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**CHARACTERIZING FINSLER SPACES WHICH ARE  
PSEUDO-RIEMANNIAN OF CONSTANT CURVATURE**

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# CHARACTERIZING FINSLER SPACES WHICH ARE PSEUDO-RIEMANNIAN OF CONSTANT CURVATURE

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Let  $M$  be an indefinite Finsler space. The bisector of two points of  $M$  is the set of points equidistant from these two points. A bisector is called flat if with any pair of points it contains the extremals joining this pair. In this paper it is shown that  $M$  is pseudo-Riemannian of constant curvature if and only if  $M$  locally has flat bisectors. Another result is that  $M$  is pseudo-Riemannian of constant curvature if and only if  $M$  can be reflected locally in each nonnull extremal.

**1. Introduction.** Blaschke [6] has shown that if  $M$  is a two dimensional definite Finsler space in which the bisector of two points is an extremal then  $M$  is a Riemannian space of constant curvature. Busemann [7] has shown that among his  $G$ -spaces the requirement that bisectors contain with each pair of points a segment joining this pair characterizes the Euclidean, hyperbolic and spherical spaces of dimension greater than one. Phadke [8] has investigated the flat bisector condition in two dimensional  $G$ -spaces which have a distance which is not necessarily symmetric. In [4] we have shown that a pseudo-Riemannian manifold locally has flat bisectors if and only if it is a space of constant sectional curvature.

In the present paper an ordinary or definite Finsler space with a symmetric distance is considered to be a special case of an indefinite Finsler space. Consequently, our arguments are valid for definite metrics as well as nondefinite metrics. The arguments are different from those of Busemann [7] because he does not make any differentiability assumptions and since a number of his arguments do not extend to indefinite metrics.

**2. Indefinite Finsler spaces.** Let  $M$  be an  $n$  dimensional connected and paracompact differentiable manifold of class  $C^\infty$ . The local coordinates of a point  $x$  will be denoted  $x^1, \dots, x^n$ . In the tangent space  $T(x)$  to  $M$  at  $x$  take the natural basis and let  $y^1, \dots, y^n$  denote the components of a vector  $Y \in T(x)$ . The coordinates of  $Y$  are  $(x, y)$ . Let  $L(x, y)$  be a continuous function defined on the tangent bundle  $T(M)$  of  $M$  which has the following properties:

- (A) The function  $L(x, y)$  is  $C^\infty$  for all  $(x, y)$  with  $y \neq 0$ .
- (B)  $L(x, ky) = k^2 L(x, y)$  for all  $k > 0$ .

(C) The metric tensor  $g_{ij}(x, y) = \frac{1}{2} \partial^2 L / \partial y^i \partial y^j$  has  $s$  negative eigenvalues and  $n - s$  positive eigenvalues for all  $(x, y)$  with  $y \neq 0$ .

(D)  $L(x, -y) = L(x, y)$ .

The function  $L(x, y)$  is called the basic metric function. It corresponds to the square of the fundamental function  $F(x, y)$  usually studied in definite Finsler spaces (compare [10]).

The manifold  $M$  together with the basic metric function  $L(x, y)$  is called an indefinite Finsler space of signature  $n - 2s$ . If  $L(x, y)$  is replaced with  $-L(x, y)$ , then  $M$  becomes a space of signature  $2s - n$ . In the special case  $s = 0$  the manifold  $M$  is a definite Finsler space. In this paper we do not exclude the case  $s = 0$ .

When  $M$  has a metric tensor  $g_{ij}(x, y)$  which does not depend on  $y$ , then  $M$  is called pseudo-Riemannian. A pseudo-Riemannian space is Riemannian when  $s = 0$  or  $n$ . If  $M$  is  $R^n$  and the metric tensor is constant, then  $M$  is called pseudo-Euclidean.

