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ON LINEAR REPRESENTATIONS OF AFFINE GROUPS. I

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The category of linear representations of an affine group is isomorphic to the category of comodules over a k -Hopf-algebra where k denotes a commutative ring. The category of C -comodules $\text{Comod-}C$ over an arbitrary k -coalgebra C is comonadic over the category $k\text{-Mod}$ of k -modules. It is complete, cocomplete and has a cogenerator. The C -comodules whose cardinality $\leq \max(\text{card}k, \aleph_0)$ generate the category $\text{Comod-}C$. $\text{Comod-}C$ is in general not abelian but can nicely be embedded into an AB_4 category. $\text{Comod-}C$ is a tensored and cotensored $k\text{-Mod}$ -category (enriched over $k\text{-Mod}$) with a canonical (E, M) -factorization which is the factorization in $k\text{-mod}$ if and only if C is flat. $\text{Comod-}C$ has free C -comodules if and only if C is finitely generated and projective. Furthermore I give numerous examples and counterexamples as well as the explicit description of all constructions, in particular of the limits in $\text{Comod-}C$ which was not known even for coalgebras over fields.

Let k be a commutative ring with a unit. $k\text{-Alg}$ shall denote a small category of models of k -algebras (cf. [5] p. XXIV). Recall that an affine k -monoid (resp. k -group) is a monoid (resp. group) in the functor category $[k\text{-Alg}, \text{Sets}]$ whose underlying functor is representable. Let M be a k -module. Then M induces an affine k -monoid $\mathcal{L}(M): k\text{-Alg} \rightarrow \text{Sets}$ by $\mathcal{L}(M)(A) = \text{End}_A(M \otimes_k A)$, $A \in k\text{-Alg}$ (cf. [5] p. 149). Let \mathcal{G} be an affine k -monoid and M a k -module. Then a monoid morphism $\varphi: \mathcal{G} \rightarrow \mathcal{L}(M)$ is called a linear representation of \mathcal{G} in M and the pair (M, φ) a $k\text{-}\mathcal{G}$ -module. The definition of morphisms between $k\text{-}\mathcal{G}$ -modules is evident. Thus one obtains the category $k\text{-}\mathcal{G}\text{-Mod}$ of linear representations of \mathcal{G} , resp. of $k\text{-}\mathcal{G}$ -modules. Since \mathcal{G} is representable we obtain the canonical isomorphisms $[k\text{-Alg}, \text{Sets}] (\mathcal{G}, \mathcal{L}(M)) \cong \mathcal{L}(M)(C) \cong k\text{-Mod} (M, M \otimes_k C)$, where C is the representing object of \mathcal{G} . The monoid structure of \mathcal{G} induces a k -coalgebra structure on C , i.e., the representing object has two k -linear mappings $\Delta: C \rightarrow C \otimes C$ and $\varepsilon: C \rightarrow k$, called comultiplication and counit, such that $\langle C, \Delta, \varepsilon \rangle$ is coassociative and counitary (cf. [19]). By the above canonical isomorphisms every monoid morphism $\varphi: \mathcal{G} \rightarrow \mathcal{L}(M)$ induces a k -linear map $\chi_M: M \rightarrow M \otimes C$ such that $M \otimes \Delta \cdot \chi_M = \chi_M \otimes C \cdot \chi_M$ and $M \otimes \varepsilon \cdot \chi_M = \text{id}_M$, and conversely. A pair $\langle M, \chi_M \rangle$ fulfilling the above properties is called a C -comodule. Let $\langle M, \chi_M \rangle$ and $\langle N, \chi_N \rangle$ be C -comodules. A k -linear mapping $f: M \rightarrow N$ is a C -comodule homo-

morphism if $\chi_N \cdot f = f \otimes C\chi_M$. Let (M, φ_M) and (N, φ_N) be $k\mathcal{C}$ -modules and $\langle M, \chi_M \rangle$, resp. $\langle N, \chi_N \rangle$ the corresponding C -comodules. Then a k -linear mapping $f: M \rightarrow N$ is a $k\mathcal{C}$ -module homomorphism $f: (M, \varphi_M) \rightarrow (N, \varphi_N)$ if and only if $f: \langle M, \chi_M \rangle \rightarrow \langle N, \chi_N \rangle$ is a C -comodule homomorphism.

Hence the category of linear representations of an affine monoid (group) is isomorphic to a category of C -comodules where C is a k -bialgebra (resp. k -Hopf algebra).

In this paper I study the elementary properties of a category of comodules over an arbitrary k -coalgebra. Categories of comodules were already studied by several authors where k is a field or the coalgebra is finite or flat (cf. [5], [7], [10], [14], [15], [17], [18], [19]). In all these cases $\text{Comod-}C$ is a Grothendieck category with a generator. But if C is not flat then $\text{Comod-}C$ need not to be abelian. This was already shown in [17]. The homomorphism theorem is no longer valid, the comodule structure on a subcomodule is in general no longer unique and so on.

But even in the case of a flat coalgebra C one didn't know as yet such elementary things as the explicit descriptions of limits.

Let C be an arbitrary coalgebra over a commutative ring k with a unit. Then the most important results of this paper are: The underlying functor $U: \text{Comod-}C \rightarrow k\text{-Mod}$ is comonadic. The category $\text{Comod-}C$ is complete, cocomplete, wellpowered and cowellpowered, has a generator and cogenerator. $\text{Comod-}C$ can be embedded (full and faithful) into an $AB4$ -category with sufficiently many injectives and projectives which in general fails to be a Grothendieck-category. This embedding is coreflective if and only if all objects in $\text{Comod-}C$ are projective and is an isomorphism if and only if $\text{Comod-}C$ is a spectral category. The functor $\lambda: \text{Comod-}C \rightarrow C^*\text{-Mod}$ (cf. [14] §1 or [19] Chap. II) is comonadic. $\text{Comod-}C$ has free comodules if and only if C is finitely generated and projective. $\text{Comod-}C$ has a proper (E, M) -factorization which is preserved by the underlying functor $\text{Comod-}C \rightarrow k\text{-Mod}$ if and only if C is flat. $\text{Comod-}C$ is well-powered and cowellpowered with respect to this factorization. By applying the techniques of V -categories I show that the $k\text{-Mod}$ -category $\text{Comod-}C$ is tensored and cotensored. If $f: C \rightarrow C'$ is coalgebra morphism then the induced k -linear functor $f^*: \text{Comod-}C \rightarrow \text{Comod-}C'$ preserves tensors and is $k\text{-Mod}$ -comonadic. The k -linear functor $- \otimes C: k\text{-Mod} \rightarrow \text{Comod-}C$ has a k -linear-right adjoint. Furthermore I give numerous examples and counterexamples as well as explicit descriptions of all constructions.

I. Comodules over arbitrary coalgebras. In the language of

monoidal categories a k -coalgebra $\langle C, \Delta, \varepsilon \rangle$ is just a comonoid in the monoidal category $(k\text{-Mod}, \otimes)$ (cf. [11] Chap. VII 3). A C -comodule $\langle M, \chi_M \rangle$ is a coaction of C on M and a C -comodule homomorphism is a morphism between coactions of C in $(k\text{-mod}, \otimes)$ (cf. [11] Chap. VII 4). This formal description gives us at once some elementary results such as the existence of a right adjoint of the underlying functor $U: \text{Comod-}C \rightarrow k\text{-Mod}$ or the creation of colimits by U .

In the sequel I will give another description of $\text{Comod-}C$ which allows us to apply the highly developed theory of monads.

