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We shall be concerned with the behavior of a mapping π from one oriented compact surface-with-boundary to another, which may fail to be a covering projection in one of two ways. Firstly, π need not be a local homeomorphism, although its interior singularities will be of a restricted type, called branch points. Secondly, boundary points may be mapped into the interior, although we shall assume the restriction of π to the boundary is injective. We shall show that π must then be a local homeomorphism except on a finite set. Moreover, we shall analyze the behavior of π near the boundary in sufficient detail to derive a formula relating Euler characteristics of the domain and of the image, with multiplicities, to the total order of branching of π . These results may be used to study ramification and ramified branch points of parametric minimal surfaces of general topological type.

For the present case of a mapping $\pi: M \rightarrow M_1$ of one surface, or topological 2-manifold, into another, we may call π a *branched immersion* provided it is locally topologically conjugate to the mapping $g_m(z) = z^m$ of the unit disk Δ in the complex plane to itself. That is, for each $p \in M$ there exists an integer $m \geq 1$, a neighborhood V of p in M , a neighborhood V_0 of $\pi(p)$ in M_1 , and homeomorphisms $h: V \rightarrow \Delta$, $h_0: V_0 \rightarrow \Delta$, with $h(p) = 0$, $h_0(\pi(p)) = 0$ and such that $g_m \circ h = h_0 \circ \pi$. The integer $m - 1$ is the order of branching (or order of ramification) of π at p , denoted $o(p)$. If $o(p) > 0$ we call p a *branch point*; if $o(p) = 1$, p is a *simple branch point*. For the definition of a branched immersion of a surface into a higher-dimensional manifold, see ([4], Definitions 1.2, 1.6).

The Euler-characteristic formula in the theorem below is a generalization of the Riemann–Hurwitz relation for the case of closed surfaces. The formula has been proved by Ahlfors under the assumption that π is a simplicial mapping with respect to appropriate triangulations of the compact surfaces-with-boundary \bar{M} and \bar{M}_1 , with an openness condition at interior edges; this amounts to requiring π to be a branched immersion up to the boundary ([1], p. 161, 168). A recent, sheaf-theoretic proof has been given by Elwin and Short under the hypothesis that the fibers are constant over components in a finite decomposition of \bar{M}_1 ([2]).

One area in which this question is of interest is in the study of ramification of minimal surfaces and of surfaces of prescribed mean

curvature vector, in conformal parameterization. A mapping $f: M \rightarrow N$ of one manifold into another is said to be *ramified* if two distinct regular points of f in M define the same germ of submanifold in N . A point of $M \cup \partial M$ is called a *ramified point* of f if the restriction of f to every neighborhood of that point is ramified; a ramified point is necessarily a singular point. If M and N are both two-dimensional and f is a branched immersion, then the notions of branch point and interior ramified point coincide. Now suppose \bar{M} is a compact surface-with-boundary, and let $f: \bar{M} \rightarrow N$ be a mapping into a manifold of arbitrary dimension, whose restriction to M is a branched immersion with the unique continuation property (see [4], p. 757), and whose restriction to ∂M is injective. Then the topological space of germs of surface defined by f at its regular points has a natural compactification \bar{M}_1 ; the fundamental theorem of branched immersions states that \bar{M}_1 is a compact oriented surface-with-boundary, and that the natural quotient mapping $\pi: \bar{M} \rightarrow \bar{M}_1$ is a branched immersion in the interior (see Theorem 4.15 of [3, I]). Branch points of π are precisely the ramified points of f . This holds in particular if f is a conformal parameterization of a surface with prescribed mean curvature vector in a riemannian manifold N , which maps the boundary injectively into N . Thus, the study of the consequences of ramification of f leads naturally to consideration of the mapping π . The results below will be applied in [3, II] to shed light on ramification of such mappings f , and in particular of the minimal surfaces of higher topological type whose existence was proven by Douglas. It should be noted, however, that if the disjoint Jordan curves comprising $f(\partial M)$ are assumed to have a sufficiently high degree of regularity, say class C^2 , then the results of the present paper may be replaced by somewhat simpler arguments exploiting recent results on the regularity of f up to the boundary.

The present work is largely self-contained, relying on a few elementary facts proved in [4]. However, the methods employed will be better understood by a reader familiar with certain concepts and techniques of [4] and of [3, I]. We point out particularly the instructive series of examples in §5 of [4].

Branched immersions between surfaces may be characterized by remarkably weak hypotheses, according to a classical theorem of Stoilow ([7], p. 121). Namely, if $\pi: M \rightarrow M_1$ is a continuous open mapping between surfaces, and π is *light*, that is, $\pi^{-1}(p_0)$ is totally disconnected for each $p_0 \in M_1$, then π is a branched immersion. Thus the result of the present paper implies that a light continuous mapping $\pi: \bar{M} \rightarrow \bar{M}_1$ between compact oriented surfaces-with-boundary, whose restriction to M is open and whose restriction to ∂M is injective, is a local homeomorphism except on a finite set in \bar{M} , at each point of which there is a well-defined order of branching.