Let  $W, Y, Z$  be three tangent vectors at  $x \in M$ . Using the natural basis let  $(x, w), (x, y)$  and  $(x, z)$  be the respective coordinate representations of these vectors. The scalar product of  $Y$  and  $Z$  with respect to  $W$  is defined by

$$W(Y, Z) = g_{ij}(x, w) y^i z^j.$$

If  $Y$  is a nonzero vector, then we say  $Y$  is perpendicular to  $Z$  when  $W(Y, Z) = 0$ . When  $Y$  is perpendicular to  $Z$  we write  $Y \perp Z$ . This relation is not, in general, symmetric. When  $M$  has dimension at least three we have shown [5] that perpendicularity is symmetric on  $M$  if and only if  $M$  is pseudo-Riemannian.

The norm squared of a vector  $Y$  is defined by  $|Y|^2 = W(Y, Y)$ . The quantity  $|Y|^2$  may be positive, negative or zero. A vector  $Y$  with  $|Y|^2 = \pm 1$  is called a unit vector. If  $|Y|^2 = 0$ , then  $Y$  is called a null vector. A vector is nonzero as long as it is not the origin of the tangent space at which it is attached.

The indicatrix  $K(x)$  consists of all of the unit vectors in  $T(x)$ . The light cone  $C(x)$  consists of the null vectors in  $T(x)$ .

If  $Y \in K(x)$ , then  $Y \perp Z$  if and only if  $Z$  is parallel to the tangent hyperplane to  $K(x)$  at  $Y$ , compare [10, p. 26].

**3. The bisector condition.** The Christoffel symbols  $\gamma'_{ik}(x, y)$  are defined in the usual way. The extremals are the solutions of the differential equations

$$\ddot{x}^j + \gamma'_{ik}(x, \dot{x}) \dot{x}^i \dot{x}^k = 0.$$

An extremal  $x(t)$  with velocity vector of length zero is called a null extremal.

A result of Whitehead [9] implies that for each point  $x$  there is a simple convex neighborhood  $U(x)$ . Given two points  $p$  and  $q$  in  $U(x)$  there is a unique extremal arc  $\alpha(p, q)$  from  $p$  to  $q$  which lies in  $U(x)$ . In  $U(x)$  the separation between two points  $p$  and  $q$  is defined by

$$d(p, q) = \int L^{1/2}(x, \dot{x}) dt.$$

The integral is taken along  $\alpha(p, q)$ . The quantity  $L^{1/2}(x, y)$  is either real and nonnegative or pure imaginary. Hence,  $d(p, q)$  is either nonnegative or imaginary. The function  $d$  is continuous on the domain  $U(x) \times U(x)$ . In indefinite metric spaces the local distance function  $d(p, q)$  is usually only defined for points sufficiently close together.

The bisector of  $p$  and  $q$  with respect to  $U(x)$  is defined by

$$B(p, q) = \{p' \in U(x) \mid d(p, p') = d(q, p')\}.$$

We say locally  $M$  has flat bisectors if for each  $x \in M$  there is a simple convex neighborhood  $U(x)$  such that for all  $p, q \in U(x)$  with  $d(p, q) \neq 0$  the bisector  $B(p, q)$  contains with any pair of points the extremals in  $U(x)$  containing this pair.

**4. The two dimensional case.** In this section and the next we always assume  $M$  satisfies the bisector condition. If  $n = 2$ , then this is the assumption that  $B(p, q)$  lies on an extremal of  $M$ .

**PROPOSITION 1.** *Let  $M$  be a two dimensional indefinite Finsler space which locally has flat bisectors. Then  $M$  is a pseudo-Riemannian space of constant curvature.*

*Proof.* If  $M$  has signature two or minus two, then the metric is definite and the proposition follows from the result of Blaschke [6] which was mentioned in the introduction.

Let  $M$  have signature zero. The metric tensor must have one negative eigenvalue and one positive eigenvalue for all  $(x, y)$  with  $y \neq 0$ . For each fixed  $x \in M$ , the light cone  $C(X)$  consists of a finite number  $m$  of lines passing through the origin of the tangent space  $T(x)$ . When  $M$  is pseudo-Riemannian, the light cone consists of two lines. When  $M$  is an indefinite Finsler space, the number of lines  $m$  may be larger than two, see [2].

