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**LINEAR OPERATORS FOR WHICH  $T^*T$  AND  $T + T^*$   
COMMUTE**

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## LINEAR OPERATORS FOR WHICH $T^*T$ AND $T + T^*$ COMMUTE

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**This paper is about the bounded linear operators  $T$  acting in a separable Hilbert space  $\mathcal{H}$  such that  $T^*T$  and  $T + T^*$  commute. It will be shown that such operators are normal if they are either compact or quasinilpotent. It is conjectured that if  $T^*T$  and  $T + T^*$  commute, then  $T = A + Q$  where  $A = A^*$ ,  $AQ = QA$ , and  $Q$  is quasinormal. This conjecture is shown to be equivalent to  $[T^*T - TT^*]T[T^*T - TT^*]$  being hermitian.**

For bounded linear operators  $X, Y$ , let  $[X, Y] = XY - YX$ . Let  $\theta = \{T: [T^*T, T + T^*] = 0\}$ . The defining condition for  $\theta$  appears in the work of Embry. She has shown that if  $\sigma(T^*) \cap \sigma(T) = \emptyset$  and  $T$  or  $T^*$  are in  $\theta$ , then  $T$  is normal [9, p. 236]. She has also shown that if  $T \in \theta$  and  $[T^*T, TT^*] = 0$ , then  $T$  is quasinormal [8, p. 459]. On the other hand if  $Q$  is quasinormal,  $A = A^*$ , and  $[A, Q] = 0$ , then  $A + Q \in \theta$ . Thus Embry's result shows that the intersection of the class (BN) =  $\{T: [T^*T, TT^*] = 0\}$  (see [4] and [5]) and  $\theta$  is trivial, i.e., the quasinormals. In particular, there are no nonquasinormal centered [11] operators in  $\theta$ . These last observations are helpful when trying to construct examples of nonquasinormal operators in  $\theta$  since (BN) includes all weighted shifts and most weighted translation operators. Using [13] it is also easy to see that if  $T^2$  is normal and  $T \in \theta$ , then  $T$  is normal.

It seems reasonable to make the following conjecture:

$$(C) \quad \theta = \{A + Q: [Q, Q^*Q] = 0, [Q, A] = 0, A^* = A\}.$$

If (C) is true, then using the canonical form for quasinormals given in [1], it is easy to see that every operator in  $\theta$  is subnormal. While we have not been able to resolve (C) we shall present several results which show that the operators in  $\theta$  behave much as if they were hyponormal. In particular, we shall show that if  $T \in \theta$  is compact or quasinilpotent, then it is normal. This will strengthen the result in [6] which asserts that if  $T \in \theta$  and  $T$  is trace class, then  $T$  is normal.

Finally, let  $B(\lambda) = (\lambda - T^*)(\lambda - T) = \lambda^2 - \lambda(T^* + T) + T^*T$ . Note that if  $T \in \theta$ , then the values of  $B(\lambda)$  form a commutative family of normal operators.

2. Main results. Recall from [6] that if  $T \in \theta$ , then  $\lambda + T \in \theta$  for real  $\lambda$ . Also if  $T \in \theta$ , then the null space of  $T$ ,  $N(T)$ , is reducing.

Finally,  $T \in \theta$  if and only if  $T^*[T^*, T] = [T^*, T]T$ .

**THEOREM 1.** *Suppose that  $T \in \theta$  and  $\lambda$  is an eigenvalue of  $T$ . Then the eigenspace of  $T$  associated with  $\lambda$  is reducing.*

*Proof.* Suppose that  $T \in \theta$  and  $\lambda$  is an eigenvalue. If  $\lambda$  is real, we are done. Suppose that  $\lambda$  is not real. Since  $N(T)$  is reducing we may also assume that  $T$  is one-to-one. Let  $\phi$  be such that  $T\phi = \lambda\phi$ . Then  $[T^*, T]\phi = (\lambda - T)T^*\phi$ . Thus  $T^*[T^*, T]\phi = [T^*, T]T\phi$  becomes  $B(\lambda)T^*\phi = 0$ . Since  $B(\lambda)$  is normal, and  $B(\lambda)^* = B(\bar{\lambda})$ , we have  $B(\bar{\lambda})T^*\phi = 0$ . Thus

$$0 = \lambda B(\bar{\lambda})T^*\phi = B(\bar{\lambda})T^*T\phi = T^*TB(\bar{\lambda})\phi,$$

so that  $B(\bar{\lambda})\phi = 0$ . But then

$$0 = B(\bar{\lambda})\phi = (\bar{\lambda} - T^*)(\bar{\lambda} - T)\phi = (\bar{\lambda} - \lambda)(\bar{\lambda} - T^*)\phi.$$

Hence  $T^*\phi = \bar{\lambda}\phi$  and the eigenspace is reducing.

That the eigenspaces of a hyponormal operator are reducing is well known. See, for example, [12, p. 420].

