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ANALYTIC MAPS OF THE OPEN UNIT DISK ONTO A GLEASON PART

KAZUO KISHI

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ANALYTIC MAPS OF THE OPEN UNIT DISK ONTO A GLEASON PART

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The purpose of this paper is to show that in certain uniform algebras all analytic maps (for the definition see $\S 2$) of the open unit disk onto a nontrivial Gleason part are mutually closely related (Theorem 2), and that these maps are isometries of the open unit disk with pseudo-hyperbolic metric onto a nontrivial Gleason part with part metric (Theorem 3).

The results of this paper are contained in §2. Some necessary preliminaries are given in §1.

1. Preliminaries. A uniform algebra A on a compact Hausdorff space X is a uniformly closed subalgebra of the algebra C(X) of complex valued continuous functions on X which contains the constants and separates the points of X. Let $\mathscr{M}(A)$ denote the maximal ideal space of A which has the Gelfand topology. Let \hat{f} be the Gelfand transform of f in A and let $\hat{A} = \{\hat{f}: f \in A\}$.

For φ and θ in $\mathcal{M}(A)$ we define

$$(1.1) \qquad G:G(\varphi,\,\theta)=\sup\left\{|\,\varphi(f)-\theta(f)|\colon f\in A,\,||\,f\,||\leq 1\right\}\,,$$

$$(1.2) \qquad \sigma \colon \sigma(\varphi, \theta) = \sup \left\{ |\varphi(f)| \colon f \in A, \, \|f\| \leq 1, \, \theta(f) = 0 \right\},$$

where $||f|| = \sup \{|f(x)|: x \in X\}$, and we write $\varphi \sim \theta$ when $G(\varphi, \theta) < 2$ (or, equivalently, $\sigma(\varphi, \theta) < 1$). Then \sim is an equivalence relation in $\mathcal{M}(A)$, and an equivalence class $P(m) = \{\varphi: \varphi \in \mathcal{M}(A), \varphi \sim m\}(\supseteq \{m\})$ is called the (nontrivial) Gleason part of $m \in \mathcal{M}(A)$ (cf. Gleason [2]). $G(\varphi, \theta)$ and $\sigma(\varphi, \theta)$ are metrics on P(m) (cf. König [5]).

If $m \in \mathscr{M}(A)$ has a unique representing measure μ_m , i.e., if m has a unique probability measure μ_m on X such that $m(f) = \int f d\mu_m$ for every $f \in A$, then every φ in P(m) also has a unique representing measure μ_{φ} . It is also known that φ in M_A belongs to P(m) if and only if there exist mutually absolutely continuous representing measures μ_{φ} , μ_m for φ , m respectively; indeed, there exists a constant c (0 < c < 1) such that $c\mu_{\varphi} \leq \mu_m$ and $c\mu_m \leq \mu_{\varphi}$.

For example, let A(D) denote the disk algebra of all continuous functions on the closed unit disk $\overline{D} = \{s: |s| \leq 1\}$ in the plane wich are analytic in the open unit disk D. Then $\mathscr{M}(A(D))$ can be identified with \overline{D} , and the open unit disk D is one part. For $t, s \in D$, we see that

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(1.3)
$$G(t, s) = \sup \{ |f(t) - f(s)| : f \in A(D), ||f|| \le 1 \},$$

(1.4)
$$\sigma(t, s) = \sup \{ |f(t)| : f \in A(D), ||f|| \leq 1, f(s) = 0 \}$$
$$= \left| \frac{t-s}{1-\overline{s}t} \right| \quad (\text{pseudo-hyperbolic metric}).$$

Throughout the rest of this paper, we do not distinguish in notations $\varphi \in \mathscr{M}(A)$ from its representing measure when φ has a unique representing measure, and we suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m. Let $A_m = \{f \in A; m(f) = 0\}$, the kernel of a complex homomorphism m. Let $H^{\infty}(m)$, H_m^{∞} be the weakstar closures of A, A_m in $L^{\infty}(m)$ respectively, and for $1 \leq p < \infty$ let $H^p(m)$, H_m^p be the closures of A, A_m in $L^p(m)$ norm respectively. If φ belongs to $P(m)(\supseteq \{m\})$, then for $1 \leq p < \infty$ the spaces $H^p(m)$ and $H^p(\varphi)$ are identical as sets of (equivalence classes of) functions; as Banach spaces, they have distinct but equivalent norms. Under the same hypothesis, the Banach algebras $H^{\infty}(m)$ and $H^{\infty}(\varphi)$ are identical.

