ANALYTIC MAPS OF THE OPEN UNIT DISK ONTO A GLEASON PART

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The purpose of this paper is to show that in certain uniform algebras all analytic maps (for the definition see §2) of the open unit disk onto a nontrivial Gleason part are mutually closely related (Theorem 2), and that these maps are isometries of the open unit disk with pseudo-hyperbolic metric onto a nontrivial Gleason part with part metric (Theorem 3).

The results of this paper are contained in §2. Some necessary preliminaries are given in §1.

1. Preliminaries. A *uniform algebra* A on a compact Hausdorff space X is a uniformly closed subalgebra of the algebra $C(X)$ of complex valued continuous functions on X which contains the constants and separates the points of X. Let $\mathcal{M}(A)$ denote the maximal ideal space of A which has the Gelfand topology. Let $\hat{f}$ be the Gelfand transform of f in A and let $\hat{A} = \{\hat{f} : f \in A\}$.

For φ and θ in $\mathcal{M}(A)$ we define

(1.1) $\qquad G \colon G(\varphi, \theta) = \sup\{|\varphi(f) - \theta(f)| \colon f \in A, \ \|f\| \leqq 1\}$,

(1.2) $\qquad \sigma \colon \sigma(\varphi, \theta) = \sup\{|\varphi(f)| \colon f \in A, \ \|f\| \leqq 1, \ \theta(f) = 0\}$,

where $\|f\| = \sup\{|f(x)| \colon x \in X\}$, and we write $\varphi \sim \theta$ when $G(\varphi, \theta) < 2$ (or, equivalently, $\sigma(\varphi, \theta) < 1$). Then $\sim$ is an equivalence relation in $\mathcal{M}(A)$, and an equivalence class $P(m) = \{\varphi \colon \varphi \in \mathcal{M}(A), \ \varphi \sim m\}(\gneqq \{m\})$ is called the *(nontrivial) Gleason part* of $m \in \mathcal{M}(A)$ (cf. Gleason [2]). $G(\varphi, \theta)$ and $\sigma(\varphi, \theta)$ are metrics on $P(m)$ (cf. König [5]).

If $m \in \mathcal{M}(A)$ has a unique representing measure μ_m, i.e., if m has a unique probability measure μ_m on X such that $m(f) = \displaystyle\int f \, d\mu_m$ for every $f \in A$, then every φ in $P(m)$ also has a unique representing measure μ_φ. It is also known that φ in M_A belongs to $P(m)$ if and only if there exist mutually absolutely continuous representing measures μ_φ, μ_m for φ, m respectively; indeed, there exists a constant c $(0 < c < 1)$ such that $c\mu_\varphi \leqq \mu_m$ and $c\mu_m \leqq \mu_\varphi$.

For example, let $A(D)$ denote the disk algebra of all continuous functions on the closed unit disk $\bar{D} = \{s \colon |s| \leqq 1\}$ in the plane wich are analytic in the open unit disk D. Then $\mathcal{M}(A(D))$ can be identified with $\bar{D}$, and the open unit disk D is one part. For $t, s \in D$, we see that

(1.3) $G(t, s) = \sup \{|f(t) - f(s)|: f \in A(D), \|f\| \leqq 1\}$,

(1.4) $\sigma(t, s) = \sup \{|f(t)|: f \in A(D), \|f\| \leqq 1, f(s) = 0\}$

$$= \left| \frac{t - s}{1 - \bar{s}t} \right| \quad \text{(pseudo-hyperbolic metric)} .$$

Throughout the rest of this paper, we do not distinguish in notations $\varphi \in \mathscr{M}(A)$ from its representing measure when φ has a unique representing measure, and we suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m. Let $A_m = \{f \in A; m(f) = 0\}$, the kernel of a complex homomorphism m. Let $H^\infty(m), H^\infty_m$ be the weak-star closures of A, A_m in $L^\infty(m)$ respectively, and for $1 \leqq p < \infty$ let $H^p(m), H^p_m$ be the closures of A, A_m in $L^p(m)$ norm respectively. If φ belongs to $P(m)(\supsetneqq \{m\})$, then for $1 \leqq p < \infty$ the spaces $H^p(m)$ and $H^p(\varphi)$ are identical as sets of (equivalence classes of) functions; as Banach spaces, they have distinct but equivalent norms. Under the same hypothesis, the Banach algebras $H^\infty(m)$ and $H^\infty(\varphi)$ are identical.

For a Dirichlet algebra Wermer [7] showed the following theorem, and Hoffman [3] generalized Wermer's result to a logmodular algebra (cf. Browder [1], Chap. IV). Functions in $H^\infty(m)$ of unit modulus are called *inner functions*.

