SET APPROXIMATION BY LEMNISCATES AND THE SPECTRUM OF AN OPERATOR ON AN INTERPOLATION SPACE

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Let B_0, B_1 be an interpolation pair of Banach spaces and T_s a bounded linear operator on the corresponding interpolation space $[B_0, B_1]_s$, $0 \leq s \leq 1$, such that the operators T_s all agree on $B_0 \cap B_1$. In this paper we extend our previous work by giving a general upper bound for the spectrum of T_s constructed from the spectra of T_0 and T_1 using a set interpolation formula which we introduce in §1 for compact sets in the plane. In §3 we show that this upper bound is essentially best possible. This requires a theorem about approximating sets with lemniscates, which we prove in §2. Finally, we show in §4 that under certain conditions the operators T_s, $0 \leq s \leq 1$, all have the same spectrum.

1. Upper bound for spectrum. In this section we introduce the notion of interpolating between two sets in the complex plane (see 1.3) and then we obtain an upper bound for the spectrum of an operator on an interpolation space by "interpolating its extreme spectra (see 1.9)."

Basic situation 1.1. B_0, B_1 are two Banach spaces continuously embedded in a topological vector space V so that $B_0 \cap B_1$ is dense in both B_0 and B_1. The corresponding spaces $[B_0, B_1]_s$, $0 \leq s \leq 1$, are the spaces defined in [2, p. 114]. By [2, §9.3], $B_j = [B_0, B_1]_j$, $j = 0, 1$. T_s, $0 \leq s \leq 1$, is a bounded linear operator on $[B_0, B_1]_s$ and all the operators T_s agree on $B_0 \cap B_1$.

General notation. For a normed linear space $B, \| \ \|_B$ will denote the norm of B. We will sometimes use $\| \ \|$ to denote $\| \ \|_B$ if no confusion results. The space of bounded linear operators on a normed linear space B with the operator norm will be denoted $O(B)$. We will call bounded linear operators, operators. If T is an operator, $\mathrm{sp}(T)$ will denote the spectrum of T and $r(T)$ the spectral radius of T, the smallest number r such that $\mathrm{sp}(T) \subset \{z : |z| \leq r\}$. If F is analytic on $\mathrm{sp}(T)$, then $F(T)$ will denote the operator $(2\pi i)^{-1} \int_\gamma (\lambda I - T)^{-1} d\lambda$ where γ is an envelope of $\mathrm{sp}(T)$ in the domain of F. If E is a subset of the complex plane, $R(E)$ will denote the rational functions with poles in the complement of E. The following is proved in [2].

LEMMA 1.2. *If T_s, $0 \leqq s \leqq 1$, is as in 1.1 then*

$$\| T_s \| \leqq \| T_0 \|^{1-s} \| T_1 \|^s.$$

If B_0 and B_1 are as in 1.1 and T_j is an operator on B_j, $j = 0, 1$, such that T_0 and T_1 agree on $B_0 \cap B_1$, then there is an operator T_s on $[B_0, B_1]_s$, $0 \leqq s \leqq 1$, such that all the operators T_s agree on $B_0 \cap B_1$.

We next introduce the notion of interpolation sets $I_s(E_0, E_1)$ corresponding to two given compact sets E_0, E_1 in the plane. These interpolation sets will be used to provide an upper bound for the spectrum of operators T_s.

If F is a complex valued function and t is a positive real number, let $L(t, F)$ denote the set of elements x in the domain of F such that $|F(x)| \leqq t$.

DEFINITION 1.3. Let E_0, E_1 be compact nonempty sets in the plane such that $E_0 \subset E_1$. For $0 \leqq s \leqq 1$ we define the set $I_s(E_0, E_1)$ to be $\cap L(r^s, F)$ where the intersection is taken over all pairs (r, F) such that: (a) r is a real number > 1, (b) F is an analytic function on an open set in the plane, (c) $L(r, F)$ is a compact subset of the domain of F and (d) E_j is a subset of the interior of $L(r^j, F)$, $j = 0, 1$.

The following lemma is an immediate consequence of the preceding definition.

LEMMA 1.4. *Suppose that E_0, E_1, F_0 and F_1 are compact subsets of the plane and that $E_0 \subset E_1$, $F_0 \subset F_1$, $E_0 \subset F_0$ and $E_1 \subset F_1$. Then,*

$$I_s(E_0, E_1) \subset I_s(F_0, F_1) \qquad 0 \leqq s \leqq 1.$$

For a complex number z and a positive real number t let $D(z, t)$ denote the set of complex numbers w such that $|z - w| < t$. Let $D(z, t]$ denote the closure of $D(z, t)$. For a set A in a topological space int A will denote the interior of A.