Let  $m > 2$  and let  $U(x)$  be a simple convex neighborhood of  $x$  such that  $B(p, q)$  is flat whenever  $p, q \in U(x)$  with  $d(p, q) \neq 0$ . Each  $p \in U(x)$  has at least three distinct null directions and there are three null extremals through  $p$  corresponding to these directions. At  $x$ , choose

three null vectors  $Y_1, Y_2$  and  $Y_3$  such that any pair  $Y_i, Y_j$  for  $i \neq j$  is a linearly independent set. Since the null directions through a point vary continuously with the point, each null vector  $Y_i$  attached at  $x$  may be extended to a continuous and nonvanishing null vector field  $Y_i$  defined on a neighborhood  $W(x)$  with  $W(x) \subset U(x)$ . For each  $p \in W(x)$ , let  $\alpha_i(p)$  where  $i = 1, 2, 3$  be a null extremal through  $p$  with tangent vector  $Y_i$  at  $p$ . Assume without loss of generality that  $W(x)$  and the extremals  $\alpha_i(p)$  have been chosen such that each extremal has its endpoints outside of  $W(x)$ . Choose  $q = x$ . For all  $p$  sufficiently close to  $q$  we have  $\alpha_i(p) \cap \alpha_j(q) \neq \emptyset$  when  $i \neq j$ , since the tangent to  $\alpha_i(p)$  converges to  $Y_i$  at  $q$  as  $p \rightarrow q$  and the tangent to  $\alpha_j(q)$  is  $Y_j$  at  $q$ . Choose a fixed  $p$  with  $\alpha_i(p) \cap \alpha_j(q) \neq \emptyset$  for  $i \neq j$  and with  $d(p, q) \neq 0$ . Let  $p_1 = \alpha_1(p) \cap \alpha_3(q)$  and  $p_2 = \alpha_2(p) \cap \alpha_3(q)$ . Since  $d(p, p_i) = d(q, p_i) = 0$ , it follows that  $p_i \in B(p, q)$  for  $i = 1, 2$ . The flat bisector condition implies  $d(p, r) = d(q, r) = 0$  for all  $r \in \alpha(p_1, p_2)$ , since  $\alpha(p_1, p_2)$  lies on the null extremal  $\alpha_3(q)$ . For each point  $r \in \alpha(p_1, p_2)$ , there is a null extremal  $\alpha(p, r)$  which determines a null direction at  $p$ . Since  $p \notin \alpha_3(q)$ , distinct points of  $\alpha(p_1, p_2)$  must determine distinct directions at  $p$ . This contradicts the fact that  $p$  has only a finite number of null directions.

Assume that  $m = 2$ . A two dimensional indefinite Finsler manifold for which  $C(x)$  always consists of two lines has been shown to be a doubly timelike surface, see [2, p. 1038]. Doubly timelike surfaces have been studied by the author in [1]. In particular, the doubly timelike surfaces which locally satisfy the flat bisector condition have been completely characterized by Theorems (IV. 36) and (VI. 17) of [1]. These two Theorems together with the differentiability of  $L(x, y)$  imply that  $M$  is a pseudo-Riemannian manifold of constant curvature.

**5. The bisector theorem.** Let  $M$  have dimension at least three and satisfy the bisector condition. If  $p, q \in U(x)$  with  $d(p, q) \neq 0$ , let  $r$  be the midpoint of  $\alpha(p, q)$  so that  $d(p, r) = d(q, r)$ . The bisector  $B(p, q)$  is a submanifold through  $r$  of codimension one. This implies that  $B(p, q)$  has an  $n - 1$  dimensional tangent space  $T_r(B(p, q))$  at  $r$ . The space  $T_r(B(p, q))$  is naturally identified with an  $n - 1$  dimensional linear subspace of the tangent space  $T(r)$ .