For a Dirichlet algebra Wermer [7] showed the following theorem, and Hoffman [3] generalized Wermer's result to a logmodular algebra (cf. Browder [1], Chap. IV). Functions in $H^{\infty}(m)$ of unit modulus are called *inner functions*.

THEOREM 1 (WERMER'S EMBEDDING THEOREM). Let A be a uniform algebra on a compact space X. Suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m on X, and that the part P of m consists of more than one point. Then we have the following.

(1) There is an inner function Z(= Wermer's embedding function) such that $ZH^2(m) = H^2_m$.

(2) If $\varphi \in P$, define $\hat{Z}(\varphi) = \int Z \, d\varphi$. Then \hat{Z} is a one-one map of the part P onto the open unit disk D in the plane. The inverse map τ of \hat{Z} is a one-one continuous map of D onto P (with the Gelfand topology).

(3) For every f in A the composed function $\hat{f} \circ \tau$ is analytic on D.

Let φ be an element of the Gleason part P(m) of m in $\mathscr{M}(A)$. Then there is a function h in $L^{\infty}(m)$ such that $\varphi(f) = \int f d\varphi = \int f h dm$ for all $f \in A$, so φ has a unique extension to a linear functional $\tilde{\varphi}$ on $H^{\infty}(m)$ which is both multiplicative and weak-star continuous. For any $f \in H^{\infty}(m) \tilde{\varphi}$ has the form

$$\widetilde{\varphi}(f) = \int f d\varphi = \int f h dm$$
.

We call $\tilde{\varphi}$ the measure extension of φ in P(m).

PROPOSITION. Let A, m, P(m) and Z be as in Wermer's embedding theorem. Let $\mathscr{P} = \mathscr{P}(m)$ be the set of measure extension $\tilde{\varphi}$ of φ in P(m). Then we have the following.

(1) \mathscr{P} is the nontrivial Gleason part of \widetilde{m} in $\mathscr{M}(H^{\infty}(m))$.

(2) The map $\hat{Z}|\mathscr{S}$ is a one-one continuous map of the part \mathscr{S} (with the Gelfand topology) onto the open unit disk D, and thus the inverse map $\tilde{\tau}$ of $\hat{Z}|\mathscr{S}$ is a homeomorphism of D onto \mathscr{P} .

Proof. If the Gelfand transform $\hat{H}^{\infty}(m) = \hat{H}^{\infty}$ of $H^{\infty}(m)$ is restricted to the maximal ideal space Y of $L^{\infty}(m)$, then \hat{H}^{∞} is a logmodular algebra on Y (see Hoffman [3], Theorem 6.4, corollary), and therefore every complex homomorphism φ of $H^{\infty}(m)$ has a unique representing measure on Y (see [3], Theorem 4.2). In particular, $\tilde{m} \in \mathscr{M}(H^{\infty}(m))$ has a unique representing (normal) measure \tilde{m} on the hyperstonean space Y. Then for every f in $L^{\infty}(m)$ we have

$$\int_x f dm = \int_y \hat{f} d\tilde{m} \; .$$

And we can identify $L^{\infty}(\tilde{m})$ with $C(Y) = \hat{L}^{\infty}(m)$ (cf. Srinivasan-Wang [6], pp. 221-223).

Now let Σ be the Gleason part of \tilde{m} . For $\tilde{\varphi}$ in \mathcal{P} , we have

(1.5)
$$\widetilde{\varphi}(f) = \int_{\mathcal{X}} f d\varphi = \int_{\mathcal{X}} f h dm = \int_{\mathcal{Y}} \widehat{f} \widehat{h} d\widetilde{m} \ (f \in H^{\infty}(m)) ,$$

where h is a function in $L^{\infty}(m)$ with a < h < b for some positive constants a and b. From this we see that $\tilde{\varphi}$ is in Σ .

Conversely if λ is an element of Σ , then λ has a unique representing measure $\widetilde{\lambda}$ on Y, and we have for every $f \in H^{\infty}(m)$

$$\lambda(f) = \int_{Y} \widehat{f} d\widetilde{\lambda} = \int_{Y} \widehat{f} rac{d\widetilde{\lambda}}{d\widetilde{m}} d\widetilde{m} \; .$$

Since $d\tilde{\lambda}/d\tilde{m}$ is an element of $L^{\infty}(\tilde{m})$, there is a function h in $L^{\infty}(m)$ such that $d\tilde{\lambda}/d\tilde{m} = \hat{h}$ a.e. $(d\tilde{m})$. Hence we have

$$\lambda(f) = \int_{Y} \widehat{f} \widehat{h} d\widetilde{m} = \int_{X} f h dm$$

and thus $\lambda \in \mathscr{P}$. So we get $\mathscr{P} = \Sigma$. Then the rest part (2) of the proposition follows from Theorem 1.