Let $\langle C, \Delta, \varepsilon \rangle$ be a coalgebra. The coalgebra structure of $\langle C, \Delta, \varepsilon \rangle$ induces a functor

$$\mathcal{C}: - \otimes C; k\text{-Mod} \longrightarrow k\text{-Mod}$$

and functorial morphisms

$$\begin{aligned} \Delta = - \otimes \Delta: \mathcal{C} &\longrightarrow \mathcal{C}^2 = - \otimes C \otimes C \\ \varepsilon = - \otimes \varepsilon: \mathcal{C} &\longrightarrow Id_{k\text{-Mod}}. \end{aligned}$$

Since $\langle C, \Delta, \varepsilon \rangle$ is a coalgebra $\langle - \otimes C, - \otimes \Delta, - \otimes \varepsilon \rangle$ clearly defines a comonad over $k\text{-Mod}$. A coalgebra $\langle M, \chi_M \rangle$ over this comonad is a pair where M is k -module and $\chi_M: M \rightarrow \mathcal{C}(M)$ is a k -morphism such that the following diagrams commutes

$$\begin{array}{ccc} \mathcal{C}^2(M) & \xleftarrow{\mathcal{C}(\chi_M)} & \mathcal{C}(M) \\ \uparrow \Delta(M) & = & \uparrow \chi_M \\ \mathcal{C}(M) & \xleftarrow{\chi_M} & M \\ & \nwarrow \varepsilon(M) & \\ M & \xleftarrow{\varepsilon(M)} & \mathcal{C}(M) \\ & \searrow & \uparrow \chi_M \\ & & M \end{array}$$

A morphism f between \mathcal{C} -coalgebras $\langle M, \chi_M \rangle$ and $\langle N, \chi_N \rangle$ is a k -morphism $f: M \rightarrow N$ such that $\chi_N \cdot f = \mathcal{C}(f) \cdot \chi_M$. Hence we obtain the following

THEOREM 1 (Notation as above). *Let $\langle C, \Delta, \varepsilon \rangle$ be a coalgebra. Then the category $\text{Comod-}C$ of C -comodules is comonadic over $k\text{-Mod}$.*

From the elementary theory of monads we obtain at once some important corollaries.

COROLLARY 2 (cf. [11], [13], [16]). *The underlying functor*

$$U: \text{Comod-}C \longrightarrow k\text{-Mod}$$

has a right adjoint $\mathcal{G}: k\text{-Mod} \rightarrow \text{Comod-}C$ defined by

$$\begin{aligned}\mathcal{G}: k\text{-Mod} &\longrightarrow \text{Comod-}C \\ M &\longmapsto \langle M \otimes C, M \otimes \Delta \rangle \\ f &\longmapsto f \otimes C\end{aligned}$$

The comonad defined in $k\text{-Mod}$ by this adjunction is the given comonad $\langle - \otimes C, - \otimes \Delta, - \otimes \varepsilon \rangle$.

COROLLARY 3. *The underlying functor $U: \text{Comod-}C \rightarrow k\text{-Mod}$ creates colimits and isomorphisms. In particular $\text{Comod-}C$ is cocomplete and the colimits are formed in $k\text{-Mod}$.*

COROLLARY 4. *U creates those limits which are preserved by $- \otimes C$. If C is flat and $T: D \rightarrow \text{Comod-}C$ is a finite diagram, then $p: \text{Diag } M \rightarrow T$ is a limit in $\text{Comod-}C$ if and only if $Up: \text{Diag } UM \rightarrow UT$ is a limit in $k\text{-Mod}$.*

Applying 21.3.6 in [16] we obtain

COROLLARY 5. *$\text{Comod-}C$ is cowellpowered.*

Since right adjoints preserve cogenerators we get

COROLLARY 6. *$\text{Comod-}C$ has a cogenerator.*

Let \mathcal{C} be a category with finite limits and finite colimits. A functor $F: C \rightarrow C'$ is called left-exact (right-exact) if F preserves finite limits (finite colimits). F is called exact if F is left-exact and right-exact.

Since $k\text{-Mod}$ is an additive category and $- \otimes C$ is additive and right-exact we obtain from Remark 21.1.11 in [16] Chap. 21 the well known

COROLLARY (cf. [7], [10]).

(1) *$\text{Comod-}C$ is an additive category.*

(2) *U and \mathcal{G} are additive functors.*

Furthermore \mathcal{G} is exact and U is right exact.

PROPOSITION 8 (Notation as above). *The following statements are equivalent:*

(i) *U is exact.*

- (ii) C is flat.
 (iii) \mathcal{S} preserves injectives.

Proof. (ii) \rightarrow (i): Since U creates finite limits and is right exact it is exact.

(i) \rightarrow (ii): Let $f: M \rightarrow N$ be an injective k -module homomorphism. Since \mathcal{S} is exact, $\mathcal{S}(f) = f \otimes C: M \otimes C \rightarrow N \otimes C$ is an equalizer in $\text{Comod-}C$. Since U is exact $f \otimes C$ is injective, i.e., C is flat.

(i) \rightarrow (iii): Well known.

(iii) \rightarrow (i): Let $m: \langle M, \chi_M \rangle \rightarrow \langle N, \chi_N \rangle$ be a monomorphism in $\text{Comod-}C$ and $f: M \rightarrow Q$ an injective extension of M in $k\text{-Mod}$. Then we obtain the following commutative diagram

$$\begin{array}{ccccc} \langle Q \otimes C, Q \otimes \Delta \rangle & \xleftarrow{f \otimes C} & \langle M \otimes C, M \otimes \Delta \rangle & \xrightarrow{m \otimes C} & \langle N \otimes C, N \otimes \Delta \rangle \\ & & \uparrow \chi_M & = & \uparrow \chi_N \\ & & \langle M, \chi_M \rangle & \xrightarrow{m} & \langle N, \chi_N \rangle \end{array}$$

Since \mathcal{S} preserves injectives, $\langle Q \otimes C, Q \otimes \Delta \rangle = \mathcal{S}(Q)$ is injective in $\text{Comod-}C$. Since $\mathcal{S}(Q)$ is injective and m is a monomorphism we obtain a comodule-homomorphism $g: \langle N, \chi_N \rangle \rightarrow \langle Q \otimes C, Q \otimes \Delta \rangle$ such that

$$\begin{array}{ccc} f \otimes C \cdot \chi_M & = & g \cdot m \\ \langle M, \chi_M \rangle & \xrightarrow{m} & \langle N, \chi_N \rangle \\ f \otimes C \cdot \chi_M \downarrow & \swarrow g & \\ \langle Q \otimes C, Q \otimes \Delta \rangle & & \end{array}$$

Since $\langle M, \chi_M \rangle$ is a C -comodule and $\varepsilon: - \otimes C \rightarrow \text{Id}_{k\text{-Mod}}$ is a functorial morphism we obtain the following equations:

$$\varepsilon_M \cdot \chi_M = \text{id}_M \quad \text{and} \quad f \cdot \varepsilon_M = \varepsilon_Q f \otimes C.$$

Thus $f = f \cdot \text{id}_M = f \cdot \varepsilon_M \cdot \chi_M = \varepsilon_Q \cdot f \otimes C \chi_M = \varepsilon_Q \cdot g \cdot m$. Hence m is injective since f is injective, i.e., U is exact.

If C is flat U creates finite limits and colimits. Since $\text{Comod-}C$ is additive and $k\text{-Mod}$ is abelian we conclude that $\text{Comod-}C$ is abelian. Since furthermore $k\text{-Mod}$ is a Grothendieck category and U preserves and reflects colimits and monomorphisms $\text{Comod-}C$ fulfills AB5' (cf. [16] 4, 6.3), i.e., we obtain the following well known result.

COROLLARY 9. *If C is flat then $\text{Comod-}C$ is a Grothendieck category. Furthermore U preserves and reflects finite limits and*

colimits. In particular a comodule homomorphism is an equalizer (coequalizer) in $\text{Comod-}C$ if and only if f is injective (surjective).