LEMMA 1.5. *Let K be a compact subset of $D(0, 1)$. To each $\epsilon > 0$ and $s, 0 < s < 1$, there corresponds a $\delta > 0$ such that for each ζ in K, $D(\zeta, \delta] \subset D(0, 1)$ and*

$$I_s(D(\zeta, \delta], \ D(0, 1]) \subset K + D(0, \epsilon).$$

Proof. For ζ in K, let λ_ζ denote the analytic function defined by $\lambda_\zeta(z) = (z - \zeta)/(1 - z\bar{\zeta})$ on the complement of $\{\bar{\zeta}^{-1}\}$. Choose u such that

$1 < u < |1/\zeta|$ for each ζ in K.   Since K is compact in $D(0, 1)$ it is clear that

(1)   corresponding to K, ϵ, s there is an $r > 1$ such that if
         $\zeta \in K$ and $|(r/u)\lambda_\zeta(z)| \leq r^s$, then $|z - \zeta| < \epsilon$.

It is clear that we can choose δ so that $D(\zeta, \delta] \subset \mathrm{int}\, L(1, (r/u)\lambda_\zeta)$ and $D(\zeta, \delta] \subset D(0, 1)$ for each ζ in K.   Let ζ be in K.   Let $F(z) = (r/u)\lambda_\zeta(z)$.   From what we have observed and Definition 1.3 it follows that

(2) $I_s(D(\zeta, \delta], \, D(0, 1] \subset L(r^s, F)$.

It follows from (1) that $L(r^s, F) \subset K + D(0, \epsilon)$; and, this together with (2) proves the lemma.

LEMMA 1.6.   *Let B_0, B_1 and T_s, $0 \leq s \leq 1$, be as in 1.1.   Suppose that E is a compact set in the plane such that $\mathrm{sp}(T_0)$, $\mathrm{sp}(T_1) \subset E$ and for each $\lambda \notin E$, $(\lambda - T_0)^{-1}$ and $(\lambda - T_1)^{-1}$ agree on $B_0 \cap B_1$.   Then $\mathrm{sp}(T_s) \subset E$, $0 \leq s \leq 1$; and, if F is analytic on E, then the operators $F(T_s)$ all agree on $B_0 \cap B_1$.*

Proof.   Fix an s in the interval $[0, 1]$.   We will first show that

(3) $\mathrm{sp}(T_s) \subset E$.

If $\lambda \notin E$, then by the restrictions on E the operators $(\lambda - T_0)^{-1}$ and $(\lambda - T_1)^{-1}$ agree on the space $B_0 \cap B_1$.   From 1.2 it follows that there is an operator Q on B_s such that Q and $(\lambda - T_0)^{-1}$ agree on $B_0 \cap B_1$; in particular, $Q(\lambda - T_s)$ and $(\lambda - T_s)Q$ are equal to the identity on $B_0 \cap B_1$.   Since both Q and T_s are bounded on B_s, we see that Q is the inverse of $\lambda - T_s$; hence, $\lambda \notin \mathrm{sp}(T_s)$.   Our argument establishes (3).

We next observe that if $\lambda \notin E$, then the operators $(\lambda - T_s)^{-1}$, $0 \leq s \leq 1$, and agree on $B_0 \cap B_1$.   To see this we first note that since $(\lambda - T_0)^{-1}$ and $(\lambda - T_1)^{-1}$ agree on $B_0 \cap B_1$, these two operators leave $B_0 \cap B_1$ invariant.   If $y \in B_0 \cap B_1$, then $y = (\lambda - T_0)x$ where $x = (\lambda - T_0)^{-1}y$, which is an element in $B_0 \cap B_1$.   Thus,

$$(\lambda - T_s)^{-1}y = (\lambda - T_s)^{-1}(\lambda - T_0)x = (\lambda - T_s)^{-1}(\lambda - T_s)x = x.$$

Since y was chosen arbitrarily in $B_0 \cap B_1$ and x does not depend on s, we have shown that the operators $(\lambda - T_s)$, $0 \leq s \leq 1$, all agree on $B_0 \cap B_1$.   Consequently, the operators $G(T_s)$, $0 \leq s \leq 1$, all agree on $B_0 \cap B_1$ if $G \in R(E)$.   Let G_n be a sequence of elements in $R(E)$ such

that G_n converges uniformly to F on some neighborhood of E. Then $G_n(T_0)$ converges to $F(T_0)$ in $O(B_0)$ and $G_n(T_1)$ converges to $F(T_1)$ in $O(B_1)$. If $x \in B_0 \cap B_1$, then $G_n(T_0)x = G_n(T_1)x$ for $n = 1, 2, \cdots$. Since $G_n(T_0)x$ converges to $F(T_0)x$ in B_0 and $F(T_1)x$ in B_1 and since B_0 and B_1 are continuously embedded in V, it follows that $F(T_0)x = F(T_1)x$. This completes the proof.