THEOREM 1 (WERMER'S EMBEDDING THEOREM). *Let A be a uniform algebra on a compact space X. Suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m on X, and that the part P of m consists of more than one point. Then we have the following.*

(1) *There is an inner function $Z(=$ Wermer's embedding function) such that $ZH^2(m) = H^2_m$.*

(2) *If $\varphi \in P$, define $\hat{Z}(\varphi) = \int Z \, d\varphi$. Then $\hat{Z}$ is a one-one map of the part P onto the open unit disk D in the plane. The inverse map τ of $\hat{Z}$ is a one-one continuous map of D onto P (with the Gelfand topology).*

(3) *For every f in A the composed function $\hat{f} \circ \tau$ is analytic on D.*

Let φ be an element of the Gleason part $P(m)$ of m in $\mathscr{M}(A)$. Then there is a function h in $L^\infty(m)$ such that $\varphi(f) = \int f \, d\varphi = \int f h \, dm$ for all $f \in A$, so φ has a unique extension to a linear functional $\tilde{\varphi}$ on $H^\infty(m)$ which is both multiplicative and weak-star continuous. For any $f \in H^\infty(m)$ $\tilde{\varphi}$ has the form

$$\tilde{\varphi}(f) = \int f \, d\varphi = \int f h \, dm .$$

We call $\widetilde{\varphi}$ the *measure extension* of φ in $P(m)$.

PROPOSITION. *Let A, m, $P(m)$ and Z be as in Wermer's embedding theorem. Let $\mathscr{P} = \mathscr{P}(m)$ be the set of measure extension $\widetilde{\varphi}$ of φ in $P(m)$. Then we have the following.*

(1) *$\mathscr{P}$ is the nontrivial Gleason part of $\widetilde{m}$ in $\mathscr{M}(H^\infty(m))$.*

(2) *The map $\hat{Z}|\mathscr{P}$ is a one-one continuous map of the part $\mathscr{P}$ (with the Gelfand topology) onto the open unit disk D, and thus the inverse map $\widetilde{\tau}$ of $\hat{Z}|\mathscr{P}$ is a homeomorphism of D onto $\mathscr{P}$.*

Proof. If the Gelfand transform $\hat{H}^\infty(m) = \hat{H}^\infty$ of $H^\infty(m)$ is restricted to the maximal ideal space Y of $L^\infty(m)$, then $\hat{H}^\infty$ is a logmodular algebra on Y (see Hoffman [3], Theorem 6.4, corollary), and therefore every complex homomorphism φ of $H^\infty(m)$ has a unique representing measure on Y (see [3], Theorem 4.2). In particular, $\widetilde{m} \in \mathscr{M}(H^\infty(m))$ has a unique representing (normal) measure $\widetilde{m}$ on the hyperstonean space Y. Then for every f in $L^\infty(m)$ we have

$$\int_X f \, dm = \int_Y \hat{f} \, d\widetilde{m} \ .$$

And we can identify $L^\infty(\widetilde{m})$ with $C(Y) = \hat{L}^\infty(m)$ (cf. Srinivasan-Wang [6], pp. 221-223).

Now let Σ be the Gleason part of $\widetilde{m}$. For $\widetilde{\varphi}$ in $\mathscr{P}$, we have

$$(1.5) \qquad \widetilde{\varphi}(f) = \int_X f \, d\varphi = \int_X f h \, dm = \int_Y \hat{f} \hat{h} \, d\widetilde{m} \quad (f \in H^\infty(m)) \ ,$$

where h is a function in $L^\infty(m)$ with $a < h < b$ for some positive constants a and b. From this we see that $\widetilde{\varphi}$ is in Σ.

Conversely if λ is an element of Σ, then λ has a unique representing measure $\widetilde{\lambda}$ on Y, and we have for every $f \in H^\infty(m)$

$$\lambda(f) = \int_Y \hat{f} \, d\widetilde{\lambda} = \int_Y \hat{f} \frac{d\widetilde{\lambda}}{d\widetilde{m}} \, d\widetilde{m} \ .$$

Since $d\widetilde{\lambda}/d\widetilde{m}$ is an element of $L^\infty(\widetilde{m})$, there is a function h in $L^\infty(m)$ such that $d\widetilde{\lambda}/d\widetilde{m} = \hat{h}$ a.e. $(d\widetilde{m})$. Hence we have

$$\lambda(f) = \int_Y \hat{f} \hat{h} \, d\widetilde{m} = \int_X f h \, dm$$

and thus $\lambda \in \mathscr{P}$. So we get $\mathscr{P} = \Sigma$. Then the rest part (2) of the proposition follows from Theorem 1.