NOTATION. When the notation \bar{M} or \bar{M}_k , etc., is used to denote a surface-with-boundary, we shall write M or M_k for the surface consisting of its interior points, ∂M or ∂M_k for its boundary. If \bar{M} is a surface-with-boundary, then an open set $U \subset M$ is itself a surface; however, its closure \bar{U} need not be a surface-with-boundary, and $\partial U = \bar{U} \setminus U$ need not be a 1-submanifold. A connected oriented compact surface-with-boundary \bar{M} may be obtained from a sphere by attaching a certain number g of handles, and removing a number of disjoint open disks; g is the *genus* of M . If M is not connected, then its genus is the sum of the genera of its connected components. The Euler characteristic $\chi(\bar{M}) = 2c - 2g - k$, where c is the number of components of M , g the genus of M , and k the number of boundary components. The restriction of a mapping $\pi: \bar{M} \rightarrow \bar{M}_1$ to a subset $U \subset M$ is denoted $\pi|U$. In the context of a branched immersion $\pi: \bar{M} \rightarrow \bar{M}_1$, we shall use the notation B_r for the set of ramified points of π in \bar{M} ; $B = B_r \cap M$ denotes the set of interior branch points, and $B_\partial = B_r \cap \partial M$ is the set of ramified boundary points.

For a continuous mapping $\pi: M \rightarrow M_1$ of one oriented surface onto another, we may define the *Brouwer degree* as an integer-valued function $\deg(\pi)$, defined at those points $p_0 \in M_1$ such that there is a compact neighborhood U of p_0 in M_1 whose pre-image $\pi^{-1}(U)$ is a compact subset of M . That is, $\deg(\pi)$ is defined on the complement of the set of limits of images of properly divergent sequences in M . If, as in the case treated in this paper, M is the interior of a compact surface-with-boundary \bar{M} and π extends continuously over \bar{M} , then $\deg(\pi)$ is defined on $M_1 \setminus \pi(\partial M)$. Now $\deg(\pi)(p_0)$ may be computed as follows: let φ be any smooth approximation to π , with respect to some pair of differentiable structures, which has p_0 as a regular value. Then $\deg(\pi)(p_0)$ is the number of points in $\varphi^{-1}(p_0)$ at which φ preserves orientation, minus the number at which φ reverses orientation. We note that if π is a branched immersion, then with respect to the appropriate orientations of M and M_1 , φ may be chosen to preserve orientation at all points (see, e.g., Lemma 2 below). Suppose $\pi_t: M \rightarrow M_1$ defines a homotopy, that is, a jointly continuous one-parameter family of mappings. Then for fixed $p_0 \in M$, $\deg(\pi_t)(p_0)$ is constant as a function of t on any interval where it is defined: the proof given in [5], pp. 27–9, for the case that ∂M and ∂M_1 are empty, may be extended without difficulty to the present case. It follows that for a mapping $\pi: M \rightarrow M_1$, $\deg(\pi)$ is constant on connected components of its (open) domain of definition. In fact, one needs only the following lemma: if U is a connected open subset of a differentiable manifold M_1 and $p, q \in U$, then there exists a homotopy of diffeomorphisms $h_t: M_1 \rightarrow M_1$, such that $h_t(U) = U$ for $0 \leq t \leq 1$, h_0 is the identity, and $h_1(p) = q$. The proof of this lemma is completely analogous to the case $U = M_1$ given by Milnor ([5], pp. 22–4).

1. Finiteness of interior branching. Our first lemma illustrates the power of the requirement of injectivity on the boundary. The lemma includes, as special cases, Lemma 6.13 of [4] and Lemma 2.6 of [3, I], and is proved in a fashion similar to the proof of the former. We shall indicate its proof here, in the interest of completeness.

LEMMA 1. *Let Δ denote the unit disk in the plane, Δ^+ , Δ^- and I its intersection with the open upper half-plane, the open lower half-plane and the horizontal axis, respectively. Suppose \bar{M} is a surface-with-boundary (not necessarily compact), and M_1 is a surface. Let $\pi: \bar{M} \rightarrow M_1$ be a continuous mapping, whose restriction to M is a branched immersion, and whose restriction to ∂M is injective. Let ∂M and I be oriented so that M and Δ^+ , respectively, lie to the left. Then for any $p \in \partial M$ there is an arbitrarily small simply-connected neighborhood $V \cup K$ of p in \bar{M} , where $V \subset M$, $K \subset \partial M$; an integer $m \geq 1$; a neighborhood V_0 of $\pi(p)$ in M_1 ; and a homeomorphism $g: V_0 \rightarrow \Delta$; such that $g \circ \pi$ maps an arc of K homeomorphically onto I in orientation-preserving fashion, and such that $\deg(g \circ \pi | V)$ is defined on $\Delta \setminus I$, has the value m on Δ^+ , and has the value $m - 1$ on Δ^- . Moreover, we may choose V_0 to be disjoint from $\pi(\partial V \cap M)$, and we may choose V so that $(V \cup K) \cap \pi^{-1}(\pi(p)) = \{p\}$.*