EXAMPLES 10. (1) Let k be a regular ring (regular in the sense of von Neumann) (cf. [2] p. 175, EX. 13). Then $\text{Comod-}C$ is a Grothendieck category for every k -coalgebra C .

Let k be a commutative, associative ring with unit. Let T be a k -module. Then $C = k \oplus T$ together $\Delta(r, t) = r \otimes 1 + 1 \otimes t + t \otimes 1 + \rho(t)$ and $\varepsilon(r, t) = r$ is a coalgebra with unit (cf. [18], where $\rho: T \rightarrow T \otimes T$ is an arbitrary coassociative k -morphism (take for example $\rho = 0$). Hence $C = k \oplus T$ is flat (projective, finitely generated, ...) if and only if T is flat (projective, finitely generated, ...).

(2) Let A be a torsion free abelian group A and $C = \mathbb{Z} \oplus A$ with the above defined structure. Then $\text{Comod-}C$ is a Grothendieck category¹.

(3) Let A be an abelian group which is not torsion free. (e.g., $\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}$). Then the coalgebra $C = \mathbb{Z} \oplus A$ with one of the above defined coalgebra structures is not flat¹.

DEFINITION 11. Let $\langle M, \chi_M \rangle$ be a C -comodule. A subcomodule $\langle N, \chi_N \rangle$ is a submodule N of M such that the inclusion $i: N \rightarrow M$ is a comodule homomorphism.

PROPOSITION 12. Let $\text{Comod-}C$ be an abelian category. Then the comodule structure on a subcomodule is unique.

Proof. Let $\langle N, \chi_1 \rangle$ and $\langle N, \chi_2 \rangle$ be subcomodules of $\langle M, \chi_M \rangle$. Since the inclusion $i: \langle N, \chi_1 \rangle \rightarrow \langle M, \chi_M \rangle$ is injective it is a monomorphism and hence an equalizer in $\text{Comod-}C$ since $\text{Comod-}C$ is abelian by assumption. Hence the identity $\langle N, \chi_2 \rangle \rightarrow \langle N, \chi_1 \rangle$ must be a comodule homomorphism. Since $U: \text{Comod-}C \rightarrow k\text{-Mod}$ creates isomorphisms we obtain $\chi_1 = \chi_2$.

EXAMPLE 13. (cf. [18]) Let $C = \mathbb{Z} \oplus \mathbb{Z}/_n\mathbb{Z}$ be the \mathbb{Z} -coalgebra with the following structure:

$$\begin{aligned}\Delta(z, \bar{q}) &= z \otimes 1 + 1 \otimes \bar{q} + \bar{q} \otimes 1 + \bar{q} \otimes \bar{1} \\ \varepsilon(z, \bar{q}) &= z. \quad (\text{cp. (11) Ex. 1})\end{aligned}$$

Then the category $\text{Comod-}C$ of $\mathbb{Z} \oplus \mathbb{Z}/_n\mathbb{Z}$ -comodules is not abelian. By applying Proposition 12 we have only to show that there exist a C -comodule $\langle M, \chi_M \rangle$ and subcomodules $\langle N, \chi_N \rangle$ and $\langle N, \chi'_N \rangle$ of

¹ Let k be a principal ideal domain. Then a k -module M is flat if and only if M is torsion free (cf. [4] §24 Prop. 3 (ii)).

$\langle M, \chi_M \rangle$ with $\chi_N \neq \chi'_N$. The following example was given in [18]. Take

$$\begin{aligned} M &= \mathbf{Q}/\mathbf{Z}; \chi_M(\bar{q}) = \bar{q} \otimes 1 \\ N &= \mathbf{Z}/n\mathbf{Z}; \chi_N(\bar{z}) = \bar{z} \otimes 1 \end{aligned}$$

and

$$N = \mathbf{Z}/n\mathbf{Z}; \chi'_N(\bar{z}) = \bar{z} \otimes 1 + \bar{1} \otimes \bar{z}.$$

Then the inclusion $i: \mathbf{Z}/n\mathbf{Z} \rightarrow \mathbf{Q}/\mathbf{Z}: \bar{z} \mapsto (\bar{z}/n)$ is a comodule homomorphism for χ_N and χ'_N . Since $\chi_N \neq \chi'_N$ we obtain that $\text{Comod-}C$ is not abelian.

Conjecture 14. $\text{Comod-}C$ is abelian if and only if C is flat.

In order to prove this conjecture one has to show that if $\text{Comod-}C$ is abelian then the comodule monomorphisms are injective (cf. Proposition 8).

In [9], P. Freyd proves the existence of free abelian categories. He does it by taking a category C and embedding it into a large ambient abelian category. He then constructs the smallest exact subcategory containing C . The external version of this construction was made by M. Alderman in [1]. He gives an explicit description of free abelian categories. I'll take up Alderman's construction and will show that the category $\text{Comod-}C$ (for every coalgebra C) can be fully and faithfully embedded into an AB -4 category with enough projectives and injectives, the free abelian category over $\text{Comod-}C$ which in general fails to be a Grothendieck category.

Let us now recall Alderman's construction. Let \mathcal{A} be an additive category. In the functor category $\mathcal{A}^{\rightarrow}$ define the following equivalence relation:

$$\begin{array}{ccccc} A' & \xrightarrow{f'} & A & \xrightarrow{f} & A'' \\ \varphi' \downarrow & & \downarrow \varphi & & \downarrow \varphi'' \equiv \psi' \\ B' & \xrightarrow{g'} & B & \xrightarrow{g} & B'' \end{array} \quad \begin{array}{ccccc} A' & \xrightarrow{f'} & A & \xrightarrow{f} & A'' \\ \psi' \downarrow & & \downarrow \psi & & \downarrow \psi'' \\ B' & \xrightarrow{g'} & B & \xrightarrow{g} & B'' \end{array}$$

iff there are maps $h_1: A \rightarrow B'$ and $h_2: A'' \rightarrow B$ such that $\varphi - \psi = g'h_1 + h_2f$, i.e., the two short complexes are homotopic. Then the resulting category $\mathcal{A}^{\rightarrow}/\equiv$ is denoted by $Ab(\mathcal{A})$. $Ab(\mathcal{A})$ is abelian ([1]). The functor $I_{\mathcal{A}}: \mathcal{A} \rightarrow Ab(\mathcal{A}): A \mapsto (0 \rightarrow A \rightarrow 0)$ is obviously full and faithful. Let now F be an additive functor from \mathcal{A} to \mathcal{B} with \mathcal{B} abelian. Then there is a unique exact functor $F^*: Ab(\mathcal{A}) \rightarrow \mathcal{B}$ such that the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{I_A} & Ab(A) \\
 & \searrow F & \downarrow F^* \\
 & & B
 \end{array}$$

commutes up to natural equivalence (cf. [1] Theorem 1.14).

Let now A be the additive category $\text{Comod-}C$.

