigma[\sigma']$ is a congruence on $R[L]$ and $R_1 = R/\sigma[L_1 \times L/\sigma]$ is a right [left] Reilly groupoid with $e\sigma[e'\sigma']$ as left [right] identity. Define K_1 by (3.3); then $(L_1, e'\sigma'; R_1, e\sigma; K_1)$ is a box-frame. Define the relation τ_1 on $L_1 \times R_1$ by (3.4) and define (\circ_1) on $K_{1\tau_1} = K_1/\tau_1$ by (3.6). Then $(L_1, e'\sigma'; R_1, e\sigma; K_{1\tau_1}(\circ_1))$ is a banded box-frame. On $T_1 = (L_1 \times R_1)/\tau_1$ define (\circ_1) by (3.7). Then $T_1(\circ_1)$ is a bisimple orthodox semigroup having $K_{1\tau_1}(\circ_1)$ as its band of idempotents, and the mapping $\theta: T_1(\circ_1) \rightarrow T(\circ)/\rho$ defined by (3.5) is an isomorphism.

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