Proof. Choose a simply-connected neighborhood U_0 of $p_0 = \pi(p)$ in M_1 . Let $V \cup K$ be tentatively chosen so that (1) $\partial V = \gamma \cup K$, where K is an arc of ∂M and γ is a Jordan arc in M connecting the two end points of K ; (2) $\pi(\bar{V}) \subset U_0$; (3) $p_0 \notin \pi(\gamma)$. Then $\pi(K)$ is a Jordan arc in U_0 passing through p_0 : it follows from the Jordan separation theorem that p_0 has an arbitrarily small neighborhood V_0 which is homeomorphic to Δ under a homeomorphism $g: V_0 \rightarrow \Delta$ which maps $\pi(K) \cap V_0$ to I in orientation-preserving fashion. We choose V_0 small enough that it is disjoint from $\pi(\bar{\gamma})$. Then $\deg(g \circ \pi | V)$ is well-defined on $\Delta \setminus I$, and its constant value m on Δ^+ is one greater than its value on Δ^- , as may be seen from the winding-number characterization of the Brouwer degree. Now since π is an open mapping on V , the cardinality of $\pi^{-1}(q_0)$ is a lower semi-continuous function of $q_0 \in M_1$ (cf. Lemma 3.26 of [4]). In particular, $V \cap \pi^{-1}(p_0)$ consists of at most $m - 1$ points. We now make the final choice of $V \cap K$, choosing V small enough that $V \cap \pi^{-1}(p_0)$ is empty, and choose V_0 accordingly.

COROLLARY 1. *Suppose \bar{M} , \bar{M}_1 are compact oriented surfaces-with-boundary, $\pi: \bar{M} \rightarrow \bar{M}_1$ a mapping which is a branched immersion in M and whose restriction to ∂M is injective. Then $\deg(\pi)$ is bounded, and its maximum and minimum differ by at most the number of components of ∂M .*

Proof. First observe that for any curve γ in M_1 , the function $\deg(\pi)$ changes along γ by exactly the intersection number of γ with $\pi(\partial M)$. In fact, the contribution from interior neighborhoods is locally unchanged, while the contribution from a boundary neighborhood changes by 1 as $\pi(\partial M)$ is crossed from right to left, according to Lemma 1. Choose $p_0 \in M_1 \setminus \pi(\partial M)$. Then since \bar{M} is compact, $\deg(\pi)(p_0)$ is finite: any smooth approximation to π is proper, so that only finitely many points are mapped to any regular value. On the other hand, for any point $q_0 \in M_1$ there is a curve γ from p_0 to q_0 which crosses each component of $\pi(\partial M)$ at most once. Therefore $\deg(\pi)(q_0)$ is at most equal to $\deg(\pi)(p_0)$ plus the number of components of ∂M .

In the proof of Proposition 1 below, it will be convenient to work with branched immersions, all of whose branch points are simple. This will be made possible by the following lemma.

LEMMA 2. *Let $\pi: V \rightarrow V_0$ be a branched immersion between surfaces V and V_0 , with exactly one branch point $q \in V$ of order $o(q) = m - 1$. Then π is homotopic to a branched immersion $\pi_1: V \rightarrow V_0$, $\pi_1 = \pi$ outside of an arbitrarily small neighborhood of q , and which has exactly $m - 1$ branch points, all of order 1.*

Proof. In an arbitrarily small neighborhood of q , π is topologically conjugate to the mapping $g(z) = z^m$ of the unit disk onto itself. It suffices, therefore, to prove the statement of the lemma for $\pi = g$, $q = 0$. Now choose $m - 1$ distinct points z_1, \dots, z_{m-1} on the unit circle in the complex plane. For $0 \leq t \leq 1$, we define an analytic function h_t by the conditions $h_t(0) = 0$ and

$$h'_t(z) = m(z - tz_1)(z - tz_2) \cdots (z - tz_{m-1}).$$

Now let $\varphi(r)$ be a smooth real-valued function for $0 \leq r \leq 1$, with $\varphi(r) = 1$ for $r \leq 1/4$ and $\varphi(r) = 0$ for $r \geq 1/2$. Define $g_t(z) = \varphi(|z|)h_t(z) + (1 - \varphi(|z|))g(z)$. Then $g_0 = h_0 = g$. Also, for $|z| < 1/4$, $g_t(z) = h_t(z)$, so that for $t < 1/4$, g_t has the $m - 1$ simple branch points tz_1, \dots, tz_{m-1} . For $|z| > 1/2$, $g_t(z) = g(z)$ and is an immersion. It may be computed that if t is sufficiently small, then g_t is an immersion on the annulus $1/4 \leq |z| \leq 1/2$ also.

Since the Brouwer degree is constant under homotopy, Lemma 2 gives an explicit formula for the degree of a branched immersion $\pi: M \rightarrow M_1$:

$$\deg(\pi)(p_0) = \sum \{o(p) + 1: p \in M, \pi(p) = p_0\}.$$

With these preliminaries at hand, we are ready to prove the finiteness of interior branching, as a first step toward the finiteness of the set of all ramified points.

PROPOSITION 1. *Suppose \bar{M}, \bar{M}_1 are compact oriented surfaces-with-boundary, $\pi: \bar{M} \rightarrow \bar{M}_1$ a continuous mapping which is a branched immersion in M and which maps ∂M injectively. Then the set $B \subset M$ of interior branch points of π is finite.*

Proof. We shall find an upper bound for the total order of branching in an appropriately chosen neighborhood of any point in ∂M . Since B is a discrete subset of M , the conclusion will then follow from the compactness of \bar{M} .