THEOREM 15. *Let C be a coalgebra. Then*

(1) *There exists an abelian category $Ab(\text{Comod-}C)$ and a full and faithful embedding*

$$I: \text{Comod-}C \longrightarrow Ab(\text{Comod-}C)$$

such that every additive functor $F: \text{Comod-}C \rightarrow B$ into an abelian category B can be factored through an exact functor $F^: Ab(\text{Comod-}C) \rightarrow B$ (up to natural equivalence).*

(2) *$Ab(\text{Comod-}C)$, the free abelian category over $\text{Comod-}C$, is an $AB4$ -category.*

(3) *The inclusion functor I preserves products and coproducts.*

(4) *The inclusion functor I preserves equalizers (coequalizers) if and only if the equalizers (coequalizers) in $\text{Comod-}C$ are coretractions (retractions).*

(5) *$Ab(\text{Comod-}C)$ has sufficiently many projectives and injectives.*

As immediate consequences of this theorem we obtain the following two theorems by applying the special adjoint functor theorem:

THEOREM 16 (Notation as above). *The following statements are equivalent.*

(i) *$\text{Comod-}C$ is a coreflective subcategory of $Ab(\text{Comod-}C)$.*

(ii) *The inclusion functor $I: \text{Comod-}C \rightarrow Ab(\text{Comod-}C)$ preserves epimorphisms.*

(iii) *Every epimorphism in $\text{Comod-}C$ is a retraction.*

(iv) *Every object in $\text{Comod-}C$ is projective.*

THEOREM 17 (Notation as above). *The following statements are equivalent:*

(i) *The inclusion $I: \text{Comod-}C \rightarrow Ab(\text{Comod-}C)$ is an isomorphism.*

(ii) *Every object in $\text{Comod-}C$ is injective.*

(iii) *Every monomorphism in $\text{Comod-}C$ is a coretraction. If (i)–(iii) are fulfilled then $\text{Comod-}C$ is a spectral category.*

REMARK 18. If $\text{Comod-}C$ is an abelian category then the

statements of the above two theorems are equivalent. But if $\text{Comod-}C$ is not abelian then these conditions need not to be equivalent.

Proof of Theorem 15. We have to prove (2), (3), (4) since the other statements were proved in [1].

(2) Let $M'_i \xrightarrow{f'i} M_i \xrightarrow{f'i} M''_i, i \in I$, be a family of $\text{Ab}(\text{Comod-}C)$ -objects. Then

$$\begin{array}{ccccc} \coprod M'_i & \xrightarrow{\coprod f'i'} & \coprod M_i & \xrightarrow{\coprod f'i} & \coprod M''_i \\ m'_i \uparrow & & m_i \uparrow & & m''_i \uparrow \\ M'_i & \xrightarrow{f'i''} & M_i & \xrightarrow{f'i} & M''_i \end{array}$$

is the coproduct of these family in $\text{Ab}(\text{Comod-}C)$ as one easily shows, where m'_i, m_i and $m''_i, i \in I$ are the corresponding coproducts of the objects M'_i, M_i and M''_i in $\text{Comod-}C$. Hence $\text{Ab}(\text{Comod-}C)$ is cocomplete, i.e., an AB -3 category. In order to show that $\text{Ab}(\text{Comod-}C)$ is an AB 4-category we have to show that for any family $\{f_i: (M_i) \rightarrow (N_i)\}$ of monomorphisms in $\text{Ab}(\text{Comod-}C)$, the morphism $\coprod f_i$ is also a monomorphism.

LEMMA 19 ([1] Theorem 1.1 or [8] Lemma 6.1).

(1) *The equalizer of*

$$\begin{array}{ccccc} M' & \xrightarrow{f'} & M & \xrightarrow{f} & M'' \\ \varphi' \downarrow & & \downarrow \varphi & & \downarrow \varphi'' \\ N' & \xrightarrow{g'} & N & \xrightarrow{g} & N'' \end{array}$$

is given by

$$\begin{array}{ccccc} M' \oplus N & \xrightarrow{\begin{pmatrix} f' & 0 \\ \varphi' & -1 \end{pmatrix}} & M \oplus N' & \xrightarrow{\begin{pmatrix} \varphi & -g' \\ f & 0 \end{pmatrix}} & N \oplus M'' \\ \downarrow (1, 0) & & \downarrow (1, 0) & & \downarrow (0, 1) \\ M' & \longrightarrow & M & \longrightarrow & M'' \end{array}$$

and the coequalizer by

$$\begin{array}{ccccc} N' & \xrightarrow{g'} & N & \xrightarrow{g} & N'' \\ \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ N' \oplus M & \xrightarrow{\begin{pmatrix} g' & \varphi \\ 0 & -f \end{pmatrix}} & N \oplus M'' & \xrightarrow{\begin{pmatrix} g & \varphi'' \\ 0 & -1 \end{pmatrix}} & N'' \oplus M'' \end{array}.$$

Since $Ab(\text{Comod-}C)$ is an abelian category we obtain at once the following criterium.

LEMMA 20. *Let*

$$(\varphi) = \begin{array}{ccccc} M' & \xrightarrow{f'} & M & \xrightarrow{f} & M'' \\ \downarrow \varphi' & & \downarrow \varphi & & \downarrow \varphi'' \\ N' & \xrightarrow{g'} & N & \xrightarrow{g} & N'' \end{array}$$

be a morphism in $Ab(\text{Comod-}C)$. Then

(1) (φ) is a monomorphism if and only if there are morphisms

$$\begin{aligned} \psi': N' &\longrightarrow M', \quad q: M \longrightarrow M' \\ q'': M'' &\longrightarrow M \text{ and } \psi: N \longrightarrow M \text{ such that} \\ f'q + \psi \cdot \varphi + q'' \cdot f &= \text{id}_M \end{aligned}$$

and

$$f' \cdot \psi' + \psi \cdot g' = 0.$$

(2) (φ) is an epimorphism if and only if there are morphisms

$$\begin{aligned} p: N &\longrightarrow N', \quad p'': N'' \longrightarrow N, \\ \delta: N &\longrightarrow M \text{ and } \delta: N'' \longrightarrow M'' \text{ such that} \\ g' \cdot p + p''g + \varphi \cdot \delta &= \text{id}_N \\ \delta''g + f \cdot \delta &= 0. \end{aligned}$$

The construction of coproducts in $Ab(\text{Comod-}C)$ and Lemma (20) 1 show immediately that $Ab(\text{Comod-}C)$ is an AB4-category.

(3) Trivial.

(4) Let $f: M \rightarrow N$ an equalizer in $\text{Comod-}C$ and assume that I preserves this equalizer

Consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \longrightarrow & 0 \\ If = (f) \downarrow & & \downarrow f & & \downarrow \\ 0 & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

Then (f) is a monomorphism in $Ab(\text{Comod-}C)$ if and only if there exists a morphism $g: N \rightarrow M$ such that $g \cdot f = \text{id}_M$, i.e., if f is a coretraction (Lemma 20.1). In the same vein one shows by applying Lemma 20.2 that f is an epimorphism if and only if f is a retraction $\text{Comod-}C$. This completes our proof.

REMARK 21. (1) $Ab(\text{Comod-}C)$ is an $AB4^*$ -Category. Let C be a coalgebra. Then $\text{Comod-}C$ is complete by Corollary 26. Now in the same vein as above one shows that $Ab(\text{Comod-}C)$ has products which are the pointwise ones. Hence $Ab(\text{Comod-}C)$ is an $AB3^*$ -category. From the construction of products and the characterization of epimorphisms by Lemma 20.2 we obtain that $Ab(\text{Comod-}C)$ is an $AB4^*$ -category.