2. Results.

DEFINITION. Let P(m) be the nontrivial Gleason part of m in

the maximal ideal space $\mathcal{M}(A)$ of a uniform algebra A. A one-one continuous map $\rho(t)$ of the open unit disk D onto P(m) (with the Gelfand topolgy) is called an *analytic map* if the composition $\hat{f}(\rho(t))$ is analytic on D, for every f in A.

Now we are in a position to prove the following theorem.

THEOREM 2. Let A be a uniform algebra on a compact space X. Suppose that $m \in \mathcal{M}(A)$ has a unique representing measure m on X, and that the part P of m consists of more than one point. Let $\tau(t)$ be an analytic map of the open unit disk D onto P which is obtained in Theorem 1. If $\rho(t)$ is an analytic map of D onto P such that $\rho(\alpha) = m$, then we have

(2.1)
$$\rho(t) = \tau \left(\beta \frac{t-\alpha}{1-\bar{\alpha}t}\right),$$

where β is a constant of modulus 1. Furthermore, $\tau(t)$ is a homeomorphism if and only if $\rho(t)$ is a homeomorphism.

Proof. Let \mathscr{P}, Z , and $\tilde{\tau}$ be as in Theorem 1 and proposition. For any $t \in D$, $\rho(t)$ has a unique representing measure $h_t dm$, where h_t is an element of $L^{\infty}(m)$. Let $\tilde{\rho}(t)$ be the measure extension of $\rho(t)$ i.e., $\tilde{\rho}(t)(f) = \int f h_t dm$ for all $f \in H^{\infty}(m)$. For each $f \in H^{\infty}(m)$ there exists a sequence $\{f_n\}$ in A such that $||f_n|| \leq ||f||$ for all n and $f_n \to f$ a.e. (dm) (Hoffman-Wermer theorem, see [1], Theorem 4.2.5). Then, by Lebesgue's dominant convergence theorem, $\rho(t)(f_n) = \int f_n h_t dm \to \tilde{\rho}(t)(f)$ for every t in D. Since $\rho(t)(f_n)(n = 1, 2, \cdots)$ are analytic in D and $|\rho(t)(f_n)| \leq ||f_n|| \leq ||f||$, we see that, for every f in $H^{\infty}(m)$, $\tilde{\rho}(t)(f)$ is analytic in D (Vitali's theorem). Hence we see that $\tilde{\rho}(t) = \hat{Z}(\tilde{\rho}(t))$, then f(t) is a one-one holomorphic map of D onto D, and $g(\alpha) = 0$. Hence we see

$$g(t)=etarac{t-lpha}{1-ar lpha t}$$
 ,

where β is a constant of modulus 1, and thus we have

$$\widetilde{\tau}\left(etarac{t-lpha}{1-\overline{lpha}t}
ight)=\widetilde{
ho}(t)$$
.

Since $\tilde{\tau}(t)|A = \tau(t)$ and $\tilde{\rho}(t)|A = \rho(t)$ we have

$$\tau\left(\beta\frac{t-\alpha}{1-\overline{\alpha}t}\right)=\rho(t)$$
.

Next we prove that $\tau(t)$ is a homeomorphism if and only if $\rho(t) = \tau(\beta(t-\alpha)/(1-\bar{\alpha}t))$ is a homeomorphism. We put $L_{\alpha}(t) = (t+\alpha)/(1+\bar{\alpha}t)$ and $\beta = e^{i\theta}$. Then $\tau(t)$ is a homeomorphism of D onto P if and only if $\hat{Z}(\varphi) \Big(=\int Zd\varphi\Big)$ is a continuous map of P onto D if and only if $L_{\alpha} \circ e^{-i\theta} \circ \hat{Z}$ is a continuous map of P onto D if and only if

$$(L_{lpha}\circ e^{-\imath heta}\circ \hat{Z})^{-\imath}(t)= au(e^{\imath heta}L_{-lpha}(t))= au\Big(e^{\imath heta}rac{t-lpha}{1-ar{lpha}t}\Big)=
ho(t)$$

is a homeomorphism, and the theorem is proved.

Next we shall prove the following theorem which generalizes a formula (6.12) in Hoffman [4], p. 105.

THEOREM 3. Let A be a uniform algebra on X. Suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m on X, and that the part P of m consists of more than one point. If $\rho(t)$ is an analytic map of the open unit disk D onto P(m), then we have

$$\sigma(
ho(t), \,
ho(s)) = \sigma(t, \, s) \; , \ G(
ho(t), \,
ho(s)) = G(t, \, s) \; .$$

For the definitions of σ , G see (1.1) ~ (1.4).