**LEMMA 2.** *If  $r$  is the midpoint of the nonnull extremal  $\alpha(p, q)$ , then  $\alpha(p, q)$  is a perpendicular to  $B(p, q)$  at  $r$ .*

*Proof.* Let  $W$  be the unit tangent to  $\alpha(p, q)$  at  $r$  and let  $Y$  be a nonzero vector at  $r$  in the hyperplane  $T_r(B(p, q))$ . Let  $a(s)$  be the solution of the extremal equations such that  $a'(0) = Y$ . For each  $s$  (sufficiently small), let  $x(t, s)$  represent the extremal  $\alpha(p, a(s))$  for

$0 \leq t \leq 1$ . Let  $\dot{x}$  denote the partial derivative of  $x(t, s)$  with respect to  $t$ . Define

$$f(x, \dot{x}) = L^{1/2}(x, \dot{x}) = [g_{ik} \dot{x}^i \dot{x}^k]^{1/2}.$$

For each fixed  $s$ , the value of  $f(x, \dot{x})$  is either real or pure imaginary. Define

$$I_1(s) = \int f(x, \dot{x}) dt = d(p, a(s))$$

where the integral is from  $t = 0$  to  $t = 1$ . Differentiation of this equation with respect to  $s$  yields

$$I_1'(s) = \int \left( \frac{\partial f}{\partial x^j} \frac{\partial x^j}{\partial s} + \frac{\partial f}{\partial \dot{x}^j} \frac{\partial \dot{x}^j}{\partial s} \right) dt.$$

Integrating by parts we obtain

$$I_1'(s) = \frac{\partial f}{\partial \dot{x}^j} \frac{\partial x^j}{\partial s} \Big|_0^1 + \int \left( \frac{\partial f}{\partial x^j} - \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{x}^j} \right) \right) \left( \frac{\partial x^j}{\partial s} \right) dt.$$

This last integral must vanish because the Euler-Lagrange equations hold along each extremal. Furthermore, the derivative of  $x^j$  with respect to  $s$  is zero at  $t = 0$ . Hence,

$$I_1'(0) = \frac{\partial f}{\partial \dot{x}^j} \frac{\partial x^j}{\partial s} \Big|_{t=1}.$$

The next equation (compare [10, p. 15]) results from the homogeneous assumption (B) together with the definition (C) of the metric tensor.

$$\frac{\partial g_{ik}}{\partial \dot{x}^j} \dot{x}^i = 0.$$

This last equation and the definition of  $f(x, \dot{x})$  imply

$$\frac{\partial f}{\partial \dot{x}^j} = \frac{g_{ij} \dot{x}^i}{f(x, \dot{x})}.$$

Consequently,

$$I_1'(0) = \frac{g_{ij} \dot{x}^i}{f(x, \dot{x})} \frac{\partial x^j}{\partial s} \Big| = |W|^{-1} W(W, Y).$$

If  $I_2(s) = d(q, a(s))$ , then

$$I_2'(0) = -|W|^{-1}W(Y).$$

The fact that  $a(s) \in B(p, q)$  implies  $I_1'(0) = I_2'(0)$ . This implies  $W \perp Y$  and establishes the Lemma.

**LEMMA 3.** *Let  $r$  be the midpoint of the nonnull extremal  $\alpha(p_1, q_1)$ . If  $p, q \in \alpha(p_1, q_1)$  and  $r$  is the midpoint of  $\alpha(p, q)$ , then  $B(p, q) = B(p_1, q_1)$ .*

*Proof.* From Lemma 2 it follows that both  $B(p, q)$  and  $B(p_1, q_1)$  consist of the union of all extremals in  $U(x)$  which pass through  $r$  and have the property that  $\alpha(p, q)$  is perpendicular to them at  $r$ .