Consider a point $p \in \partial M$. Applying Lemma 1, we may find a simply-connected neighborhood $V \cup K$ of p in \bar{M} , and a neighborhood V_0 of $\pi(p)$ in M_1 which is separated into two simply-connected components V_0^+ and V_0^- by $\pi(K)$, such that $\deg(\pi|V)$ has the constant values m on V_0^+ and $m-1$ on V_0^- , V_0 is disjoint from $\pi(\partial V \cap M)$, and such that $(V \cup K) \cap \pi^{-1}(\pi(p)) = \{p\}$. Let $W \cup L$ be a neighborhood of p with $\pi(W \cup L) \subset V_0$, $W \subset V$ and $L \subset K$. We shall show that the total order O_w of branching of π in W is at most $(m-1)^2$.

We first apply Lemma 2 to see that without loss of generality, it may be assumed that π has only simple branch points in W . Namely, B is discrete; in an arbitrarily small neighborhood of each branch point q of order $o(q) > 1$, we replace π by a branched immersion homotopic to it, having exactly $o(q)$ simple branch points in this neighborhood, and we leave π unchanged outside this neighborhood. Further, we may readily modify π so that for each branch point $q \in V$, $\pi(q) \notin \pi(K)$; and so that for distinct branch points $q, q' \in V$, $\pi(q) \neq \pi(q')$. Observe that these modifications do not change the total order of branching of π in W . Let O_w^+ be the number of branch points $q \in W$ of π (as now modified) with $\pi(q) \in V_0^+$; thus $O_w = O_w^+ + O_w^-$.

We shall first find an estimate for O_w^+ . Write $\nu = O_w^+$: there are distinct simple branch points p_1, \dots, p_ν in W with distinct images $P_j = \pi(p_j) \in V_0^+$, $1 \leq j \leq \nu$. Choose a point $Q \in V_0^+ \setminus \pi(B)$, and let the m points of $V \cap \pi^{-1}(Q)$ be denoted q_1, \dots, q_m . For each P_j , $1 \leq j \leq \nu$, choose a closed curve $\gamma_j: [0, 1] \rightarrow V_0^+ \setminus \pi(B)$, $\gamma_j(0) = \gamma_j(1) = Q$, such that γ_j has winding number 1 around P_j but has winding number 0 around $\pi(p')$ for every branch point $p' \in V$ other than p_j . We choose the curves $\gamma_1, \dots, \gamma_\nu$ to be disjoint except at Q . Now since $\pi(\partial V)$ is disjoint from V_0^+ , it may be seen that $\pi: V \rightarrow M_1$ is locally a covering map over $\gamma_j([0, 1])$. Thus for each k , $1 \leq k \leq m$, there is a unique lifting $\delta_k: [0, 1] \rightarrow V$ with $\pi \circ \delta_k = \gamma_j$ and $\delta_k(0) = q_k$. Because p_j is a simple branch point, it follows that there are two integers $r = r(j)$, $s = s(j)$, with

$1 \leq r < s \leq m$, such that $\delta_r(1) = q_s$, $\delta_s(1) = q_r$, and for $r \neq k \neq s$, $\delta_k(1) = q_k$. In fact, we may observe that P_j has exactly $m - 1$ pre-images in V under π , namely p_j plus $m - 2$ distinct regular points, since $\deg(\pi|V)(P_j) = m$. But on an appropriate small punctured neighborhood of p_j , π is a two-to-one covering map onto its image.

Now suppose, for contradiction, that $O_w^+ = \nu > m(m - 1)/2$. There are exactly $m(m - 1)/2$ ways to choose distinct pairs r, s with $1 \leq r < s \leq m$. Thus our supposition implies that the same pair is chosen twice. That is, for a certain pair i, j of numbers with $1 \leq i < j \leq \nu$, we have $r(i) = r(j)$ and $s(i) = s(j)$. We shall write simply $r = r(i) = r(j)$ and $s = s(i) = s(j)$. Let a closed curve $\delta: [0, 2] \rightarrow V$ be defined by $\delta(t) = \delta_r(t)$ and $\delta(1 + t) = \delta_s(t)$ for $0 \leq t \leq 1$: this construction will be denoted $\delta = \delta_r + \delta_s$. Observe that $\pi \circ \delta = \gamma_i + \gamma_j$.

For clarity in the following discussion, we shall assume there is a simple arc γ passing through Q such that the closed curve $\gamma_i + \gamma_j: [0, 2] \rightarrow V_0^+$ traverses γ twice, once simply in each direction, and otherwise is disjoint from γ . This may be achieved without changing the homotopy classes of γ_i and γ_j in $V_0^+ \setminus \pi(B)$ with base point Q . Now since γ_i and γ_j are disjoint except at Q , there is a closed curve $\tilde{\gamma}: [0, 2] \rightarrow (V_0^+ \setminus \pi(B)) \cup \{\pi(p)\}$, $\tilde{\gamma}(0) = \tilde{\gamma}(2) = \pi(p)$, which is disjoint from $\gamma_i((0, 1))$ and $\gamma_j((0, 1))$, and which meets Q exactly once at $Q = \tilde{\gamma}(1)$, with $\tilde{\gamma}(t)$ crossing from one side of γ to the other at $t = 1$. Since $\tilde{\gamma}$ misses $\pi(B)$, π is locally a covering projection over $\tilde{\gamma}((0, 2))$. Therefore, there is a unique lifting $\tilde{\delta}: (0, 2) \rightarrow V$ with $\pi \circ \tilde{\delta} = \tilde{\gamma}$ and $\tilde{\delta}(1) = q_r$. Note that $\tilde{\delta}$ leaves every compact subset of V as $t \rightarrow 0$ and as $t \rightarrow 2$, since $\pi(p') \neq \pi(p)$ for $p' \in V$. Meanwhile V is simply-connected, which implies that $\tilde{\delta}$ has intersection number zero with any closed curve in V . But $\tilde{\delta}$ intersects the closed curve δ exactly once, at $\tilde{\delta}(1) = q_r$, a regular point of π , at which point $\tilde{\delta}$ crosses from one side of δ to the other: that is, the intersection number of $\tilde{\delta}$ with δ is ± 1 . This contradiction shows that $O_w^+ \leq m(m - 1)/2$.