(2) $Ab(\text{Comod-}C)$ is, in general, not a Grothendieck category. Take Z with the trivial coalgebra structure. Then $\text{Comod-}Z$ is isomorphic to $Z\text{-Mod}$, the category of abelian groups. Assume $Ab(\text{Comod-}Z) = Ab(Z\text{-Mod})$ is a Grothendieck category. Since $Ab(Z\text{-Mod})$ is an $AB3^*$ -category by 21.1, $Ab(Z\text{-Mod})$ is a C_2 -category (Mitchell [12]), i.e., for any set (M_i) of objects in $Ab(Z\text{-Mod})$ the canonical morphism

$$m: \coprod M_i \longrightarrow \prod M_i$$

is a monomorphism. Take now $M_n = Z$ for $n \in N$. Then the canonical morphism

$$I(m) = \begin{array}{ccccccc} 0 & \longrightarrow & \coprod_N Z & = & Z^{(N)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow m & & \downarrow \\ 0 & \longrightarrow & \prod_N Z & = & Z^N & \longrightarrow & 0 \end{array}$$

is the image of the canonical morphism $m: Z^{(N)} \rightarrow Z^N$. Then $I(m)$ is a monomorphism in $Ab(Z\text{-Mod})$ if and only if the canonical morphism $m: Z^{(N)} \rightarrow Z^N$ is a coretraction. Consider now the canonical projection $p: Z^N \rightarrow Z^N/Z^{(N)}$ and the element $\bar{x} = (2^n; n \in N) \in Z^N$. Then the image $p(\bar{x})$ is obviously divisible by every power of 2. Since an element $(x_i; i \in I)$ in Z^I is divisible if and only if all components x are divisible in Z we obtain that $Z_N/Z^{(N)}$ cannot be embedded in a product Z^I . Hence the monomorphism $m: 0 \rightarrow Z^{(N)} \rightarrow Z^N$ is not split, i.e., no coretraction and therefore $I(f)$ is no monomorphism in $Ab(Z\text{-Mod})$. Hence $Ab(\text{Comod-}Z)$ is not a Grothendieck category.

Next I will prove that $\text{Comod-}C$ has a generator where C is an arbitrary coalgebra. The existence of a generator in $\text{Comod-}C$ where C is flat was proved by Saavedra [15] 2.07. But his proof cannot be generalized. The following proof uses Barr's results in [3] and is in fact an imitation of his proof of the existence of a set of generators in the category of coalgebras over a commutative ring.

A submodule $U \subset M$ of a module M is called a *pure submodule* of M provided that for any module N $U \otimes N \rightarrow M \otimes N$ is a monomorphism.

PROPOSITION 22 (Barr [3] 1.3). *Given $U \subset M$ there is an $U^* \subset M$*

such that $U \subset U^*$ such that U^* is a pure submodule of M , and such that

$$\text{card}(U^*) \leq \max(\text{card}(U), \text{card}(k), \chi_0)^2.$$

THEOREM 23. Let $\langle M, \chi \rangle$ be a C -comodule, U a submodule of M . Then there is a subcomodule $M' \subset M$ such that $U \subset M'$ and

$$\text{card}(M') \leq \max(\text{card } U, \text{card } k, \aleph_0).$$

Proof. Let $\langle M, \chi \rangle$ be a C -comodule. A k -submodule U of M is called χ -invariant if $\chi(U) \subset i \otimes C(U \otimes C)$ where $i: U \rightarrow M$ is the inclusions. Let U be a submodule of M . For each $u \in U$ choose a representation

$$\chi(u) = \sum_{i=1}^n m_i \otimes C_i.$$

Let U' be the submodule generated by all m_i and the elements of U . Then $U \subset U' \subset M$, $\chi(U) = \sum_{i=1}^n m_i \otimes C_i \in i \otimes C(U' \otimes C)$ and $\text{card}(U') \leq \max(\text{card } U, \text{card } k, \chi_0)$.

Now iterate the above process in order to get a sequence

$$U \subset U' \subset U'' \subset \dots \subset U^{(n)} \subset \dots$$

such that $\chi(U^{(n)}) \subset i \otimes C(U^{(n+1)} \otimes C)$. Define $\hat{U} = \bigcup_{n \in \mathbb{N}} U^{(n)}$. Then \hat{U} is a submodule of M such that $U \subset \hat{U}$ such that \hat{U} is χ -invariant and such that $\text{card}(\hat{U}) \leq \max(\text{card } U, \text{card } k, \chi_0)$. Next we define the following sequence of submodule of M

$$U_n = U_{n-1}^* \quad \text{when } n \text{ is odd}$$

and

$$U_n = \hat{U}_{n-1} \quad \text{when } n \text{ is even,}$$

where U_{n-1}^* is "the" pure submodule of M containing U_{n-1} (\rightarrow Proposition 22). Then let $M' = \bigcup U_n$. Then $M' \subset M$ is a pure submodule of M which is χ -invariant. Hence $\chi(M') \subset M' \otimes C$ and $\langle M', \chi \rangle$ is a subcomodule of $\langle M, \chi \rangle$. The cardinality conclusion is obvious.

THEOREM 24. The C -comodule whose cardinality $\leq \max(\text{card } k, \aleph_0)$ generate the category $\text{Comod-}C$. In particular $\text{Comod-}C$ has a generator.

Proof. Let $f, g: \langle M, \chi_M \rangle \rightrightarrows \langle N, \chi_N \rangle$ be two different comodule homomorphisms. Then there exists an element $m \in M$ such that

² $\text{card}(X)$ means the cardinality of the set X .

$f(m) \neq g(m)$. Then by Theorem 22 there exists a subcomodule M' containing the submodule generated by m ;

$\langle m \rangle \subset M' \subset M$. Furthermore $\text{card } \langle m \rangle \leq \text{card } k$. Hence $\text{card } M' \leq \max(\text{card } k, \chi_0)$ and $f_i \neq g_i: \langle M', \chi_{M'} \rangle \xrightarrow{i} \langle M, \chi_M \rangle \xrightarrow{f} \langle N, \chi_N \rangle$.

EXAMPLE 25. Let $C = \mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$. Then the "set" of denumerable $\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$ -comodules generates the category $\text{Comod-}\mathbb{Z}\mathbb{Q}/\mathbb{Z}$.

Since $\text{Comod-}C$ is cocomplete, cowellpowered and has a generator we obtain by applying the special functor theorem [cf. [13] p. 114 Corollary].

COROLLARY 26. *The category $\text{Comod-}C$ is complete. Moreover $\text{Comod-}C$ is locally presentable in the sense of Gabriel-Ulmer.³*

This Corollary shows only the existence of arbitrary limits in $\text{Comod-}C$ but gives us no explicit description. Our next step will be therefore to describe explicitly the limits. This was not known even in the case where k is a field. We apply Linton's techniques of constructing colimits in an Eilenberg-Moore category over Sets (cf. [14] Chap. 21)

Construction of limits in $\text{Comod-}C$ 27. Let I be a small category and $D: I \rightarrow \text{Comod-}C$ be a diagram. Let $(\lim UD, \varphi)$ be the limit of UD in $k\text{-Mod}$ and $(\lim(- \otimes C \cdot U \cdot D, \psi)$ the limit of $- \otimes CU \cdot D$ in $k\text{-Mod}$. If I is void then $\lim D$ is the zero comodule. Now let I be nonvoid. Let $\eta: Id_{\text{Comod-}C} \rightarrow - \otimes C \cdot U$ be the functorial morphism defined by

$$\begin{array}{ccc} \chi = \eta(\langle M, \chi \rangle): \langle M, \chi \rangle & \longrightarrow & \langle M \otimes C, M \otimes A \rangle \\ M & \xrightarrow{\chi} & M \otimes C \\ \chi \downarrow & & \downarrow M \otimes A \\ M \otimes C & \xrightarrow{M \otimes \chi} & M \otimes C \otimes C \end{array}$$

Then there is exactly one k -morphism

$$\eta^*: \lim(UD) \longrightarrow \lim(- \otimes C \cdot UD)$$

such that the following diagram commutes:

$$\begin{array}{ccc} UD & \xrightarrow{\varphi} & \text{Diag}(\lim UD) \\ U*\eta^*D \downarrow & = & \downarrow \text{Diag}(\eta^*) \\ - \otimes C \cdot U \cdot D & \xrightarrow{\psi} & \text{Diag}(\lim - \otimes CUD) \end{array}$$

³ The set of generators in $\text{Comod-}C$ is \aleph_1 -presentable-(Ulmer).

where Diag is the diagonal functor.