Let  $W$  and  $Y$  be nonzero vectors attached at  $x$  with coordinate representations  $(x, w)$  and  $(x, y)$  respectively. Then  $W \perp Y$  if and only if  $g_{ij}(x, w)w^i y^j = 0$ . Since the metric tensor is nonsingular the vector  $W$  is always perpendicular to a hyperplane containing the origin of  $T(x)$ . This holds even if  $|W|^2 = 0$  (as long as  $W \neq 0$ ). This hyperplane varies continuously with  $W$  and may actually contain  $W$ .

**LEMMA 4.** *If  $M$  is an indefinite Finsler space which locally has flat bisectors, then perpendicularity is symmetric on  $M$ .*

*Proof.* The nonnull vectors are dense in the set of nonzero vectors and a vector  $W$  is perpendicular to a hyperplane which varies continuously with  $W$ . Consequently, it is only necessary, to verify that  $W \perp Y$  implies  $Y \perp W$  for nonnull vectors  $W$  and  $Y$ .

Let  $\alpha(p, q)$  be a nonnull extremal with midpoint  $r$  and unit tangent  $W$  at  $r$ . Let  $Y$  be a nonnull vector at  $r$  with  $W \perp Y$ . Using the notation of Lemma 2, we let  $a(s)$  be an extremal with  $a(0) = r$  and  $a'(0) = Y$ . The extremal  $\alpha(p, q)$  has an arclength representation  $b(u)$  where  $-|d(p, r)| \leq u \leq |d(p, r)|$  and  $b'(0) = W$ . Choose some fixed  $s_0$  different from zero and let  $x(t, u)$  represent the extremal  $\alpha(a(s_0), b(u))$  for  $0 \leq t \leq 1$ . The partial derivative of  $x$  with respect to  $t$  will be denoted by  $\dot{x}$ . Define

$$I_0(u) = \int f(x, \dot{x}) dt = d(a(s_0), b(u)).$$

The arguments used in the proof of Lemma 2 yield

$$I_0'(0) = \frac{\partial f}{\partial \dot{x}^i} \frac{\partial x^i}{\partial u} \bigg|_{t=1} = |Y|^{-1}Y(Y, W).$$

Lemma 3 implies that  $I_0(-u) = I_0(u)$ . It follows that  $I'_0(0) = 0$ . Hence,  $|Y|^{-1}Y(Y, W) = 0$ . This implies  $Y \perp W$  and establishes the Lemma.

**THEOREM 5.** *Let  $M$  be an indefinite Finsler space. Locally  $M$  has flat bisectors if and only if  $M$  is pseudo-Riemannian of constant sectional curvature.*

*Proof.* If  $M$  has dimension two, then Proposition 1 yields the result.

In [5] we have shown that an indefinite Finsler space of dimension at least three has symmetric perpendicularity if and only if it is pseudo-Riemannian. In [4] we have shown that a pseudo-Riemannian manifold locally has flat bisectors if and only if it is a space of constant curvature. These two results together with the conclusion of Lemma 4 that  $M$  has symmetric perpendicularity complete the proof of the Theorem.

**6. Reflections in extremals.** In this section another theorem characterizing pseudo-Riemannian spaces of constant curvature is proven.

Let  $f$  be a diffeomorphism of  $M$  onto itself and let  $f_*$  denote the derivative map induced on the tangent bundle. The map  $f$  is an isometry if for all  $x \in M$  and  $W, Y, Z \in T(x)$  we have

$$W(Y, Z) = f_*(W)(f_*Y, f_*Z).$$

When  $f$  is a diffeomorphism of some open set  $U_1$  of  $M$  onto an open set  $U_2$  of  $M$  which satisfies the above equality, the map  $f$  is called a local isometry. When  $f$  is a local isometry different from the identity and such that  $f^2$  is the identity, then  $f$  is an involution.

Let  $x$  be an interior point of the nonnull extremal  $\alpha$ . A reflection in  $\alpha$  near  $x$  is said to exist, if there is a neighborhood  $V(x)$  and a local isometry  $f$  defined on  $V(x)$  such that  $f$  is an involution and the set of fixed points of  $f$  is exactly  $\alpha \cap V(x)$ .