Similarly, since $\deg(\pi|V)$ has the constant value $m - 1$ on V_0^- , it may be shown that $O_w^- \leq (m - 1)(m - 2)/2$. Therefore, the total order of branching of π in W ,

$$O_w = O_w^+ + O_w^- \leq (m - 1)^2,$$

and, in particular, π has at most $(m - 1)^2$ branch points in W .

2. Behavior near ramified boundary points. We now turn our attention to the boundary ramified set B_∂ . Having established the finiteness of the interior branch set B , we may restrict attention to a neighborhood of any given boundary point which is disjoint from B , that is, on whose interior part π is a local homeomorphism. Under the

hypothesis that the restriction of π to the boundary is injective, the behavior of π near any boundary point can be described quite precisely. The following proposition will be applied to an appropriate neighborhood $U \cup K$ of a boundary point, where $U \subset M$ and K is an arc of ∂M .

PROPOSITION 2. *Suppose $U \cup K$ is an oriented surface with boundary K , and let M_1 be an oriented surface. Let $\pi: U \cup K \rightarrow M_1$ be a continuous mapping which is a local homeomorphism on U and which maps K injectively. Denote $S' = U \cap \pi^{-1}(\pi(K))$. Consider any point $p \in K$. Then there is a neighborhood $V \cup K_1$, $V \subset U$ and K_1 an arc of K , which may be chosen arbitrarily small; an integer $m \geq 1$; and a Jordan curve γ_0 in M_1 , which bounds a disk D_0 ; with the following properties. (1) $P = \pi(p) \in D_0$. (2) $\gamma = V \cap \pi^{-1}(\gamma_0)$ consists of a single Jordan arc, with endpoints a and b on K ; the union of γ with the arc of K between a and b bounds a disk $D = V \cap \pi^{-1}(D_0)$. (3) $S' \cap D$ is the disjoint union of a family of $2m - 2$ disjoint Jordan arcs $\sigma_1, \dots, \sigma_{m-1}, \tau_1, \dots, \tau_{m-1}$, each tending to p at one end and to distinct points of γ at the other. (4) γ_0 meets $\pi(K)$ in exactly two points, $A = \pi(a)$ and $B = \pi(b)$. (5) Finally, $\pi(K)$ separates D_0 into two disks, D_0^+ and D_0^- , so that $\deg(\pi|D)$ has the constant values m on D_0^+ and $m - 1$ on D_0^- .*

Proof. We first refer to Lemma 1, with $\bar{M} = U \cup K$, to see that there exists a simply-connected, relatively compact neighborhood $V \cup K_1$ of p in $U \cup K$, $V \subset U$ and $K_1 \subset K$, an integer $m \geq 1$, and a simply-connected neighborhood V_0 of P in M_1 , V_0 disjoint from $\pi(\partial V \cap U)$, so that V_0 is divided into components V_0^+ and V_0^- by the Jordan arc $\pi(K)$ and $\deg(\pi|V)$ has the constant values m on V_0^+ and $m - 1$ on V_0^- . Moreover, we may assume $(V \cup K_1) \cap \pi^{-1}(\pi(p)) = \{p\}$. Let γ_0 be any closed Jordan curve in V_0 which has P in its interior D_0 , and which meets $\pi(K)$ in exactly two points, A and B , at each of which γ_0 crosses between V_0^+ and V_0^- . Since π is a local homeomorphism on V , we see that $\gamma = V \cap \pi^{-1}(\gamma_0)$ is the disjoint union of Jordan curves and arcs. Observe that the only limit points of γ in \bar{V} are the unique points a and b on K with $\pi(a) = A$, $\pi(b) = B$, since $\pi(U \cap \partial V)$ is disjoint from V_0 . Since \bar{V} is compact, it follows that $\gamma \cup \{a, b\}$ is compact.

Let γ_1 be any connected component of γ , and choose $q \in \gamma_1$ with $\pi(q) \notin \pi(K)$. Beginning from the point $\pi(q)$ on γ_0 , we may construct curves $\delta_0, \tilde{\delta}: [0, 1] \rightarrow M_1$ with $\delta_0(0) = \tilde{\delta}_0(0) = \pi(q)$, $\delta_0(1) = P$ and $\tilde{\delta}_0(1) \notin \pi(K)$, so that $\delta_0((0, 1))$ and $\tilde{\delta}_0((0, 1))$ are disjoint from $\gamma_0 \cup \pi(K)$. Let $\delta: [0, t_0] \rightarrow V$ and $\tilde{\delta}: [0, \tilde{t}_0] \rightarrow V$ be the unique maximal liftings of δ_0 and $\tilde{\delta}_0$, that is, with $\delta(0) = \tilde{\delta}(0) = q$, $\pi \circ \delta = \delta_0$ and $\pi \circ \tilde{\delta} = \tilde{\delta}_0$. Now $\delta_0((0, 1))$ lies in the interior of γ_0 and hence in V_0 . Thus $\delta(t)$ remains in a compact subset of $V \cup K_1$ as $t \rightarrow t_0$; by a standard argument,