Let $\lim UD = M$ and $\lim - \otimes C \cdot U \cdot D = N$. Then there exists exactly one k -morphism $\varphi^*: M \otimes C \rightarrow N$ such that $- \otimes C * \varphi = \psi \cdot \text{Diag}(\varphi^*)$. We claim that η^* is a monomorphism. Consider

$$\begin{array}{ccccc} \text{Diag}(X) & \xrightarrow[\text{Diag}(g)]{\text{Diag}(f)} & \text{Diag}(M) & \xrightarrow{\text{Diag}(\eta^*)} & \text{Diag}(N) \\ & & \downarrow \varphi & = & \downarrow \psi \\ & & UD & \xrightarrow{U*\eta^*D} & - \oplus C \cdot U \cdot D \end{array}$$

where $f, g: X \rightarrow M$ are k -morphisms with $\eta^* \cdot f = \eta \cdot g$. Since $(U, - \otimes C)$ is an adjoint functor pair $U*\eta$ is a coretraction and hence also $U*\eta^*D$. Thus we obtain $\varphi \text{Diag}(f) = \varphi \text{Diag}(g)$ and hence $f = g$ since φ is a universal morphism.

Consider now the cofree comodules $\langle M \otimes C, M \otimes \Delta \rangle$ and $\langle N \otimes C, N \otimes \Delta \rangle$ and the comodule homomorphisms

$$\varphi^* \otimes C \cdot M \otimes \Delta, \eta^* \otimes C: M \otimes C \longrightarrow N \otimes C.$$

Let $\langle K, \chi_K \rangle \xrightarrow{m} \langle M \otimes C, M \otimes \Delta \rangle \xrightarrow[\varphi^* \otimes C \cdot M \otimes \Delta]{\eta^* \otimes C} \langle N \otimes C, N \otimes \Delta \rangle$ be an equalizer of $(\eta^* \otimes C, \varphi^* \otimes M \otimes \Delta)$. Then $\langle K, \chi_K \rangle$ is the limit of D in $\text{Comod-}C$.

This is now shown in several steps (cf. [16] 21. 2. 10).

EXAMPLE 28. Let C be a flat coalgebra. Then the finite limits and in particular the equalizers in $\text{Comod-}C$ are formed in $k\text{-Mod}$. We want now to compute the products in $\text{Comod-}C$. Let $\langle M_i, \chi_i \rangle$; $i \in I$, be a family of C -comodules. Denote by $\prod M_i$ the product of the underlying k -modules and by $\prod M_i \otimes C$ the product of the k -modules $M_i \otimes C$. Then we obtain two canonical morphisms η^* and φ^* defined by the universal property of $\prod M_i \otimes C$:

$$\begin{array}{ccc} M_i \otimes C & \xleftarrow{\text{can}} & \prod M_i \otimes C \\ \downarrow \chi_i & = & \downarrow \prod \chi_i = \eta^* \\ M_i & \xleftarrow{\text{can}} & \prod M_i \end{array}$$

and

$$\begin{array}{ccc} M_i \otimes C & \xleftarrow{\text{can}} & \prod M_i \otimes C \\ \uparrow & = & \uparrow \varphi^* \\ M_i \otimes C & \xleftarrow{\text{can} \otimes C} & (\prod M_i) \otimes C \end{array}$$

with $\varphi^*((m_i) \otimes c) = (m_i \otimes c)$ and $\eta^*(m_i) = (\chi_i(m_i))$. Then the equalizer of

$$(\Pi M_i) \otimes C \xrightarrow[\varphi^* \otimes C \cdot (\Pi M_i) \otimes \Delta]{\eta^* \otimes C} (\Pi M_i \otimes C) \otimes C$$

is the product of the family $\langle M_i, \chi_i \rangle$ in $\text{Comod-}C$, i.e.,

$$\begin{aligned} \prod_{\text{Comod-}C} \langle M_i, \chi_i \rangle &= \left\{ \sum_{\text{finite}} \bar{m}_k \otimes C_k \in (\Pi M_i) \otimes C; \sum_{\text{finite}} (\chi_i(m_i^k)) \otimes C_k \right. \\ &= \left. \sum_{\text{finite}} \sum_{(C_k)} (m_i^k \otimes C_{k(1)}) \otimes C_{k(2)} \right\} \end{aligned}$$

where $\bar{m}_k = (m_i^k)_i \in I$ and $\Delta C_k = \sum_{(C_k)} C_{k(1)} \otimes C_{k(2)}$ with the comodule structure induced by the comodule structure $(\Pi M_i) \otimes \Delta$ and $(\Pi M_i \otimes \varepsilon(\Pi M_i) \otimes C$. The projections p_i are given by the following assignments.

$$p_i: \prod_{\text{Comod-}C} \langle M_i, \chi_i \rangle \longrightarrow \langle M_i, \chi_i \rangle \sum_{\text{finite}} (m_i^k \otimes C_k \longmapsto \varepsilon(C_k) \cdot m_i^k.$$

Let us now consider the functorial morphism (functorial in C)

$$\lambda: k\text{-Mod}(M, N \otimes C) \longrightarrow k\text{-Mod}(C^* \otimes M, N)$$

defined by $\lambda(f)(c^* \otimes m) = (1 \otimes c^*)f(m)$ where $C^* = k\text{-Mod}(C, k)$. If C is a coalgebra then C^* is a k -algebra with the multiplication

$$f * f'(c) = \sum_{(c)} f(c_{(1)}) \cdot f'(c_{(2)})$$

and unit $e(c) = \varepsilon(c)$. (cf. [14]) Let C be a coalgebra and $\langle M, \chi: M \rightarrow M \otimes C \rangle$ a comodule. Then M is a C^* -left module with multiplication: $\lambda(\chi): C^* \otimes M \rightarrow M$. The assignments

$$\begin{aligned} \lambda: \text{Comod-}C &\longrightarrow C^*\text{-Mod} \\ \langle M, \chi \rangle &\longmapsto \langle M, \lambda(\chi) \rangle \\ f &\longmapsto f \end{aligned}$$

define a functor (cf. [14]).

THEOREM 29. $\lambda: \text{Comod-}C \rightarrow C^*\text{-Mod}$ is comonadic. In particular λ has a right adjoint.

Proof. Since $\text{Comod-}C$ is cocomplete, cocomplete and has a generator, λ has a right-adjoint if and only if λ preserves colimits (special adjoint functor theorem). Let

$$\langle M_i, \chi_i \rangle \xrightarrow{m_i} \langle \text{colim } M_i, \chi \rangle$$

be a colimit diagram in $\text{Comod-}C$. Then $\lambda(\chi): C^* \otimes \text{colim } M_i \rightarrow \text{colim } M_i$ is a colimit of $\langle M_i, \lambda(\chi_i) \rangle$, $i \in I$, as one easily computes.