If every nonnull extremal may be reflected near each interior point, then we say  $M$  may be locally reflected in each nonnull extremal.

Let  $f$  be a reflection in  $\alpha$  near  $x$ . The tangent map  $f_*$  is a linear map of  $T(x)$  onto  $T(x)$  which preserves the metric induced on  $T(x)$ . Hence,  $f_*$  maps the indicatrix  $K(x)$  onto itself and the light cone  $C(x)$  onto itself. If  $W$  is a nonzero vector tangent to  $\alpha$  at  $x$ , then  $f_*W = W$  and

$$W(W, Z) = W(W, f_*Z)$$

for all  $Z \in T(x)$ . This implies that if  $W$  is perpendicular to the  $(n-1)$  dimensional linear subspace  $H$  of  $T(x)$  then  $f_*H = H$ .



Let  $(M, g)$  be a pseudo-Riemannian space of constant sectional curvature. It is known (see [11, p. 69]) that each  $x \in M$  must have a neighborhood which is isometric to an open set of one of the model spaces  $S_s^n$ ,  $R_s^n$  or  $H_s^n$ . When  $s = 0$ , these model spaces are the classical models for spaces of constant curvature. The space  $S_0^n$  is an  $n$  dimensional sphere, the space  $R_0^n$  is  $n$  dimensional Euclidean space and  $H_0^n$  is an  $n$  dimensional hyperbolic space. The groups of motions of all of the model spaces are well known, compare [11, pp. 65–66]. In particular, each of the model spaces may be reflected over any nonnull geodesic  $G$ . This reflection may have more than  $G$  as its set of fixed points, however, the geodesic  $G$  will have a neighborhood  $U$  such that the fixed points of  $U$  are all on  $G$ . It follows that any pseudo-Riemannian space of constant curvature may be locally reflected in any nonnull extremal. In general, pseudo-Riemannian spaces of constant curvature cannot be reflected over null extremals.

**PROPOSITION 6.** *If  $M$  is a two dimensional indefinite Finsler space which may be locally reflected in all nonnull extremals, then  $M$  is pseudo-Riemannian of constant curvature.*

*Proof.* If the metric on  $M$  is definite the result is well known, see [7, p. 350].

Assume the metric is not definite and let  $W$  be a nonnull vector in  $T(x)$ . There is a local reflection  $f$  in the extremal  $\alpha$  determined by  $W$ . Furthermore,  $f_* W = W$  and  $f_*$  is an involutoric motion on  $T(x)$ . Letting  $W$  vary, it follows that there exist infinitely many motions of  $T(x)$  holding the origin fixed. The metric on  $T(x)$  is Minkowskian and it is known [3, p. 533] that a two dimensional Minkowskian space has an infinite group of motions holding one point fixed if and only if the metric is the ordinary two dimensional Lorentz metric. Letting  $x$  vary, it follows that  $M$  is pseudo-Riemannian.

Let  $\alpha(p, q)$  be a nonnull extremal from  $p$  to  $q$ . For each positive integer  $k$ , there is a set of equally spaced points  $\{p_0, p_1, \dots, p_k\}$  on  $\alpha(p, q)$  with  $d(p, p_m) = md(p, q)/k$  where  $m = 1, 2, \dots, k$ . Each extremal  $\alpha(p_i, p_{i+1})$  has a midpoint  $r_i$ . Let  $\alpha^\perp(r_i)$  be the nonnull extremal perpendicular at  $r_i$  to  $\alpha(p_i, p_{i+1})$ . Let  $F_i$  be the local reflection over  $\alpha^\perp(r_i)$ . The map  $F_i$  takes points of  $\alpha(p_i, p_{i+1})$  to points of  $\alpha(p_{i+1}, p_i)$ . For sufficiently large  $k$  each  $F_i$  may be defined on all of  $\alpha(p_i, p_{i+1})$  and this map interchanges  $p_i$  and  $p_{i+1}$ . Consequently, the composite map

$$F = F_k \circ F_{k-1} \circ \dots \circ F_1$$

is a local isometry taking  $p$  to  $q$  whenever  $k$  is sufficiently large. It follows that  $M$  has the same curvature at  $p$  and  $q$ .