Comment. In view of Lemma 1.6 it is natural to ask whether the set E in 1.6 can always be taken to be $\text{sp}(T_0) \cup \text{sp}(T_1)$. We have not been able to answer this question. The best that we have done is the content of the next lemma.

LEMMA 1.7. *Let B_0, B_1, T_0 and T_1 be as in 1.1. Then the set $H = \{\lambda : \lambda \notin \text{sp}(T_0) \cup \text{sp}(T_1)$ and $(\lambda - T_0)^{-1}, (\lambda - T_1)^{-1}$ agree on $B_0 \cap B_1\}$ contains the unbounded component of the complement of $\text{sp}(T_0) \cup \text{sp}(T_1)$; and, the set H is a union of components of the complement of $\text{sp}(T_0) \cup \text{sp}(T_1)$.*

Proof. Suppose λ is in the unbounded component of the complement of $E = \text{sp}(T_0) \cup \text{sp}(T_1)$. There is a sequence p_n of polynomials which converges uniformly to $(\lambda - z)^{-1}$ for z in a neighborhood of E. It follows that $p_n(T_j)$ converges to $(\lambda - T_j)^{-1}$ in the operator norm of B_j, $j = 0, 1$. If $x \in B_0 \cap B_1$, then $p_n(T_j)x$ converges to $(\lambda - T_j)^{-1}x$ in B_j. Therefore, $p_n(T_j)x$ converges to $(\lambda - T_j)^{-1}x$ in the topology of V, $j = 0, 1$. Since T_0 and T_1 agree on $B_0 \cap B_1$, $p_n(T_0)x = p_n(T_1)x$ for each n. Thus, $(\lambda - T_0)^{-1}x = (\lambda - T_1)^{-1}x$. If λ is a point in some bounded component V of the complement of E such that $(\lambda - T_0)^{-1}$ and $(\lambda - T_1)^{-1}$ agree on $B_0 \cap B_1$ and if λ_1 is also in V, then there is a sequence of polynomials $p_n (n = 1, 2, \cdots)$ such that $p_n(1/(\lambda - z))$ converges uniformly to $1/(\lambda_1 - z)$ for z in some neighborhood of E. The same reasoning as above shows that $(\lambda_1 - T_0)^{-1}$ and $(\lambda_1 - T_1)^{-1}$ agree on $B_0 \cap B_1$. This completes the proof.

LEMMA 1.8. *Let B_0, B_1 and T_s, $0 \leq s \leq 1$, be as in 1.1. Then*

$$r(T_s) \leq r(T_0)^{1-s} r(T_1)^s, \qquad 0 \leq s \leq 1.$$

Proof. If $\epsilon > 0$, then there corresponds on n_0 such that

$$\|(T_j)^n\| \leq (r(T_j) + \epsilon)^n \qquad j = 0, 1; n \geq n_0.$$

Since the operators $(T_s)^n$, $0 \leq s \leq 1$, all agree on $B_0 \cap B_1$, it follows from 1.2 that

$$\|(T_s)^n\| \le \|(T_1)^n\|^s, \qquad 0 \le s \le 1; \ n = 1, 2, \cdots.$$

Since ϵ was chosen arbitrarily, the conclusion of the lemma follows from our observations and the fact that $r(T_s) = \lim \|(T_s)^n\|^{1/n}$.

THEOREM 1.9. *Suppose that B_0, B_1 and T_s, $0 \le s \le 1$, are as in 1.1 and that* $\mathrm{sp}(T_0) \subset \mathrm{sp}(T_1)$. *Let E denote the complement of $\{\lambda: \lambda \notin \mathrm{sp}(T_1)$ and $(\lambda - T_0)^{-1}, (\lambda - T_1)^{-1}$ agree on $B_0 \cap B_1\}$. Then*

$$\mathrm{sp}(T_s) \subset I_s(\mathrm{sp}(T_0), E).$$

Proof. Suppose that the theorem is false; that is, suppose that there is a λ in $\mathrm{sp}(T_s)$ which is not in $I_s(\mathrm{sp}(T_0), E)$. In particular, there is a pair (r, F) corresponding to the sets $\mathrm{sp}(T_0)$ and E as in 1.3 such that $\lambda \notin L(r^s, F)$. However, since $E \subset$ domain of F and 1.6 implies that $\lambda \in E$, we conclude that