Hence λ preserves colimits and thus has a right adjoint. Next I'll show that λ creates equalizer of λ -contractible pairs. Let $f, g: \langle A, \chi_A \rangle \rightrightarrows \langle B, \chi_B \rangle$ be a pair of λ -contractible $\text{Comod-}C$ morphisms and $m: K \rightarrow A$ be an equalizer of $f, g: \langle A, \lambda(\chi_A) \rangle \rightrightarrows \langle B, \lambda(\chi_B) \rangle$ in $C^*\text{-Mod}$. Then there exist C^* -module homomorphisms $h: \langle B, \lambda(\chi_B) \rangle \rightarrow \langle A, \lambda(\chi_A) \rangle$ and $k: \langle A, \lambda(\chi_A) \rangle \rightarrow K$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & A & \xrightarrow{1_A} & A \\
 & \nearrow h & & \nwarrow f & \\
 & & B & & \\
 m \uparrow & & \uparrow g & & \uparrow m \\
 & & A & & \\
 & \nwarrow k & & \nearrow m & \\
 K & \xrightarrow{1_K} & K & & K
 \end{array}$$

Since functors preserve equalizers of contractible pairs, $K \xrightarrow{m} A \xrightleftharpoons[f]{g} B$ is an equalizer of the contractible pair (f, g) in $k\text{-Mod}$. Since $U: \text{Comod-}C \rightarrow k\text{-Mod}$ is comonadic, K carries a comodule structure χ_K such that $\langle K, \chi_K \rangle \xrightarrow{m} \langle A, \chi_A \rangle \xrightleftharpoons[f]{g} \langle B, \chi_B \rangle$ is an equalizer diagram in $\text{Comod-}C$. Hence λ creates equalizers of λ -contractible pairs and hence is comonadic.

REMARKS 30. (1) The fact that λ creates equalizers of λ -contractible pairs follows also from the following:

LEMMA. Let $f, g: \langle A, \chi_A \rangle \rightrightarrows \langle B, \chi_B \rangle$ be a pair of comodule homomorphisms and $K \xrightarrow{m} A \xrightleftharpoons[f]{g} B$ the equalizer of f, g in $k\text{-Mod}$. If m is a coretraction in $k\text{-Mod}$ then K carries a comodule structure χ_K such that

$$\langle K, \chi_K \rangle \xrightarrow{m} \langle A, \chi_A \rangle \xrightleftharpoons[f]{g} \langle B, \chi_B \rangle$$

is an equalizer diagram in $\text{Comod-}C$.

Let m be an equalizer of a λ -contractible pair f, g . Then m is a coretraction in $k\text{-Mod}$ and hence an equalizer in $\text{Comod-}C$, i.e., λ creates equalizers of λ -contractible pairs.

(2) The fact that λ is comonadic follows immediately from the following Dubuc-triangle

$$\begin{array}{ccc}
 \text{Comod-}C & \xrightarrow{\lambda} & C^*\text{-Mod} \\
 & \searrow U \quad \quad \swarrow V & \\
 & = & \\
 & \swarrow & \searrow \\
 & k\text{-Mod} &
 \end{array}$$

where U and V are the underlying functors. Since U and V are comonadic and $\text{Comod-}C$ has equalizer, λ is also comonadic (cf. [20] Proposition 6.11).

(3) If C is finite (\equiv finitely generated and projective) then $\lambda: \text{Comod-}C \rightarrow C^*\text{-Mod}$ is an isomorphism of categories (cf. [14]).

The next proposition solves the problem of the existence of free comodules i.e. answers the following question: For which coalgebras C does the forgetful functor $V: \text{Comod-}C \rightarrow \text{Sets}$ have a left-adjoint?

PROPOSITION 31. *The following statements are equivalent:*

- (i) *The forgetful functor $V: \text{Comod-}C \rightarrow \text{Sets}$ has a left-adjoint.*
- (ii) *C is finite i.e. finitely generated and projective.*
- (iii) *$- \otimes C: k\text{-Mod} \rightarrow k\text{-Mod}$ preserves limits.*
- (iv) *$\lambda: \text{Comod-}C \rightarrow C^*\text{-Mod}$ has a left-adjoint.*
- (v) *$U: \text{Comod-}C \rightarrow k\text{-Mod}$ preserves limits.*

If one of these conditions is fulfilled then $\lambda: \text{Comod-}C \rightarrow C^\text{-Mod}$ is an isomorphism.*

Proof. The equivalences (i) \leftrightarrow (iii) \leftrightarrow (iv) \leftrightarrow (v) are categorical routine. The equivalence (iii) \leftrightarrow (ii) follows from the well-known fact that $- \otimes C$ preserves limits if and only if C is finitely presented and flat or equivalently if C is finitely generated and preprojective. If one of these conditions is fulfilled then λ is an isomorphism by (30.3).

Description of the free C -comodules 32. Let C be a finitely generated and projective coalgebra. The above proposition gives us the following explicit description of the free C -comodules: Let X be an arbitrary set. Then the free C -comodule FX generated by X is given by $FX \cong \bigoplus_x C^*$ where C^* has the "canonical" C -comodule structure.

COROLLARY 33. *Notation as above. The functor $\lambda: \text{Comod-}C \rightarrow C^*\text{-Mod}$ is an isomorphism if and only if C is finitely generated and projective.*

Next we consider factorizations in $\text{Comod-}C$. Let us first recall some of the basic notions and propositions (cf. [20]). Let A be a

category. For two A -morphisms $e: A \rightarrow B$ and $m: C \rightarrow D$ we write $e \downarrow m$ if every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ \downarrow & \swarrow w & \downarrow \\ C & \xrightarrow{m} & D \end{array}$$

can be made commutative by a unique morphism $w: B \rightarrow C$. Let P be any class of A -morphisms. Then p^\uparrow resp. p^\downarrow shall denote the following classes of A -morphisms.

$$p^\uparrow = \{e; e \downarrow m \text{ for all } m \in P\}$$

$$p^\downarrow = \{m; e \downarrow m \text{ for all } e \in P\}.$$

A pair (E, M) of classes E and M of A -morphisms is a *prefactorization* in A if $E = M^\uparrow$ and $M = E^\downarrow$. A prefactorization (E, M) is called a *factorization in A* if every morphism f in A is of the form $f = m \cdot e$ with $m \in M$ and $e \in E$. A factorization (E, M) is proper if every $e \in E$ is an epimorphism and every $m \in M$ is a monomorphism. Hence a proper factorization on A is the same thing as a bicategorical structure in the sense of Isbell. We say that a category A has a M -factorization if A has a (M^\uparrow, M) -factorization. Let K and L be categories with factorizations M_K resp. M_L . A functor $F: K \rightarrow L$ is said to *top reserve* M_K -factorizations if $F(M_K) \subset M_L$ and $F(M_K^\uparrow) \subset M_L^\uparrow$. F is said to *reflect* M_L -factorizations if $F^{-1}(M_L) \subset M_K$ and $F^{-1}(M_L^\uparrow) \subset M_K^\uparrow$. Let $H_K \subset \text{Mor } K$ with $\text{Iso}(K) \subset H_K$ and $H_K \text{ Iso}(K) \subset H_K$. A functor $F: K \rightarrow L$ is said to *create H_K -factorizations from M_L -factorizations* if for all $f \in \text{Mor } K$ with

$$Ff = m_L e_L, m_L \in M_L, e_L \in M_L^\uparrow$$

there is a unique factorization $f = m_K \cdot e_K$ in K with $F_{m_K} = m_L$, $F_{e_K} = e_L$, $m_K \in H_K$, $e_K \in K^\uparrow$.

PROPOSITION 34. *Let K be a cocomplete, cowellpowered category. Then K has an (epi, extremal mono)-factorization i.e., a factorization (E, M) where E is the class of all epimorphisms and M is the class of all extremal monomorphisms (Isbell-Kennison).*

Hence the category $\text{Comod-}C$ has at least one proper factorization.

PROPOSITION 35. *Let (E, M) be a proper factorization in $\text{Comod-}C$. Then the following statement are equivalent.*

(i) *The underlying functor $U: \text{Comod-}C \rightarrow k\text{-Mod}$ preserves the*

factorization.

- (ii) U is exact.
- (iii) C is flat.