To conclude that  $M$  has the same curvature at all points we observe that any pair of points of  $M$  may be joined by a path consisting of a finite sequence of nonnull extremals. This establishes the Proposition.

**LEMMA 7.** *Let  $W$  be a unit vector at  $x$  which is tangent to  $\alpha$  and let  $f$  be a reflection in  $\alpha$  near  $x$ . Then  $W \perp Z$  implies  $f_*Z = -Z$ .*

*Proof.* Let  $W$  be perpendicular to  $Z$ . Then  $W$  is also perpendicular to  $f_*Z$  since  $f_*$  preserves the metric on  $T(x)$ . Assume  $f_*Z \neq -Z$  and let  $Y = Z + f_*Z$ . Then  $Y$  is nonzero. Also,  $f_*Y = f_*Z + f_*^2Z = f_*Z + Z = Y$  and  $W \perp Y$ .

If  $|Y|^2 \neq 0$ , let  $\beta$  be the extremal through  $x$  with tangent  $Y$  at  $x$ . Then  $f$  leaves  $\beta$  pointwise fixed near  $x$  which contradicts the assumption that  $f$  only leaves  $\alpha \cap V(x)$  fixed.

If  $|Y|^2 = 0$ , let  $P$  be the two dimensional linear subspace of  $T(x)$  spanned by  $Y$  and  $W$ . The map  $f_*$  is the identity on  $P$  since  $f_*Y = Y$  and  $f_*W = W$ . For sufficiently small positive  $\epsilon$ , the vector  $X = W + \epsilon Y$  is a nonnull vector in  $P$ . Letting  $\beta$  be an extremal tangent to  $X$  at  $x$ , it follows as before that  $f$  leaves  $\beta$  pointwise fixed near  $x$ . This last contradiction establishes the Lemma.

**THEOREM 8.** *If  $M$  is an indefinite Finsler space, then  $M$  may be reflected locally in each nonnull extremal if and only if  $M$  is a pseudo-Riemannian space of constant curvature.*

*Proof.* Because of Proposition 6, we only consider  $n \geq 3$ .

Let  $W$  be a nonnull vector tangent to  $\alpha$  at  $x$ . Assume that  $f$  is a local reflection in  $\alpha$  and that  $Z$  is any vector with  $W \perp Z$ . Let  $(x, w)$  and  $(x, z)$  be the respective coordinate representations of  $W$  and  $Z$ . Lemma 7 and the fact that  $f_*$  must preserve the metric induced on the tangent space  $T(x)$  yield  $g_{ij}(x, w + \epsilon z) = g_{ij}(x, w - \epsilon z)$  for all real  $\epsilon$ . This implies the derivative of  $g_{ij}(x, w + \epsilon z)$  with respect to  $\epsilon$  must vanish at  $\epsilon = 0$ . The function  $g_{ij}(x, y)$  is homogeneous of degree zero in  $y$  because of conditions (B) and (C). Thus, the derivative of  $g_{ij}(x, w + \epsilon w)$  with respect to  $\epsilon$  must vanish at  $\epsilon = 0$ . We conclude that

$$\frac{\partial g_{ij}(x, w)}{\partial \dot{x}^k} = 0$$

for all  $k = 1, 2, \dots, n$ . This equation must hold for all nonnull vectors  $W$ .

Since the nonnull vectors at  $x$  are dense in  $T(x)$ , we find  $g_{ij}(x, \dot{x})$  is independent of  $\dot{x}$ . Hence,  $M$  is pseudo-Riemannian.