one may show that $t_0 = 1$. Further, since $(V \cup K_1) \cap \pi^{-1}(P) = \{p\}$, δ has a continuous extension to $[0, 1]$ given by $\delta(1) = p$. On the other hand, as $t \rightarrow \tilde{t}_0$, $\tilde{\delta}(t)$ tends to $\partial V \cap U$, and $\tilde{t}_0 < 1$. This shows that p may be reached from one side of γ_1 , and $\partial V \cap U$ from the other, by means of paths which do not cross γ .

Now if any component γ_1 of γ is closed, then it separates V into an interior and exterior by the Jordan curve theorem, and both p and $\partial V \cap U$ would be in the exterior. This contradiction shows that γ has no closed components in V . However, γ is a one-dimensional submanifold of V , and $\gamma \cup \{a, b\}$ is compact. Thus every component of γ is an arc from a to b . Each such arc must separate \bar{V} into two components, one containing p and the other containing $\partial V \cap U$. Finally, if there were two components γ_1 and γ_2 of γ , then since γ_1 and γ_2 are disjoint, γ_2 must lie in one component or the other of $V \setminus \gamma_1$. If γ_2 lies in the component containing p , then every path from γ_1 to p crosses γ_2 , contradicting the result of the above paragraph. Otherwise, every path from γ_2 to p crosses γ_1 , which is again a contradiction. This shows that γ consists of a single Jordan arc from a to b . Let D be the open set in V bounded by γ and the arc of K between a and b .

Now consider a point q moving from a to b along γ : since π is a local homeomorphism in V , $\pi(q)$ must move along γ_0 in a strictly monotone fashion. For any $Q \in \gamma_0 \cap V_0^+$, there are precisely m points in $V \cap \pi^{-1}(Q)$, and these must all lie on γ , so that Q is crossed exactly m times. Similarly, a point $Q' \in \gamma_0 \cap V_0^-$ is crossed exactly $m - 1$ times. It follows that A and B are crossed exactly m times, counting a and b . Thus we may write $\pi^{-1}(A) \cap V = \{a_1, \dots, a_{m-1}\}$ and $\pi^{-1}(B) \cap V = \{b_1, \dots, b_{m-1}\}$, where these points occur along γ in alternating order: $a, b_1, a_1, b_2, \dots, a_{m-1}, b$.

Let $\sigma, \tau: [0, 1] \rightarrow V_0$ be homeomorphisms into the Jordan arc $\pi(K)$, with $\sigma(0) = A$, $\tau(0) = B$, $\sigma(1) = \tau(1) = P$. Then σ and τ may be lifted uniquely to give maximal curves $\sigma_k: [0, s_k) \rightarrow V$, $\tau_k: [0, t_k) \rightarrow V$, with $\sigma_k(0) = a_k$, $\tau_k(0) = b_k$, $\pi \circ \sigma_k = \sigma$ and $\pi \circ \tau_k = \tau$, $1 \leq k \leq m - 1$. Note that these $2m - 2$ arcs are disjoint Jordan arcs, since π is a local homeomorphism on V . Denote $\sigma_0, \tau_0: [0, 1] \rightarrow K$ the unique liftings, $\pi \circ \sigma_0 = \sigma$, $\pi \circ \tau_0 = \tau$. We shall show $s_k = t_k = 1$, $1 \leq k \leq m - 1$. First observe that $\sigma_k(t)$ leaves every compact subset of V as $t \rightarrow s_k$. However, since $\sigma([0, 1]) \subset D_0$, $\sigma_k([0, s_k)) \subset D$, so the only possible cluster point of $\sigma_k(t)$ would be on the arc of K between a and b . Any such cluster point is mapped by π to $\sigma(s_k)$, so by the injectivity of π on K , the only possible cluster point is $\sigma_0(s_k)$. If $s_k < 1$, then $\sigma_k([0, s_k))$ separates D into two components, such that the arc of γ between a and a_k cannot be connected to $\tau_0([0, 1])$ by a path in D unless that path crosses $\sigma_k([0, s_k))$. Meanwhile, τ_1 connects b_1 to $\tau_0(t_1) = \tau_1(t_1)$; but b_1 lies on the arc of γ between a and a_k , so that τ_1 must cross σ_k , say at $\tau_1(t) = \sigma_k(s)$. But then

$\tau(t) = \pi \circ \tau_1(t) = \pi \circ \sigma_k(s) = \sigma(s)$, which can only happen if $t = s = 1$. Therefore $s_k = 1$ for $1 \leq k \leq m-1$, and similarly $t_k = 1$.