(4) $$|F(\lambda)| > r^s.$$

Define the operators $U_t = F(T_t)$, $0 \le t \le 1$, in the usual manner of spectral theory. Since $\lambda \in \mathrm{sp}(T_s)$ it follows from the spectral mapping theorem that

(5) $$F(\lambda) \in \mathrm{sp}(U_s).$$

From 1.6 it follows that the operators U_t, $0 \le t \le 1$, all agree on $B_0 \cap B_1$; consequently, we conclude from 1.8 that

(6) $$r(U_s) \le r(U_0)^{1-s} r(U_1)^s.$$

By the choice of F, $\mathrm{sp}(T_j) \subset \{z: |F(z)| \le r^j\}$, $j = 0, 1$. From this and the spectral mapping theorem we obtain

(7) $$\mathrm{sp}(U_j) \subset \{z: |z| \le r^j\}, \qquad j = 0, 1.$$

If we combine (5) thru (7) we get $|F(\lambda)| \le r^s$, which contradicts (4). This completes the proof.

2. Set approximation by lemniscates. The main purpose of this section is the proof of the approximation theorem, Theorem 2.6.

By a simple set we mean a compact set with dense interior and with a boundary consisting of a finite number of disjoint simple closed curves which are also regular analytic curves (see [1, p. 226]). For notation we

use ∂E and int E to denote the boundary and the interior, respectively, of a set E in the plane.

REMARK 2.1.   By a harmonic triple in this paper we mean what is defined in [**4**, 5.1] except that we do not require that the interior of E_1 be connected.   The main result about harmonic triples that we will need is [**4**, 5.3]; and this result is still valid because it can be applied to the closure of each component of the interior of E_1.

LEMMA 2.2.   *Let E_0 and E_1 be compact subsets of the plane such that $E_0 \subset E_1$ and each component of the complement of E_0 intersects the complement of E_1.   To each $\epsilon > 0$ there corresponds a harmonic triple (D_0, D_1, ω) such that: (a) D_0 and D_1 are simple sets and (b) $E_j \subset \text{int } D_j \subset E_j + D(0, \epsilon)$, $j = 0, 1$.*

In order to prove this lemma we will use the following topological lemma.

LEMMA 2.3.   *Let E_0 and E_1 be sets which satisfy the conditions of Lemma 2.2.   To each $\epsilon > 0$ there corresponds a pair of sets D_0, D_1 which satisfy* (a) *and* (b) *of Lemma 2.2, $D_0 \subset \text{int } D_1$ and each component of the complement of D_0 intersects the complement of D_1.*

Given sets E_0, E_1 as in Lemma 2.1 let D_0, D_1 be the sets corresponding to ϵ in Lemma 2.3.   To complete the proof of Lemma 2.2 we need only show that there is an ω such that (D_0, D_1, ω) is a harmonic triple.   Let U be a component of the set int $D_1 \backslash D_0$.   The set U is a subset of a component V of the complement of D_0.   If $\partial U \subset D_0$, then $\partial U \subset \partial V$; however, this would imply that $U = V$, since V is connected.   But, $V \neq U$ because V contains a point in the complement of D_1.   Let x be a point in ∂U that is not in D_0.   In particular, $x \in V$.   Since U is a component of int $D_1 \backslash D_0$, it follows that x is not in int D_1.   Thus $x \in \partial D_1$.   The boundary of U is the union of two disjoint subsets A and B where $A \subset \partial D_0$ and $B \subset \partial D_1$.   We have argued above that B is nonempty; however, it is possible that $A = \varnothing$.   If y is in A or B, then y must be contained in an arc that lies in A or B since U is a component of int $D_1 \backslash D_0$.   It follows from the general theory on the Dirichlet problem that there is a continuous function ω_0 on the closure of U which is identically 1 on B, identically 0 on A and harmonic on U.   We will now define ω on D_1: Let ω be defined on the closure of each component of int $D_1 \backslash D_0$ in the same way that ω_0 was defined on the closure of U and define ω on D_0 to be identically 0.   We claim that (D_0, D_1, ω) is harmonic triple (see 2.1); in fact, this is clear except, possibly for (iii) of [**4**, 5.1].   Suppose $x \in D_1 \backslash D_0$.   If $x \in \partial D_1$, then

$\omega(x) = 1$.   Since each component of int $D_1 \backslash D_0$ has a boundary point in ∂D_1 (We showed this above.) It follows from the minimum principle that ω does not assume the value 0 on int $D_1 \backslash D_0$. These observations verify that condition (iii) of [4, 5.1] is met.