**THEOREM 2.** *If  $T \in \theta$  and  $T$  is quasinilpotent, then  $T = 0$ .*

*Proof.* Suppose that  $T \in \theta$  and  $\sigma(T) = \{0\}$ . We may assume that  $T$  is one-to-one if  $T$  is not zero. If  $T^*T(T + T^*) = 0$ , we are done. Suppose then that  $T^*T(T + T^*) \neq 0$ . Since  $\sigma(T) = \{0\}$ ,  $B(\lambda)$  is invertible for all  $\lambda \neq 0$ . Let  $E(\cdot)$  be the spectral measure associated with the commutative Banach  $*$ -algebra generated by  $T^*T$  and  $T + T^*$ . Then there exist  $E$  measurable functions  $g, h$  such that

$$T^*T = \int_{\mathcal{A}} g(s)E(ds), \quad T^* + T = \int_{\mathcal{A}} h(s)E(ds)$$

and  $\mathcal{A}$  is a compact subset of the plane. (In fact  $\mathcal{A} \subseteq \sigma(T^*T) \times \sigma(T^* + T)$ .) Since  $(T^*T)(T + T^*) \neq 0$ , there exists  $s_0 \in \mathcal{A}$ ,  $s_0$  in the support of  $E$ , such that  $g(s_0), h(s_0)$  are in the  $E$ -essential ranges of  $g, h$ , respectively, and both  $g(s_0), h(s_0)$  are nonzero. The polynomial  $\lambda^2 + h(s_0)\lambda + g(s_0)$  has at least one nonzero root. Call it  $\lambda_0$ . Then

$$B(\lambda_0) = \int_{\mathcal{A}} \left( \lambda_0^2 + h(s)\lambda_0 + g(s) \right) E(ds)$$

is not invertible which is a contradiction. Hence  $T = 0$ .

As an immediate consequence of Theorems 1 and 2 we get:

COROLLARY 1. *If  $T \in \theta$  and  $T$  is compact, then  $T$  is normal.*

Our next result has two interesting corollaries.

THEOREM 3. *Suppose that  $N$  is normal,  $B \in \theta$ , and  $[N, B] = 0$ . Then  $N + B \in \theta$  if and only if, relative to the same orthogonal decomposition of the underlying Hilbert space,  $N = N_1 \oplus N_2$ ,  $B = B_1 \oplus B_2$ ,  $N_1 = N_1^*$  and  $B_2$  is normal.*

*Proof.* The only if part is clear. Suppose then that  $T = N + B \in \theta$  where  $N$  is normal,  $[N, B] = 0$ , and  $B \in \theta$ . Note that  $[N, B^*] = 0$  by Fuglede's theorem. Then  $[T^*, T] = [B^*, B]$ , so that  $T^*[T^*, T] = [T^*, T]T$  becomes  $(N^* - N)[B^*, B] = 0$ . Let  $P$  be the orthogonal projection onto the null space of  $N^* - N$ . Then  $PN = NP$  and  $PB = BP$  since  $P$  is a measurable function of  $N$ . Thus the range of  $P$  reduces both  $N$  and  $B$ , so that  $N = N_1 \oplus N_2$ ,  $B = B_1 \oplus B_2$  relative to  $R(P) \oplus R(I - P)$ . But  $N_1^* = N_1$  by definition of  $P$  and  $B_2$  is normal since  $P[B^*, B] = [B^*, B]$ .

COROLLARY 2. *If  $T \in \theta$ ,  $\lambda + T \in \theta$ , and  $\lambda$  is not real, then  $T$  is normal.*

COROLLARY 3. *If  $T \in \theta$  and  $T$  is completely nonnormal, then there does not exist any nonhermitian normal operator  $N$  such that  $[T, N] = 0$  and  $T + N \in \theta$ .*

3. **Block matrix representation.** If Conjecture (C) is true, then if  $T \in \theta$  and  $T$  is completely nonnormal,  $T$  must have a lower triangular block matrix representation with all zero entries except on the diagonal and first subdiagonal. All diagonal entries are the same self-adjoint operator  $A$ , and all subdiagonal entries are the same positive operator  $P$ . This decomposition follows easily from the work of Brown on quasinormal operators [1].

It is easy to compute what subspace the first block corresponds to. It is the closure of the range of  $T^*T - TT^*$ . Morrel has developed a decomposition for operators  $T$  which have a subspace of  $N[T^*T - TT^*]$  invariant [10]. Applying this to  $T \in \theta$  yields a lower triangular block representation for  $T$  provided that  $T^*T - TT^*$  is not one-to-one. If this approach is to verify Conjecture (C) then it will be necessary and sufficient to show that  $[T^*T - TT^*]T[T^*T - TT^*]$  is hermitian.