Proof. Since (ii) and (iii) are equivalent by Proposition 8 and since the implication (iii) \rightarrow (i) is trivial we have only to prove (i) \rightarrow (iii). Let E_k resp. M_k be the class of all epimorphisms resp. monomorphisms in $k\text{-Mod}$. Since U preserves the factorization and U reflects isomorphisms we obtain that $E = U^{-1}(E_k)$ and $M = U^{-1}(M_k)$. Since $U(E) \subset E_k$ and $- \otimes C$ is right adjoint to U we get $(M_k) \otimes C \subset M$. Hence we get for the functor $- \otimes C: k\text{-Mod} \rightarrow k\text{-Mod}$

$$(M_k) \otimes C = U(- \otimes C)(M_k) \subset (M) \subset M_k$$

i.e., $- \otimes C$ preserves monomorphisms.

COROLLARY 36. *The underlying functor $U: \text{Comod-}C \rightarrow k\text{-Mod}$ creates factorizations from E_k -factorizations in $k\text{-Mod}$ if and only if C is flat.*

Proposition 35 shows that, if C is not flat, then an arbitrary C -comodule homomorphism can not be factorized through a surjective comodule homomorphism and an injective comodule homomorphism. In particular the canonical (epi-mono)-factorization of a comodule homomorphism in $k\text{-Mod}$ cannot be lifted to a factorization in $\text{Comod-}C$. In the sequel (E, M) shall always denote the proper factorization (epi, extremal mono) on $\text{Comod-}C$. Words as epimorphism, monomorphism, generator, wellpowered \dots are used in a sense relative to (E, M) .

PROPOSITION 37. *$\text{Comod-}C$ is wellpowered relative to the factorization (epi, extremal mono).⁴*

Proof. In the same vein as the proof for Proposition 10.6.3 in [16].

For the rest of this paper we will use the property that the category $k\text{-Mod}$ is a symmetrical monoidal closed category with respect to the tensor product, and that $\text{Comod-}C$ is an enriched category over $k\text{-Mod}$. In the following we will study the left adjoints of the $k\text{-Mod}$ -representable functors called tensors and cotensors. They provide a characterisation of certain constructions which is not available in an ordinary set based approach. Cotensors will play an important role in duality theory (i.e. Gelfand theory)

⁴ $\text{Comod-}C$ is even wellpowered with respect to all monos.

as it will be shown in part II of the present work. We use the language in [6].

$\text{Comod-}C$ is a $k\text{-Mod}$ -category. The internal Hom-functor $[,]: \text{Comod-}C^{\text{op}} \times \text{Comod-}C \rightarrow k\text{-Mod}$ is given by $[M, N] = \text{Comod-}C(M, N)$. The pair of adjoint functors $\text{Comod-}C \rightleftarrows k\text{-Mod}$ is a pair of $k\text{-Mod}$ -functors. In the sequel we call $k\text{-Mod}$ -functors k -linear functors.

PROPOSITION 38. *The category $\text{Comod-}C$ is tensored i.e. for every k -module M and every C -comodule X the functor $\text{Comod-}C \rightarrow k\text{-Mod}: Y \mapsto k\text{-Mod}(M, \text{Comod-}C(X, Y))$ is representable over $k\text{-Mod}$.*

Proof. Let $M \in k\text{-Mod}$ and $X \in \text{Comod-}C$. The $M \otimes X$ is a C -comodule. The rest follows from the canonical k -linear isomorphism

$$\text{Comod-}C(M \otimes X, Y) \cong k\text{-Mod}(M, \text{Comod-}C(X, Y)).$$

COROLLARY 39. *The cofree k -linear functor $- \otimes C: k\text{-Mod} \rightarrow \text{Comod-}C$ has a k -linear right adjoint functor represented by the k -linear functor $\text{Comod-}C(C, -)$.*

PROPOSITION 40. *The category $\text{Comod-}C$ is cotensored i.e. for every $M \in k\text{-Mod}$ and $X \in \text{Comod-}C$ the functor $\text{Comod-}C^{\text{op}} \rightarrow k\text{-Mod}: Y \mapsto k\text{-Mod}(M, \text{Comod}(Y, X))$ is representable.*

Proof. Since $\text{Comod-}C$ is a tensored category $\text{Comod-}C$ is cotensored if and only if for every k -module M the k -linear functor $F_M: M \otimes -: \text{Comod-}C \rightarrow \text{Comod-}C$ has a k -linear right adjoint. Let $N \otimes X$ be a tensor with $N \in k\text{-Mod}$ and $X \in \text{Comod-}C$ as above. Then $F_M(N \otimes X) = M \otimes (N \otimes X) \cong N \otimes (M \otimes X) \cong N \otimes F_M(X)$. Hence F_M is a tensor preserving functor in the sense of [6]. Since F_M preserves colimits, F_M has a right adjoint by the Special Adjoint Functor Theorem. Since F_M preserves tensors the right adjoint $\overline{\text{Comod-}C}(M, -)$ is a k -linear functor and the representation $\text{Comod-}C(X, \overline{\text{Comod-}C}(M, X)) \cong \text{Comod-}C(M \otimes X, Y) \cong k\text{-Mod}(M, \text{Comod}(X, Y))$ is k -linear.

COROLLARY 41. *$\text{Comod-}C$ is $k\text{-Mod}$ -complete and $k\text{-Mod}$ -cocomplete.*

Let $f: C \rightarrow C'$ be a coalgebra morphism. Then f induces a functor $f^*: \text{Comod-}C \rightarrow \text{Comod-}C'$ by the assignment $(M, \chi_M) \mapsto (M, 1 \otimes f\chi_M)$. Then f^* is obviously a k -linear functor. By [15] 21.2.1 the mapping $f \mapsto f^*$ induces a bijection between $\text{Coalg}(C, C')$ and the "set" of all functors $\varphi: \text{Comod-}C \rightarrow \text{Comod-}C'$ with $U_c = U_{c'}\varphi$.

PROPOSITION 42. *Let $f: C \rightarrow C'$ be a coalgebra morphism. Then*

- (1) f^* preserves tensors.
- (2) f^* has a k -linear right adjoint f_* .

Proof. The assertion 1 is trivial. Since f^* preserves colimits it has a right adjoint by the Special Adjoint Functor Theorem. Since f^* preserves tensors the right adjoint is k -linear.

*Description of the functor f_** 43. Let M be a C -right comodule and N a C -left comodule. The tensor coproduct of M and N under C denoted by $M \otimes^C N$ is given by the following equalizer digram in $k\text{-Mod}$.

$$M \otimes^C N \longrightarrow M \otimes N \begin{array}{c} \xrightarrow{\chi_M \otimes M} \\ \xleftarrow{M \otimes \chi_N} \end{array} M \otimes C \otimes N$$

Then if $f: C \rightarrow C'$ is a coalgebra morphism between flat coalgebras C and C' the functor $f_*: \text{Comod-}C' \rightarrow \text{Comod-}C$ is given by the following assignment $f_*(M, \chi_M) = (M \otimes^C C, 1_M \otimes^C \Delta)$.

Final Observation 44. In the same vein as I studied the category of comodules for a fixed coalgebra one can study the category Comod of all comodules i.e. pairs $((M, \chi_M), C)$ where (M, χ_M) is a comodule over C . One obtains similar results. The starting point for the study of this category is the following theorem

THEOREM 45. *The underlying functor*

$$U: \text{Comod} \longrightarrow k\text{-Mod} \times k\text{-Coalg}: ((M, \chi_M), C) \longmapsto (M, C)$$

is comonadic.

This note was written during my visit to the University of California at San Diego. I would like to thank in particular Professor Helmut Röhrl for his hospitality and the stimulating discussions on this paper. Furthermore I am indebted to Professor Bodo Pareigis for stimulating the study of comodules over an arbitrary coalgebra.

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