LEMMA 2.4.   *Suppose that* (D_0, D_1, ω) *is a harmonic triple and that* D_0, D_1 *are simple sets. Then to each* $\epsilon, \eta > 0$, *there corresponds an f in* $R(D_1)$ *and a positive real number b such that*

$$|\omega(z) - b \log|f(z)\|| < \epsilon, \quad \text{if} \quad \omega(z) \geqq \eta.$$

Proof. Because of Runge's theorem it clearly suffices to prove Lemma 2.4 for the case where D_1 is a connected simple set, so we will assume that D_1 is a connected simple set.   Let $\omega_n, n = 1, 2, \cdots$, be the function defined on the domain of ω as follows: $\omega_n(z) = 1/n$ if $\omega(z) \leqq 1/n$ and $\omega_n(z) = \omega(z)$ if $\omega(z) > 1/n$.   Locally, each ω_n is either a harmonic function or the maximum of two harmonic functions.   Thus, each ω_n is subharmonic; and, since the sequence ω_n decreases pointwise to ω, we see that ω is also subharmonic.   Let U denote the interior of E_1.   Since ω is subharmonic on the domain U, the Riesz theorem [5, p. 48] implies that

$$(1) \qquad \omega(z) = u(z) + \int \log|z - a| \, d\mu(a), \qquad z \in U$$

where u is harmonic on U and μ is a positive Borel measure, supported on D_0.   One can easily show that there is a positive integer n and a finite number of points a_j in D_0 such that

$$(2) \quad \left| \int_{D_0} \log|z - a| \, d\mu(a) - n^{-1} \sum_j \log|z - a_j| \right| < \epsilon, \quad \text{for} \quad \omega(z) \geqq \eta.$$

We will now establish the following.

2.5.   There is a continuous function u_1 on D_1 such that: u_1 is harmonic on U, $|u(z) - u_1(z)| < \epsilon$ for z in D_1 and all the periods of the harmonic conjugate of u_1 are rational multiples of 2π.

We may assume that the boundary of D_1 has at least two components.   Let γ_{n+1} denote the boundary component of D_1 which also forms the boundary of the unbounded component of the complement of D_1 and let $\gamma_1, \cdots, \gamma_n$ denote the other boundary components of D_1.   Let ω_j denote the harmonic measure of $\gamma_j, j = 1, \cdots, n$, with respect to the region U [1, p. 245].   Let $a_{k,j}$ denote the period of the harmonic conjugate of ω_j corresponding to the curve γ_k (see [1, p. 162]).   We define u_1 by:

$$u_1 = u + \Sigma \; \epsilon_j \omega_j \qquad\qquad (j = 1, 2, \cdots, n)$$

for an appropriate choice of the numbers ϵ_j. Since the matrix $(a_{k,j})$, $0 \leq k, j \leq n$, is a nonsingular matrix [1, p. 246] and since the period of the harmonic conjugate of u_1 corresponding to the curve γ_k is $p_k + \Sigma_j \; \epsilon_j a_{k,j}$, where p_k is the period of the harmonic conjugate of u corresponding to γ_k, it follows that we can choose the numbers ϵ_j in such a way that $- \epsilon < \epsilon_j < \epsilon$ for each j and the numbers $p_k + \Sigma_j \; \epsilon_j a_{k,j}$, $k = 1, \cdots, n$, are all rational multiples of 2_π. The conclusion of (2.5) follows from the above remarks and the maximum principle.

We now return to the proof of Lemma 2.4. Since the boundary curves of D_1 are analytic curves and ω is constant on each component of the boundary of D_1 it follows from the reflection principle that ω has an extension to a function which is harmonic on a neighborhood of D_1. Since the integral in (1) is clearly harmonic on a neighborhood of the boundary of D_1, it follows that u has an extension which is harmonic on a neighborhood of D_1. Consequently, u_1 also has an extension which is harmonic on a neighborhood of D_1. By 2.5 we can choose an integer m such that the periods of the harmonic conjugate h of mu_1 are all multiples of 2_π. Let $g = e^{mu_1 + ih}$. From what we have noted above we conclude that g has an extension which is analytic on a neighborhood of D_1. Since g has no zeros in D_1 we can use Runge's theorem to choose an element g_1 in $R(D_1)$ such that

$$(3) \qquad\qquad |m^{-1} \log |g_1(z)| - u_1(z)| < \epsilon, \; z \in D_1.$$

Let $p(z) = \Pi_j \; (z - a_j)$ where the a_j are the numbers that appear in (2). From (1), (2), 2.5, and (3) we obtain

$$|\omega(z) - (mn)^{-1} \log |p(z)^m g_1(z)^n|| < 3\epsilon, \; \omega(z) \geq \eta.$$

This completes the proof of Lemma 2.4.