We shall show next that $D = V \cap \pi^{-1}(D_0)$. Observe that $\pi(D)$ is connected and disjoint from γ_0 , and that $P \in \pi(D)$; therefore, $\pi(D) \subset D_0$, and hence $D \subset V \cap \pi^{-1}(D_0)$. Conversely, suppose $q \in V$ and $\pi(q) \in D_0$. Then $\pi(q)$ is one endpoint of an arc $\eta_0: (0, 1) \rightarrow D_0 \setminus \pi(K)$, $\eta_0(0) = \pi(q)$, $\eta_0(1) = P$. η_0 has a unique lifting $\eta: [0, 1) \rightarrow V$ with $\eta(0) = q$, as may be seen via a standard argument. But $\eta(t) \rightarrow p$ as $t \rightarrow 1$, since $(V \cup K) \cap \pi^{-1}(P) = \{p\}$. Meanwhile η_0 is disjoint from γ_0 , so that η cannot cross γ . Therefore $q \in D$.

It remains to show that $S' \cap D$ is the union of the $2m-2$ disjoint Jordan arcs $\sigma_k((0, 1))$ and $\tau_k((0, 1))$, $1 \leq k \leq m-1$. First observe that since the function $\deg(\pi|_V)$ is lower semi-continuous, a point $Q \in \pi(K)$ can have at most $m-1$ pre-images in V . Now consider $q \in S' \cap D$. Since $q \in V$, we have $\pi(q) \neq P$. Thus we may write either $\pi(q) = \sigma(t)$ or $\pi(q) = \tau(t)$ for some t , $0 < t < 1$. In either case, the $m-1$ distinct points $\sigma_1(t), \dots, \sigma_{m-1}(t)$ or $\tau_1(t), \dots, \tau_{m-1}(t)$ are all in $V \cap \pi^{-1}(\pi(q))$, which, according to the degree argument, contains at most $m-1$ points. Therefore q is one of these.

DEFINITION. For $\pi: U \cup K \rightarrow M_1$ as in Proposition 2, and for any $p \in K$, observe that the integer m is characterized by the number of arcs of $\pi^{-1}(\pi(K))$ which converge to p , and is therefore independent of the choice of V and γ_0 . We define the *order of ramification* of π at p to be $o(p) = m-1$. Thus $o(p) > 0$ if and only if p is a ramified point of π , as follows from Proposition 2.

COROLLARY 2. Suppose $\pi: U \cup K \rightarrow M_1$ satisfies the hypotheses of Proposition 2. Then the set B_a of ramified boundary points is discrete.

Proof. According to Proposition 2, any point $p \in B_a$ has a neighborhood $D \cup K$ such that $D \cap \pi^{-1}(\pi(K))$ consists of a nonempty union of disjoint Jordan arcs tending to p and having no other limit points on the boundary. But for $p' \in K$ sufficiently close to p , $D \cup K$ is a neighborhood of p' , so that p' is not the end point of any arc in $\pi^{-1}(\pi(K))$, and therefore $p' \notin B_a$.

We are now ready to prove our main result. It may be observed that the description of the behavior of π given in Propositions 1 and 2 can be used to satisfy the hypotheses used by Elwin and Short in [2] to prove an Euler-characteristic formula similar to the one given below. For the sake of completeness, we shall give a proof relying only on elementary topological methods.

THEOREM. Suppose \bar{M} , \bar{M}_1 are compact oriented surfaces-with-boundary, $\pi: \bar{M} \rightarrow \bar{M}_1$ a continuous surjective mapping which is a branched immersion in M , and whose restriction to ∂M is injective. Then (i) the set $B, \subset \bar{M}$ of ramified points of π is finite; (ii) the function $\deg(\pi)$ on M_1 has an upper bound μ ; and (iii) the Euler-characteristic formula

$$(*) \quad \chi(\bar{M}) + \sum_{p \in B_r} o(p) = \sum_{i=1}^{\mu} \chi(\bar{M}_i)$$

holds, where for $p \in B = B_r \cap M$, $o(p)$ is defined in the introduction; for $p \in B_s = B_r \cap \partial M$, $o(p)$ is defined following Proposition 2; and for $i \geq 1$, $M_i = \{p_0 \in M_1: \deg(\pi)(p_0) \geq i\}$.

Proof. Conclusion (ii) follows from Corollary 1. To obtain conclusion (i), we first use Proposition 1 to see that $B = B_r \cap M$ is finite. Now for any $p \in B_s = B_r \cap \partial M$, there is a neighborhood $U \cup K$ of p in \bar{M} disjoint from B and which therefore satisfies the hypotheses of Corollary 2, so that p is an isolated point of B_s . Thus B_s is discrete, and hence finite, since ∂M is compact.

In order to verify formula (*), we first modify π , if necessary, on a small neighborhood of each interior branch point, so that for $p, q \in B$, $\pi(p) \neq \pi(q)$; and for $p \in B$, $\pi(p) \notin \pi(\partial M)$. The modified mapping still satisfies all hypotheses and has the same order of branching at corresponding branch points. We list the boundary ramified points $B_s = \{p_1, \dots, p_n\}$ and the interior branch points $B = \{q_1, \dots, q_\nu\}$.

