LUMPY SUBSETS IN LEFT-AMENABLE LOCALLY COMPACT SEMIGROUPS

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This note adapts to locally compact, topologically left-amenable semigroups S the characterization given by Mitchell in discrete semigroups of those subsets T of S substantial enough that at least one topologically left-invariant mean in S is concentrated in T.

Introduction. Mitchell (1965) showed that for a discrete left-amenable semigroup S the following two properties of a subset T of S are equivalent:

(1) There is at least one left-invariant mean α on the space $m(S)$ of bounded functions on S which is concentrated on T; that is, $\alpha(\xi_T) = 1$.

(2) T is left-thick; that is, for each finite subset σ of S there is an s in S such that $\sigma s = \{ts \mid t \in \sigma\}$ is contained in T.

My paper (Day, 1969, Th. 7.2) gave a version of this for locally compact *groups*, which Wong (preceding paper) points out has a misprint; $K \backslash Es$ should have been $K \cap Es$ in condition (2) of that theorem.

In a locally compact *group* G there are (at least) three conditions equivalent to left amenability of G; that is, existence of a left-invariant mean α on the bounded measurable functions on G. The pertinent three are called in Day (1968) conditions (lsau), (lsam), and (lsaπ) and will be defined in the next section. In locally compact semigroups each implies the next, but they are not known to be equivalent except in groups. In the preceding paper Wong chose a condition (W) equivalent to (lsau) and generalized Mitchell's theorem to semigroups satisfying (W). This note picks the simpler middle property (lsam) and proves a Mitchelloid theorem for such semigroups. For the weakest of the properties (lsaπ) some implications are given but no equivalent condition is yet known.

NOTATION. C_0 is the space of continuous functions vanishing at infinity on S, a locally compact semigroup, and M is the space of regular Borel measures on S. [It is well known that M is like the conjugate space of C_0; see Hewitt and Ross (1963).] $P \subset M$ is the set of probability measures and $P_c \subset P$ is the set of probability measures with compact support, so P_c is dense in P. $P^{**} \subset M^{**}$ is the set of means on M^*; as usual, the canonical image $Q(P_c)$ of P_c into M^{**} is w^*-dense in P^{**}.

C_0 and M^* are abstract M-spaces and M and M^{**} are abstract

L-spaces. M is a Banach algebra under the usual convolution operation: For each x in C_0,

$$[\nu * \mu](x) = [\nu \times \mu](x^\vee),$$

where $x^\vee$ is defined on $S \times S$ by: For each (s, t) in $S \times S$,

$$x^\vee(s, t) = x(st).$$

Arens multiplication can then be defined (see, for example, Day (1969)) to make M^{**} also a Banach algebra for which the natural isometric embedding Q of M into M^{**} becomes an algebra isomorphism.

If T is a Borel subset of S, the *characteristic functional* χ_T of T is that element of M^* defined by: $\chi_T(\mu) = \mu(T)$ for each μ in M.   To state that a Γ in P^{**} is *concentrated on* T is to say that $\Gamma(\chi_T) = 1$.

A locally compact semigroup S will be called *left-amenable* if

(laπ)   there exists a Γ in P^{**} such that $Q_{\delta_s} * \Gamma = \Gamma$ for each s in S.

   S will be called *topologically left-amenable* if

(lam)   there exists Γ in P^{**} such that $Q\mu * \Gamma = \Gamma$ for each μ in P_c, or equivalently for each μ in P, or equivalently $\Delta * \Gamma = \Gamma$ for each Δ in P^{**}

For locally compact *groups* these and the three conditions below are equivalent (Day 1968, 1969), but for locally compact semigroups we know only the obvious (lam) implies (laπ).

Among the "strong" left-amenability conditions described for locally compact groups in Day (1968, 1969) are three pertinent to this paper.   We adapt them for locally compact semigroups:

(lsau)   (Uniform (on compact sets) left strong amenability) There exists a net (μ_n) of elements of P_c such that for each compact set K in S, $\|\delta_s * \mu_n - \mu_n\|$ tends to zero uniformly for s in K.

(lsam)   (topological left strong amenability) There exists a net (μ_n) in P_c such that $\|\nu * \mu_n - \mu_{n'}\|$ tends to zero for each ν in P_c.

(lsaπ)   (pointwise left strong amenability) There exists a net (μ_n) in P_c such that $\|\delta_s * \mu_n - \mu_n\|$ tends to zero for each s in S.

Clearly (lsau) is equivalent to the condition of the preceding paper:

(W)   There exists a net (μ_n) in P_c such that for each compact set
        K contained in S, $\| \nu * \mu_n - \mu_n \|$ tends to zero uniformly
        over all ν in P_c which are supported in K.