2. Results.

DEFINITION. Let $P(m)$ be the nontrivial Gleason part of m in

the maximal ideal space $\mathscr{M}(A)$ of a uniform algebra A. A one-one continuous map $\rho(t)$ of the open unit disk D onto $P(m)$ (with the Gelfand topolgy) is called an *analytic map* if the composition $\hat{f}(\rho(t))$ is analytic on D, for every f in A.

Now we are in a position to prove the following theorem.

THEOREM 2. *Let A be a uniform algebra on a compact space X. Suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m on X, and that the part P of m consists of more than one point. Let $\tau(t)$ be an analytic map of the open unit disk D onto P which is obtained in Theorem 1. If $\rho(t)$ is an analytic map of D onto P such that $\rho(\alpha) = m$, then we have*

$$(2.1) \qquad \rho(t) = \tau\Big(\beta \frac{t-\alpha}{1-\bar{\alpha}t}\Big),$$

where β is a constant of modulus 1. Furthermore, $\tau(t)$ is a homeomorphism if and only if $\rho(t)$ is a homeomorphism.

Proof. Let $\mathscr{P}$, Z, and $\tilde{\tau}$ be as in Theorem 1 and proposition. For any $t \in D$, $\rho(t)$ has a unique representing measure $h_t dm$, where h_t is an element of $L^{\infty}(m)$. Let $\tilde{\rho}(t)$ be the measure extension of $\rho(t)$ i.e., $\tilde{\rho}(t)(f) = \int f h_t dm$ for all $f \in H^{\infty}(m)$. For each $f \in H^{\infty}(m)$ there exists a sequence $\{f_n\}$ in A such that $\|f_n\| \leqq \|f\|$ for all n and $f_n \to f$ a.e. (dm) (Hoffman-Wermer theorem, see [1], Theorem 4.2.5). Then, by Lebesgue's dominant convergence theorem, $\rho(t)(f_n) = \int f_n h_t dm \to \tilde{\rho}(t)(f)$ for every t in D. Since $\rho(t)(f_n)(n = 1, 2, \cdots)$ are analytic in D and $|\rho(t)(f_n)| \leqq \|f_n\| \leqq \|f\|$, we see that, for every f in $H^{\infty}(m)$, $\tilde{\rho}(t)(f)$ is analytic in D (Vitali's theorem). Hence we see that $\tilde{\rho}(t)$ is an analytic map of D onto $\mathscr{P}$. If we set $g(t) = (\tilde{\tau}^{-1} \circ \tilde{\rho})(t) = \hat{Z}(\tilde{\rho}(t))$, then $f(t)$ is a one-one holomorphic map of D onto D, and $g(\alpha) = 0$. Hence we see

$$g(t) = \beta \frac{t-\alpha}{1-\bar{\alpha}t},$$

where β is a constant of modulus 1, and thus we have

$$\tilde{\tau}\Big(\beta \frac{t-\alpha}{1-\bar{\alpha}t}\Big) = \tilde{\rho}(t).$$

Since $\tilde{\tau}(t)|A = \tau(t)$ and $\tilde{\rho}(t)|A = \rho(t)$ we have

$$\tau\Big(\beta \frac{t-\alpha}{1-\bar{\alpha}t}\Big) = \rho(t).$$

Next we prove that $\tau(t)$ is a homeomorphism if and only if $\rho(t) = \tau(\beta(t-\alpha)/(1-\bar{\alpha}t))$ is a homeomorphism. We put $L_\alpha(t) = (t+\alpha)/(1+\bar{\alpha}t)$ and $\beta = e^{i\theta}$. Then $\tau(t)$ is a homeomorphism of D onto P if and only if $\hat{Z}(\varphi)\left(=\int Zd\varphi\right)$ is a continuous map of P onto D if and only if $L_\alpha \circ e^{-i\theta} \circ \hat{Z}$ is a continuous map of P onto D if and only if

$$(L_\alpha \circ e^{-i\theta} \circ \hat{Z})^{-1}(t) = \tau(e^{i\theta}L_{-\alpha}(t)) = \tau\left(e^{i\theta}\frac{t-\alpha}{1-\bar{\alpha}t}\right) = \rho(t)$$

is a homeomorphism, and the theorem is proved.

Next we shall prove the following theorem which generalizes a formula (6.12) in Hoffman [4], p. 105.