THEOREM 2.6. *Suppose that E_0, E_1 are two compact sets in the plane, that $E_0 \subset E_1$ and that each component of the complement of E_0 intersects the complement of E_1. To each $\epsilon_1 > 0$ there corresponds an F in $R(E_1)$ and an $r > 1$ such that*

$$E_j \subset \text{int } L(r^j, F) \subset E_j + D(0, \epsilon_1), \qquad j = 0, 1.$$

Proof. First choose a harmonic triple (D_0, D_1, ω) corresponding to ϵ_1 as in Lemma 2.2. In particular,

$$(4) \qquad\qquad D_j \subset E_j + D(0, \epsilon_1), \qquad j = 0, 1.$$

Choose $\epsilon, \eta > 0$ so that

(5) $$\{z: \omega(z) \leqq 2\epsilon + \eta\} \subset E_0 + D(0, \epsilon_1),$$

(6) $$\epsilon + \eta < 1 - \epsilon \quad \text{and}$$

(7) $$E_1 \subset \text{int}\{z: \omega(z) \leqq 1 - 2\epsilon\}.$$

Corresponding to ϵ and η we choose b and f as in Lemma 2.4. For convenience, let $h = b \log|f|$. Let $H_0 = \{z: h(z) \leqq \epsilon + \eta\}$. Since h is subharmonic on $\{z: \omega(z) < \eta\}$ and $h \leqq \epsilon + \eta$ on the set $\{z: \omega(z) = \eta\}$, it follows that $h \leqq \epsilon + \eta$ on $\{z: \omega(z) \leqq \eta\}$. In particular, $D_0 \subset H_0$. Thus,

(8) $$E_0 \subset \text{int } H_0.$$

Since h is superharmonic on the complement of D_1 and $h \geqq 1 - \epsilon$ on $\{z: \omega(z) = 1\}$, we have

(9) $$h(z) \geqq 1 - \epsilon, \quad z \not\in D_1.$$

Suppose $z \in H_0$, so $h(z) \leqq \epsilon + \eta$. Since $\epsilon + \eta < 1$ least. Thus either $\omega(z) \leqq \eta$ or $\omega(z) \leqq 2\epsilon + \eta$. Consequently, from (5) we obtain

(10) $$H_0 \subset E_0 + D(0, \epsilon_1).$$

Let $H_1 = \{z: H(z) \leqq 1 - \epsilon\}$. We have already noted that if $\omega(z) \leqq \eta$, then $h(z) \leqq \epsilon + \eta$. From this and (6) it follows that $h(z) \leqq 1 - \epsilon$ if $\omega(z) \leqq 1 - 2\epsilon$. Thus

$$H_1 \supset \{z: \omega(z) \leqq 1 - 2\epsilon\}.$$

We conclude from (7) that

(11) $$E_1 \subset \text{int } H_1.$$

From (9) and the definition of H_1 we have $H_1 \subset D_1$; consequently,

(12) $$H_1 \subset E_1 + D(0, \epsilon_1).$$

We define F and r as follows: $F = e^{-(\epsilon+\eta)/b} f$ and $r = e^{[(1-\epsilon)-(\epsilon+\eta)]/b}$. Evidently, $r > 1$, $F \in R(E_1)$ and $H_j = L(r^j, F)$ for $j = 0, 1$. The conclusion of the theorem now follows from (8), (10), (11), and (12).

3. The upper bound is best possible. The purpose of this section is to establish Theorem 3.1. This theorem gives a class of operators for which the upper bound given in Theorem 1.9 is attained; therefore, it shows, in particular, that an upper bound for $sp(T_s)$ which depends only on $sp(T_0)$ and $sp(T_1)$ can be no better than the one given in Theorem 1.9 provided that $sp(T_0)$ and $sp(T_1)$ satisfy the one further condition imposed in Theorem 3.1.

If E is a compact set in the plane, then $C_R(E)$ will denote all functions which are the uniform limit of sequences in $R(E)$.