Standard methods from earlier papers show that (laπ) is equivalent
to (lsaπ) and that (lam) is equivalent to (lsam).   Also (W) implies (lsam)
implies (lsaπ).   In locally compact semigroups no more is known so
Mitchell's theorem has several possible extensions.   Wong's theorem
works for semigroups with the property (W); for such semigroups he
defines a version of left-thickness and proves it equivalent to the property
of concentrating some topologically left-invariant mean.   This paper
assumes only that the semigroup is topologically left amenable and uses a
different internal property of subsets of S.   As a guide observe that
Mitchell's property (2) of left-thickness is equivalent in discrete semi-
groups to: For each ν in P with compact (= finite) support there is s in S
with $[\nu * \delta_s](T) = 1$.   We choose an approximation of this for the first of
the definitions below; the second is a direct analogue of Mitchell's and is
much too strong.   Even in Mitchell's case the word "lumpy" is more
descriptive of the circumstances than is "thick".

DEFINITION.   A Borel set T contained in S is called *topologically
left-lumpy* if for each ν in P_c and each $\epsilon > 0$ there is s in S such that
$[\nu * \delta_s](T) > 1 - \epsilon$. T is *left-lumpy* if for each compact set K in S there is
s in S such that $Ks \subset T$.

THEOREM (Mitchelloid, Mark III).   *For each Borel set T of a
topologically left-amenable, locally compact semigroup S the following
conditions on T are equivalent:*
    (3)   *There is a topologically left-invariant mean Γ in P^{**} such that Γ
is concentrated on T; that is, $\Gamma(\chi_T) = 1$.*
    (4)   *T is topologically left-lumpy.*
    (5)   *For each ν in P_c and each $\epsilon > 0$ there is a μ in P_c such that
$[\nu * \mu](T) > 1 - \epsilon$.*

Proof.   (5) implies (4).   If $\nu \in P_c$ and if $[\nu * \delta_s](T) \leqq 1 - \epsilon$ for all s
in S, then for each μ in P_c, $[\nu * \mu](T) = \mu(x)$, where $x(s) = [\nu * \delta_s](T)$
for all s in S.   But then $[\nu * \mu](T) \leqq 1 - \epsilon$ for all μ in P_c.   This is
contrapositive of (5) implies (4).
    (4)   implies (3).   If $\sigma = (\nu_1, \cdots, \nu_k)$ is a finite subset of P_c and if
$\epsilon > 0$, by condition (4) there is a $s = s_{\epsilon, \sigma}$ in S such that

$$[(\nu_1 + \cdots + \nu_k)/k * \delta_s](T) > 1 - \epsilon/k;$$

so for all $i \leqq k$, $[\nu_i * \delta_s](T) > 1 - \epsilon$.

If $(\epsilon, \sigma) > (\epsilon', \sigma')$ means that $\epsilon < \epsilon'$ and $\sigma \supseteq \sigma'$, then w^*-compactness of P^{**} implies that the net $(Q\delta_{s_{\epsilon,\sigma}})$ has a w^*-convergent subnet $(Q\delta_{s_m})$ with limit Γ in P^{**}. Then for each ν in P_c

$$1 \geq [Q\nu * \Gamma](\chi_T) = [Q\nu](\Gamma * \chi_T) = [\Gamma * \chi_T](\nu) = \Gamma(\chi_T * \nu)$$

$$= \lim_m [Q\delta_{s_m}](\chi_T * \nu) = \lim_m [\chi_T * \nu](\delta_{s_m})$$

$$= \lim_m \chi_T(\nu * \delta_{s_m}) = \lim_m [\nu * \delta_{s_m}](T).$$

But

$$[\nu * \delta_{s_{\epsilon,\sigma}}](T) > 1 - \epsilon \quad \text{if } \nu \text{ is in } \sigma,$$

so that the limit is 1 and $[Q\nu * \Gamma](\chi_T) = 1$ for all ν in P_c. By w^*-density of QP_c in P^{**} and the w^*-continuity of $*$ in its first variable, $[\Delta * \Gamma](\chi_T) = 1$ for all Δ in P^{**}. In particular, Δ can be chosen to be a topologically left-invariant mean. Then $\Phi = \Delta * \Gamma$ is a topologically left-invariant mean such that $\Phi(\chi_T) = 1$; that is, (3) holds if (4) does.