THEOREM 3. *Let A be a uniform algebra on X. Suppose that $m \in \mathcal{M}(A)$ has a unique representing measure m on X, and that the part P of m consists of more than one point. If $\rho(t)$ is an analytic map of the open unit disk D onto $P(m)$, then we have*

$$\sigma(\rho(t), \rho(s)) = \sigma(t, s) ,$$
$$G(\rho(t), \rho(s)) = G(t, s) .$$

For the definitions of σ, G see (1.1) $\sim$ (1.4).

Proof. Let $Z, \mathscr{P}, \tau$ and $\tilde{\tau}$ be as Theorem 1 and proposition. Let $\tilde{\tau}(t) = \tilde{\varphi}, \tilde{\tau}(s) = \tilde{\theta}, \tau(t) = \varphi$ and $\tau(s) = \theta$. From Lemma 4.4.4 in Browder [1], we see that

$$f \in H_{\tilde{\theta}}^\infty = \left\{f : f \in H^\infty(m) = H^\infty(\theta), \tilde{\theta}(f) = \int f d\theta = 0\right\}$$

if and only if $f \in (Z-s)H^\infty(m)$, and from this we easily get $H_{\tilde{\theta}}^\infty = \{(Z-s)/(1-\bar{s}Z)\}H^\infty(m)$. So we have

$$\sigma(\tilde{\varphi}, \tilde{\theta}) = \sup\{|\tilde{\varphi}(f)| ; f \in H^\infty(m), \|f\| \leqq 1, \tilde{\theta}(f) = 0\}$$

$$= \sup\left\{|\tilde{\varphi}(f)| : f \in \frac{Z-s}{1-\bar{s}Z}H^\infty(m), \|f\| \leqq 1\right\}$$

$$= \sup\left\{\left|\tilde{\varphi}\left(\frac{Z-s}{1-\bar{s}Z}\right)\tilde{\varphi}(g)\right| : g \in H^\infty(m), \|g\| \leqq 1\right\}$$

$$= \left|\frac{t-s}{1-\bar{s}t}\right| = \sigma(t, s) .$$

Since the closures of A_θ and $H_{\tilde{\theta}}^\infty$ in $L^2(m)$ are the same $H_\theta^2 = \{f : f \in H^2(m) = H^2(\theta), \int f d\theta = 0\}$, we have the following equalities from the result which is stated as "the perhaps surprising equality" in

Browder [1], p. 134. $\left(\text{Note that } \widetilde{\varphi}(f) = \int_Y \widehat{f}\widehat{h}\,d\widetilde{m} = \int_Y \widehat{f}\,d\widetilde{\varphi} \text{ (see (1.5))}\right.$
and $\left. \int_X |f|^2 d\varphi = \int_Y |\widehat{f}|^2 d\widetilde{\varphi}, \text{ for any } f \in H_\theta^\infty.\right)$

$$\sigma(\varphi, \theta) = \sup\{|\varphi(f)|: f \in A_\theta, \|f\| \le 1\}$$
$$= \sup\left\{|\varphi(f)|: f \in A_\theta, \int |f|^2 d\varphi \le 1\right\}$$
$$= \sup\left\{|\varphi(f)|: f \in H_\theta^2, \int |f|^2 d\varphi \le 1\right\}$$
$$= \sup\left\{|\widetilde{\varphi}(f)|: f \in H_{\widetilde{\theta}}^\infty, \int |f|^2 d\varphi \le 1\right\}$$
$$= \sup\{|\widetilde{\varphi}(f)|: f \in H_{\widetilde{\theta}}^\infty, \|f\| \le 1\}$$
$$= \sigma(\widetilde{\varphi}, \widetilde{\theta}).$$

Hence we have

$$\sigma(\tau(t), \tau(s)) = \sigma(\widetilde{\tau}(t), \widetilde{\tau}(s)) = \sigma(t, s).$$

If $\rho(t)$ is an analytic map of D onto $P(m)$, then by Theorem 2 we have $\rho(t) = \tau(\beta(t - \alpha)/(1 - \bar{\alpha}t))$, where β is a constant of modulus 1. Therefore we have

$$\sigma(\rho(t), \rho(s)) = \sigma\left(\beta\frac{t - \alpha}{1 - \bar{\alpha}t}, \beta\frac{s - \alpha}{1 - \bar{\alpha}s}\right) = \sigma(t, s).$$

The following equality is proved by König [5], which holds for φ, θ in the maximal ideal space $\mathscr{M}(A)$ of any uniform algebra A.

$$2\log\frac{2 + G(\varphi, \theta)}{2 - G(\varphi, \theta)} = \log\frac{1 + \sigma(\varphi, \theta)}{1 - \sigma(\varphi, \theta)}.$$

Using this we get

$$G(\rho(t), \rho(s)) = G(t, s).$$

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