THEOREM 3.1. *Suppose that E_0, E_1 are compact subsets of the plane, $E_0 \subset E_1$, the complement of E_1 meets each component of the complement of E_0 and that the zero function is the only function in $C_R(E_1)$ which vanishes identically on E_0. Let $B_k = C_k(E_k)$, $k = 0, 1$, $V = B_0$ and identify B_0, B_1 with their natural embeddings in V. Let T_s be the bounded linear operators on $[B_0, B_1]_s$, $0 \leq s \leq 1$, such that $T_s f\ (z) = zf(z)$, $z \in E_0$, $f \in B_1$. Then*

$$sp(T_s) = I_s(E_0, E_1), \qquad 0 \leq s \leq 1.$$

Proof. It is clear in this case that the set E of Theorem 1.9 is E_1 and that $E_0 = sp(T_0)$. Thus, $sp(T_s) \subset I_s(E_0, E_1)$.

Now suppose that $z \in I_s(E_0, E_1)$. Since the conclusion of the theorem is obvious in case $s = 0$ or 1, we assume $0 < s < 1$. To show that $z \in sp(T_s)$ we first show that

(1) $|x(z)| \leq \|x\|_{B_s}, \qquad x \in R(E_1).$

Let $\epsilon > 0$. Since $B_0 \cap B_1 = B_1$, it follows from [4, 2.5] that there is an f in $\mathscr{F} = \mathscr{F}(B_0, B_1)$ (see [2] for definition of $\mathscr{F}$) such that $f(s) = x$,

(2) $\|x\|_{B_s} + \epsilon \geq \|f\|_{\mathscr{F}}$

and $f(\xi) = \Sigma\, c_j(\xi)x_j$ where the sum is finite, each $x_j \in B_1$ and each c_j is continuous on $0 \leq \operatorname{Re} \xi \leq 1$, vanishes at ∞ and is analytic on $0 < \operatorname{Re} \xi < 1$. It is clear from the definition of $C_R(E_1)$ that each x_j can be replaced by a y_j in $R(E_1)$ such that for the corresponding function $f_1(\xi) = \Sigma\, c_j(\xi)y_j$ we have

(3) $\|f(\xi) - f_1(\xi)\|_{B_1} < \epsilon, \qquad 0 \leq \operatorname{Re} \xi \leq 1.$

From (3) and the fact that the norm of B_1 dominates the norm of B_0 on B_1 we have

(4) $$\|f - f_1\|_{\mathscr{F}} < \epsilon.$$

Given a pair (r, F) which satisfies (a) through (c) of 1.3 let $\mathscr{F}_1$ denote the space

$$\mathscr{F}(C_R(L(1, F)), C_R(L(r, F))).$$

From Theorem 2.6 and the fact that each of the functions y_j (there are only a finite number of functions y_j) is in $R(E_1)$ it is clear that (r, F) can be chosen so that $f_1 \in \mathscr{F}_1$ and

(5) $$(1 + \epsilon)\|f_1\|_{\mathscr{F}} \geqq \|f_1\|_{\mathscr{F}_1}.$$

If we let E_k in [**4**, 5.3] (see 2.1 above) be $L(r^k, F)$, $k = 0, 1$ and define ω by: $\omega(z) = 0$, $z \in E_0$ and $\omega(z) = \log|F(z)|$ for $z \in E_1 \backslash E_0$; then we can conclude from [**4**, 5.3] that

(6) $$\|f_1\|_{\mathscr{F}_1} \geqq |f_1(z, s)|.$$

From Theorem 2.6 it is clear that $E_1 \supset I_s(E_0, E_1)$. In particular, $z \in E_1$. From this and (3) we obtain

(7) $$|x(z) - f_1(z, s)| \leqq \|x - f_1(s)\|_{B_1} < \epsilon.$$

Since ϵ is arbitrary, (1) now follows by combining (2), (4), (5), (6) and (7).

Now suppose that $x \in R(E_1)$ such that $x(z) \neq 0$. From (1) we obtain

$$|x(z)| = |x(z) - (z - T_s)y(z)| \leqq \|x - (z - T_s)y\|_{B_s}, \quad y \in R(E_1).$$

Since $R(E_1)$ is dense in B_s, this last inequality shows that the range of $(z - T_s)$ is $\neq B_s$. Thus, $x \in \mathrm{sp}(T_s)$, which completes the proof.

4. A criterion for constant spectra. This section is devoted to Theorem 4.1, which gives a criterion for $\mathrm{sp}(T_s) = \mathrm{sp}(T_0)$, $0 \leqq s < 1$.