(3) implies (5). If Γ is a topologically left-invariant mean such that $\Gamma(\chi_T) = 1$, let (μ_n) be a net of elements of P_c such that $w^*\text{-}\lim_n Q\mu_n = \Gamma$. Then for each ν in P_c, because an Arens product is $w^*\text{-}w^*$-continuous in its second variable when the first variable is a $Q\nu$,

$$w^*\text{-}\lim_n (Q\nu * Q\mu_n - Q\mu_n) = Q\nu * \Gamma - \Gamma = 0.$$

But $Q\mu_n(\chi_T)$ tends to $\Gamma(\chi_T) = 1$, so

$$[Q_\nu * Q\mu_n](\chi_T) = [Q(\nu * \mu_n)](\chi_T) = \chi_T(\nu * \mu_n) = [\nu * \mu_n](T),$$

so $[\nu * \mu_n](T)$ tends to 1 for each ν in P_c. Hence for each ν and ϵ some μ_n satisfies (5).

REMARKS. (A) Although for locally compact groups it is well known that left-amenability implies strong left-amenability uniformly on compact sets and hence implies topological left-amenability (Day 1969, Th. 13.3), the space used there is not the full space M of regular Borel measures but the smaller spaces M_a of measures absolutely continuous with respect to Haar measure on G. Wong (preceding paper, Th. 5.2) points out that for a locally compact group G there is a topologically left-invariant Γ in P^{**} if and only if there is one in $P^{**} \cap M_a^{**}$.

(B) Wong also shows in Th. 5.2 that for a locally compact group G the Reiter condition

(Isau$_1$) There is a net (φ_n) in $P_c \cap M_a$ such that $\|\delta_s * \varphi_n - \varphi_n\|$ tends to zero uniformly in each compact subset K of S,

is equivalent when $S = G$ to his condition (W). We have already remarked that (W) is equivalent to the condition (lsau) above in every locally compact semigroup.

(C)   In those locally compact semigroups where (W) holds, Wong's Theorem 4.1 and my theorem above both characterize sets T for which some topologically left-invariant mean is concentrated on T; hence in such semigroups his *topologically left-thick sets are the same as* my *topologically left-lumpy sets*.

(D)   In a left-amenable locally compact semigroup no adequate characterization has been found for those sets on which left-invariant means can be concentrated; however something can be done like parts of the earlier theorem.

THEOREM (Mitchelloid, Mark IIIa). *In a left amenable locally compact semigroup S each of the following conditions on a Borel set T implies the next in line.*

(6) T *is left lumpy.*

(5)   (*as in first theorem*).

(4) T *is topologically left lumpy.*

(7)   *There is a left-invariant mean Φ in P^{**} such that $\Phi(\chi_T) = 1$.*

(8)   *There is a net (μ_n) in P_c such that $\lim_n \| \varphi * \mu_n - \mu_n \| = 0$ for each finitely supported mean φ on S_0 and $\lim_n [\varphi * \mu_n](T) = 1$.*

(2)   (*Mitchell's left thick*) *For each finite set $\sigma \subset S$ there is s in S such that $\sigma s \subset T$.*

Proof. Left lumpy is stronger than (W), since it assures that $[\nu * \delta_s](T) = 1$ if ν is supported in K.   (W) implies (5) implies (4).   The proof that (4) implies (3) need only be modified to choose Δ as a left-invariant mean to give a proof of (7).   The proof that (7) implies (8) is like that of (3) implies (5), using finitely supported means instead of compactly supported ones on the left.   For (8) implies (2) let σ be a finite subset of S and let k be the number of elements of σ. Set $\varphi = (\Sigma_{s \in \sigma} \delta_s)/k$. If (μ_n) satisfies (7), then $[\varphi * \mu_n](T)$ tends to 1.   As in the proof of (5) implies (4) there must be an s_n in the support of μ_n such that $[\varphi * \delta_{s_n}](T)$ tends to 1.   But as soon as $[\varphi * \delta_{s_n}](T) > 1 - 1/k$, $[\varphi * \delta_{s_n}](T) = 1$, so $\sigma s_n \subset T$.

We note that (7) and (8) are equivalent, but the proof that (2) implies (7) would require something like left-amenability of S regarded as a discrete semigroup, which is not a common property of left-amenable locally compact semigroups.

Open problems.   (A) Except in groups it is not known whether in a locally compact semigroup left-amenability implies topological left-

amenability or whether topological left-amenability implies (W) which is equivalent to (lsau).

(B)   A characterization is still needed for those subsets of a locally compact left-amenable semigroup on which a left-invariant mean can be concentrated; left-lumpy is much too strong and Mitchell's left-thick is much too weak.   It should be noted, however, that left ideals in S are left-lumpy and therefore support left-invariant means.

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