Consider a nondegenerate two dimensional linear subspace  $E$  of  $T(x)$  with sectional curvature  $K(x, E)$ . Let  $E$  be spanned by vectors  $Y$  and  $Z$ . The two dimensional sections of  $T(x)$  have a natural topology induced from the Grassmann manifold of 2-planes in  $T(x)$ . If  $Y_i \rightarrow Y$  and  $Z_i \rightarrow Z$ , then the subspace spanned by  $Y_i$  and  $Z_i$  converges to  $E$ .

If  $f$  is the reflection in the nonnull extremal  $\alpha$  through  $x$ , then  $K(x, E) = K(x, f_*E)$ . In general, given two arbitrary sections  $E_1$  and  $E_2$  at  $x$  there may not be a reflection  $f$  such that  $E_2 = f_*E_1$ . In fact, it may happen that the metric is definite on one section and indefinite on the other.

Let  $Y'$  be a vector attached at  $x$  and let  $E'$  denote the section spanned by  $Y'$  and  $Z$ . If  $Y'$  is chosen sufficiently close to  $Y$ , then there is a reflection  $f$  in some nonnull extremal  $\alpha$  such that  $E' = f_*E$ . It follows easily that all sections sufficiently close to  $E$  have the same curvature. This implies that two nondegenerate sections  $E_1$  and  $E_2$  will have the same curvature if there is a continuous family of nondegenerate sections from  $E_1$  to  $E_2$ . It follows that the sectional curvature  $K(x, E)$  is independent of  $E$ . However, when  $n \geq 3$  the sectional curvature is only constant at each  $x$  when the curvature is independent of  $x$ , see [11, p. 57]. Therefore,  $M$  is a space of constant curvature.

Theorems 5 and 8 yield our final Proposition.

**PROPOSITION 9.** *If  $M$  is an indefinite Finsler space, then the following conditions are equivalent.*

- (i)  *$M$  is pseudo-Riemannian of constant curvature.*
- (ii) *Locally  $M$  has flat bisectors.*
- (iii)  *$M$  may be reflect locally in each nonnull extremal.*

**REMARK.** If  $M$  has a definite Finsler metric, then Theorems 5 and 8 may be established without using the assumption of condition (D) that the metric be symmetric. Furthermore, by making some modifications of the arguments in [3] and in the proof of Theorem 8, we may establish Theorem 8 for indefinite metrics without assuming condition (D).

## REFERENCES

1. J. K. Beem and P. Y. Woo, *Doubly timelike surfaces*, Mem. Amer. Math. Soc. No. 92, 1969.
2. ———, *Indefinite Finsler spaces and timelike spaces*, Canad. J. Math., **22** (1970), 1035–1039.
3. ———, *Motions in two dimensional indefinite Finsler spaces*, Indiana Univ. Math. J., **21** (1971), 551–555.
4. ———, *Pseudo-Riemannian manifolds with totally geodesic bisectors*, Proc. Amer. Math. Soc., **49** (1975), 212–215.

5. ———, *Symmetric perpendicularity for indefinite Finsler metrics on Hilbert manifolds*, *Geometriae Dedicata*, **4** (1975), 45–49.
6. W. Blaschke, *Geometrische Untersuchungen zur Variationsrechnung, I. Über Symmetralen*, *Math. Z.*, **6** (1920), 281–285.
7. H. Busemann, *The Geometry of Geodesics*, New York, Academic Press, 1955.
8. B. B. Phadke, *Flatness of bisectors and the symmetry of distance*, *J. Geometry*, **4** (1974), 35–51.
9. J. H. C. Whitehead, *Convex regions in the geometry of paths — addendum*, *Quart. J. Math. Oxford Ser.*, **4** (1933), 226–227.
10. H. Rund, *The Differential Geometry of Finsler Spaces*, Berlin, Springer-Verlag, 1959.
11. J. A. Wolf, *Spaces of Constant Curvature*, McGraw-Hill, 1967.

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