THEOREM 4.1. *Suppose that T_j is a bounded linear operator on B^j, $j = 0, 1$, that T_0 and T_1 agree on $B_0 \cap B_1$ and that to each $\epsilon > 0$ there correspond subspaces $Q_{1,j}, \cdots, Q_{n,j}$ of B_j, $j = 0, 1$ with the following properties: (a) each space $Q_{k,j}$ is T_j invariant; (b) B_j is the direct sum of the spaces $Q_{k,j}$, $j = 0, 1$; (c) $Q_{k,0} \cap Q_{k,1}$ is dense in both $Q_{k,0}$ and $Q_{k,1}$ for all j; (d) $\mathrm{sp}(T_0|Q_{k,0}) \subset \mathrm{sp}(T_1|Q_{k,1})$ for each k; (e) $\mathrm{sp}(T_0|Q_{k,0})$ is contained in $D(z, \epsilon)$ for some z in $\mathrm{sp}(T_0)$ depending on k, $k = 1, 2, \cdots, n$. Then*

$$\text{sp}(T_s) = \text{sp}(T_0) \quad for \quad 0 \leq s < 1.$$

Proof. Since $\text{sp}(aT_s) = \text{sp}(T_s)$ for any complex number a, it clearly suffices to prove the theorem assuming $\|T_0\| \leq 1/2$ and $\|T_1\| \leq 1/2$. Fix s and let $\epsilon > 0$. Choose δ corresponding to ϵ, s and $K = \text{sp}(T_0)(\text{sp}(T_0) \subset D(0, 1/2])$ in Lemma 1.5. Let $Q_{k,j}$ be the spaces corresponding to δ as in the theorem. Let $U_{k,j}$ be the restriction of the operator T_j to the space $Q_{k,j}$. Let $Q_{k,s}$ denote the space $[Q_{k,0}, Q_{k,1}]_s$, where $Q_{k,0}$ and $Q_{k,1}$ are regarded as embedded in V (see 1.1) in the natural manner. If $x \in Q_{k,0} \cap Q_{k,1}$, then $T_s x = T_0 x = T_1 x$; consequently, $T_s x \in Q_{k,0} \cap Q_{k,1}$. This shows that $Q_{k,s}$ is an invariant subspace of T_s. Let $U_{k,s}$ denote the restriction of T_s to $Q_{k,s}$. For each k it is clear from (d) that the hypotheses of Theorem 1.9 are satisfied in the case of the operators $U_{k,j}$, $j = 0, 1$; thus, we conclude that

$$\text{sp}(U_{k,s}) \subset I_s(\text{sp}(U_{k,0}), E_k)$$

where E_k is the set corresponding to $U_{k,j}$ as in Theorem 1.9. Because of the choice of subspaces $Q_{k,j}$ we have $\text{sp}(U_{k,0}) \subset D(z_k, \delta)$ for some $z_k \in \text{sp}(T_0)$. Since $\|T_k\| \leq 1/2$, $\text{sp}(T_k) \subset D(0, 1]$; this together with Theorem 1.7 implies that $E_k \subset D(0, 1]$. From these observations, Lemma 1.4, Lemma 1.5 and the way in which δ was chosen it follows that

$$\text{sp}(U_{k,s}) \subset \text{sp}(T_0) + D(0, \epsilon).$$

Since $\text{sp}(T_s) = \cup \text{sp}(U_{k,s})$, $k = 1, 2, \cdots, n$, it follows that

$$\text{sp}(T_s) \subset \text{sp}(T_0) + D(0, \epsilon).$$

The conclusion of the theorem follows from this and the fact that ϵ was chosen arbitrarily.

REMARK. A special case of Theorem 4.1 can be proved by slightly extending a method used by Hirschman [3] for determining the spectrum of certain multipliers. The special case occurs if, in addition to the assumptions in Theorem 4.1, we assume that each operator U_k, which is obtained by restricting T_0 to $Q_{k,0}$, has the property that the norm of $\beta I - U_k$ is equal to its spectral radius for each complex number β. Suppose $\lambda \notin \text{sp}(T_0)$. Let d be the distance from λ to $\text{sp}(T_0)$. Let $0 < s < 1$ and choose ϵ so that $\epsilon^{1-s}(2\|T_1\|/d)^s \leq 1/2$. Choose the subspaces $Q_{k,j}$ corresponding to ϵ. Fix k and let S_j be the operator obtained by restricting T_j to $Q_{k,j}$ and let S_s be the corresponding operators on the interpolation spaces. Let $\beta \in \text{sp}(S_0)$. Then $\lambda I - S_s = (\lambda - \beta)I + \beta I - S_s = (\lambda - \beta)(I - A_s)$ where, in general, $A_t =$

$(\beta - \lambda)^{-1}(\beta I - S_t)$.   Now $\|A_s\|_s \leq \|A_0\|^{1-s} \|A_1\|^s \leq 1/2$.   Thus, $\lambda I - S_s$ has an inverse.   Since k was arbitrary, we see that $\lambda \notin \mathrm{sp}(T_s)$.

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