THEOREM 4. *Suppose that  $T \in \theta$  is completely nonnormal. If  $[T^*, T]T[T^*, T]$  is hermitian, then  $T = A + Q$  where  $[A, Q] = 0$ ,  $A =$*

$$A^*, [Q, Q^*Q] = 0.$$

*Proof.* Suppose that  $T \in \theta$  is completely nonnormal and  $[T^*, T]T[T^*, T]$  is hermitian. If  $[T^*, T]$  is one-to-one we have  $T = T^*$  and are done. Assume then that  $[T^*, T]$  is not one-to-one. Since  $T$  is nonnormal we have  $[T^*, T] \neq 0$ . Thus from [10] we get that

$$(1) \quad \begin{bmatrix} A_1 & 0 & 0 & \cdot \\ B_1 & A_2 & 0 & \cdot \\ 0 & B_2 & A_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

on  $\mathcal{H} = \sum_{i=0}^l \oplus H_i$ ,  $H_0 = \overline{R([T^*, T])}$ ,  $l \leq \infty$ ,  $\dim H_i \geq \dim H_{i+1}$ . By assumption  $A_1 = A_1^*$ . But then  $[T^*, T] = B_1^* B_1$  so that  $B_1$  is one-to-one. Using the fact that  $H_0 = \overline{R([T^*, T])}$  one gets by direct computation from (1) that

$$(2) \quad B_i^* A_{i+1} = A_i B_i^*, \quad A_{i+1}^* A_{i+1} + B_{i+1}^* B_{i+1} = B_i B_i^* + A_{i+1} A_{i+1}^*$$

for  $i = 1, 2, \dots$  where  $A_{i+1} = B_{i+1} = 0$  if  $l < \infty$ . Furthermore, by definition of the  $H_i$  we have  $B_i$  has dense range so that  $B_i^*$  is one-to-one. Now since  $T^*[T^*, T] = [T^*, T]T$  we have that  $A_1 B_1^* B_1 = B_1^* B_1 A_1$ , or  $B_1^* A_2 B_1 = B_1^* A_2^* B_1$ . Since  $B_1$  is one-to-one with dense range we get that  $A_2 = A_2^*$ . But then from (2), we see that  $B_2^* B_2 = B_1 B_1^*$  and  $B_2$  is one-to-one. Thus from  $B_2^* A_3 = A_2 B_2^*$  we get that  $B_2^* A_3 B_2 = A_2 B_2^* B_2 = A_2 B_1 B_1^* = B_1 A_1 B_1^* = B_1 B_1^* A_2$ . Hence  $A_3 = A_3^*$  and  $[A_2, B_2^* B_2] = 0$ . Suppose now that  $A_i = A_i^*$ ,  $[A_i, B_i^* B_i] = 0$ ,  $B_{i+1}^* B_{i+1} = B_i B_i^*$ , and  $B_i$  is one-to-one with dense range for  $i \leq k$ . Then  $B_{k+1}$  is one-to-one with dense range. Also  $B_k^* A_{k+1} B_k = A_k B_k^* B_k$  and hence  $A_{k+1}^* = A_{k+1}$ . Thus  $B_{k+2}^* B_{k+2} = B_{k+1} B_{k+1}^*$  so that  $B_{k+2}$  is one-to-one with dense range. But then  $A_{k+1} B_{k+1}^* B_{k+1} = A_{k+1} B_k B_k^* = B_k A_k B_k^* = B_k B_k^* A_{k+1}$ . Hence  $[A_{k+1}, B_{k+1}^* B_{k+1}] = 0$ .

If  $l < \infty$ , then the  $l$ th equation is  $A_{l+1}^* A_{l+1} = B_l B_l^* + A_{l+1} A_{l+1}^*$ . As before we get  $A_{l+1}^* = A_{l+1}$  and hence  $B_l = 0$ . But then  $B_i = 0$  for all  $i$  which is a contradiction of the nonnormality of  $T$ . Thus  $l = \infty$ . Now let

$$A = \begin{bmatrix} A_1 & 0 & & \\ 0 & A_2 & & \\ & & \cdot & \\ & & & \cdot \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 & \cdot \\ B_1 & 0 & 0 & \cdot \\ 0 & B_2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Then  $B^* A = A B^*$  from (2). But  $A = A^*$  so that  $[B, A] = 0$ . Hence  $B = T - A \in \theta$ . However  $B^*[B^*, B] = 0$  so that  $B^*(B^* B) = (B^* B) B^*$  and  $B$  is quasinormal as desired.

3. **Comments.** The conclusion of Theorem 1, that eigenspaces are reducing, appears in the work of Berberian. Using Theorem 1, it follows immediately from [3, p. 276] that if  $T \in \theta$ ,  $\sigma(T)$  is countable, and  $T$  is reduction-isoloid [3, p. 277], then  $T$  is normal.

In studying nonnormal operators one usually picks off a normal summand and studies the completely nonnormal operator that is left. Theorem 1 tells us that any condition which provides for eigenvalues is incompatible with the complete nonnormality of a  $T \in \theta$ . Thus one can prove results such as [2, p. 190], [3, p. 277].

**THEOREM 5.** *If  $T \in \theta$  is completely nonnormal and  $T$  is also  $(G_1)$  or restriction convexoid, then  $\sigma(T)$  has no isolated points.*

Finally, we note that the restriction of a  $T \in \theta$  to an invariant subspace is not necessarily in  $\theta$ . The quasinormal operator in [7] whose restriction to an invariant subspace is not quasinormal is an example.

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