Proof. Let Z, \mathcal{P}, τ and $\tilde{\tau}$ be as Theorem 1 and proposition. Let $\tilde{\tau}(t) = \tilde{\varphi}, \tilde{\tau}(s) = \tilde{\theta}, \tau(t) = \varphi$ and $\tau(s) = \theta$. From Lemma 4.4.4 in Browder [1], we see that

$$f \in H^{\infty}_{\widetilde{ heta}} = \left\{ f \colon f \in H^{\infty}(m) = H^{\infty}(heta), \ \widetilde{ heta}(f) = \int f d heta = 0
ight\}$$

if and only if $f \in (Z - s)H^{\infty}(m)$, and from this we easily get $H^{\infty}_{\tilde{\theta}} = \{(Z - s)/(1 - \bar{s}Z)\}H^{\infty}(m)$. So we have

$$egin{aligned} &\sigma(\widetilde{arphi},\,\widetilde{ heta}) = \sup\left\{ |\widetilde{arphi}(f)|;\,f\in H^{\infty}(m),\,||\,f\,||\leq 1,\,\widetilde{ heta}(f)=0
ight\} \ &=\sup\left\{ |\widetilde{arphi}(f)|;\,f\in rac{Z-s}{1-\overline{s}Z}H^{\infty}(m),\,||\,f\,||\leq 1
ight\} \ &=\sup\left\{ \left|\widetilde{arphi}\Big(rac{Z-s}{1-\overline{s}Z}\Big)\widetilde{arphi}(g)\Big|\colon g\in H^{\infty}(m),\,||\,g\,||\leq 1
ight\} \ &=\left|rac{t-s}{1-\overline{s}t}
ight|=\sigma(t,\,s)\;. \end{aligned}$$

Since the closures of A_{θ} and $H^{\infty}_{\tilde{\theta}}$ in $L^2(m)$ are the same $H^2_{\theta} = \left\{f: f \in H^2(m) = H^2(\theta), \int f d\theta = 0\right\}$, we have the following equalities from the result which is stated as "the perhaps surprising equality" in

Browder [1], p. 134. (Note that
$$\widetilde{\varphi}(f) = \int_{Y} \widehat{f} h d\widetilde{m} = \int_{Y} \widehat{f} d\widetilde{\varphi}$$
 (see (1.5))
and $\int_{X} |f|^{2} d\varphi = \int_{Y} |\widehat{f}|^{2} d\widetilde{\varphi}$, for any $f \in H_{\theta}^{\infty}$.)
 $\sigma(\varphi, \theta) = \sup \{ |\varphi(f)| : f \in A_{\theta}, ||f|| \leq 1 \}$
 $= \sup \{ |\varphi(f)| : f \in A_{\theta}, \int |f|^{2} d\varphi \leq 1 \}$
 $= \sup \{ |\varphi(f)| : f \in H_{\theta}^{2}, \int |f|^{2} d\varphi \leq 1 \}$
 $= \sup \{ |\widetilde{\varphi}(f)| : f \in H_{\theta}^{\infty}, \int |f|^{2} d\varphi \leq 1 \}$
 $= \sup \{ |\widetilde{\varphi}(f)| : f \in H_{\theta}^{\infty}, \int |f|^{2} d\varphi \leq 1 \}$
 $= \sup \{ |\widetilde{\varphi}(f)| : f \in H_{\theta}^{\infty}, \||f|| \leq 1 \}$
 $= \sigma(\widetilde{\varphi}, \widetilde{\theta})$.

Hence we have

$$\sigma(\tau(t), \tau(s)) = \sigma(\tilde{\tau}(t), \tilde{\tau}(s)) = \sigma(t, s)$$
.

If $\rho(t)$ is an analytic map of *D* onto P(m), then by Theorem 2 we have $\rho(t) = \tau(\beta(t-\alpha)/(1-\overline{\alpha}t))$, where β is a constant of modulus 1. Therefore we have

$$\sigma(\rho(t), \rho(s)) = \sigma\left(\beta \frac{t-\alpha}{1-\overline{\alpha}t}, \beta \frac{s-\alpha}{1-\overline{\alpha}s}\right) = \sigma(t, s) .$$

The following equality is proved by König [5], which holds for φ , θ in the maximal ideal space $\mathcal{M}(A)$ of any uniform algebra A.

$$2\lograc{2+G(arphi, heta)}{2-G(arphi, heta)} = \lograc{1+\sigma(arphi, heta)}{1-\sigma(arphi, heta)}$$

Using this we get

 $G(\rho(t), \rho(s)) = G(t, s)$.

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