For each $p_i \in B_s$, taken in order, we may apply Proposition 2 in a neighborhood of p_i disjoint from the finite set B , to see that there is a simply-connected neighborhood $D_i \cup K_i$ of p_i in \bar{M} , with the following properties. (1) D_i is bounded by the arc K_i of ∂M and a single Jordan arc in M . (2) The image $\pi(D_i \cup K_i)$ is an open disk in M_1 , bounded by a Jordan curve which meets $\pi(\partial M)$ in exactly two points; $\pi(D_i \cup K_i)$ is separated into two simply-connected components by $\pi(\partial M)$, on one of which $\deg(\pi|D_i)$ has the constant value $o(p_i) + 1$, and the constant value $o(p_i)$ on the other. (3) $D_i \cup K_i$ is small enough that $\pi(D_i \cup K_i)$ is disjoint from $\pi(D_j \cup K_j)$ for $1 \leq j < i$, from $\pi(p_j)$, $i < j \leq n$, and from $\pi(B)$.

We next take each interior branch point $q_k \in B$ in order. There is a simply-connected, open neighborhood E_k of q_k in M , bounded by a single Jordan curve in M , with the following properties. (1) $\pi(E_k)$ is a disk in M_1 , bounded by a Jordan curve in M_1 . (2) $\deg(\pi|E_k)$ has the constant value $o(q_k) + 1$ on $\pi(E_k)$. (3) $\pi(E_k)$ is disjoint from $\pi(D_i \cup K_i)$, $1 \leq i \leq n$, from $\pi(E_j)$, $1 \leq j < k$, and from $\pi(q_j)$, $k < j \leq \nu$. Namely, there are neighborhoods V of q_k in M and V_0 of $\pi(q_k)$ in M_1 , and homeomorphisms $g: V \rightarrow \Delta$, $g_0: V_0 \rightarrow \Delta$ onto the unit disk, such that $g_0(\pi(p)) = (g(p))^m$ for all $p \in V$, where $m = o(q_k) + 1$. Therefore, we may choose

$E_k = \{p \in V : |g(p)| < \epsilon\}$ for a sufficiently small $\epsilon > 0$. Observe that $\pi(D_1 \cup K_1), \dots, \pi(D_n \cup K_n), \pi(E_1), \dots, \pi(E_\nu)$ are disjoint closed disks in M_1 .

We now define a topological surface-with-boundary $\bar{\Sigma} = \bar{M} \setminus \bigcup_{j=1}^n D_j \setminus \bigcup_{k=1}^\nu E_k$. Then π is a local homeomorphism on $\bar{\Sigma}$, although not an open mapping in general. For convenience, we let M' denote the disjoint union $M_1 + \dots + M_\mu$, where $M_i = \{p \in M_1 : \deg(\pi) \geq i\}$. Then $\chi(M') = \sum_{i=1}^\mu \chi(M_i)$. Similarly, $\Sigma' = \Sigma_1 + \dots + \Sigma_\mu$, where $\Sigma_i = \{p \in M_1 : \deg(\pi|_\Sigma) \geq i\}$. M' and Σ' may be thought of as the leaves of the branched coverings π and $\pi|_\Sigma$, respectively. Thus a regular value $p_0 \in M_1$ of π appears once in M' for each point of the fiber $\pi^{-1}(p_0) \cap M$.

Observe that $\chi(\bar{\Sigma}) = \chi(\bar{\Sigma}')$. In fact, we may triangulate $\bar{\Sigma}_1$ in such a way that $\bar{\Sigma}_2, \dots, \bar{\Sigma}_\mu$ are subcomplexes, and give $\bar{\Sigma}$ the triangulation induced by the local homeomorphism π . Then a simplex of $\bar{\Sigma}_1$ occurs in $\bar{\Sigma}'$ exactly as many times as there are simplices in $\bar{\Sigma}$ mapped onto it. Now $\bar{\Sigma}'$ is obtained from \bar{M}' by removing certain interior disks and boundary half-disks: for each $p_j \in B_\partial$, $o(p_j)$ interior disks and one boundary half-disk is removed, while for each $q_k \in B$, $o(q_k) + 1$ interior disks are removed. This gives a total of $0 + \nu$ interior disks and n boundary half-disks, where $0 = \sum_{p \in B} o(p)$ is the total order of ramification of π . Therefore, one may compute $\chi(\bar{\Sigma}') = \chi(\bar{M}') - (0 + \nu)$. In fact, $\bar{\Sigma}'$ and the closure T of its complement in \bar{M}' are simplicial subcomplexes, so that

$$\chi(\bar{M}') + \chi(\bar{\Sigma}' \cap T) = \chi(\bar{\Sigma}') + \chi(T)$$

(cf. [6], pp. 189–90). But $\chi(T) = 0 + n + \nu$ and $\chi(\bar{\Sigma}' \cap T) = n$. Similarly, one may compute $\chi(\bar{\Sigma}) = \chi(\bar{M}) - \nu$. Therefore

$$\chi(\bar{M}) + 0 = \chi(\bar{\Sigma}) + 0 + \nu = \chi(\bar{M}') = \sum_{i=1}^\mu \chi(\bar{M}_i).$$

We have not treated the question of a topological characterization of mappings which are branched immersions up to the boundary and which map the boundary injectively. A mapping $\pi: M \rightarrow M_1$ between surfaces may be called a branched immersion up to the boundary if $\pi|_{M_1}$ is a branched immersion and moreover, for every boundary point p , there is an integer $m = o(p) + 1$, a neighborhood V of p in \bar{M} , a neighborhood V_0 of $\pi(p)$ in M_1 and homeomorphisms $g: V \rightarrow \Delta^+ \cup I$, $g_0: V_0 \rightarrow \Delta$, such that for all $q \in V$, $g_0(\pi(q)) = (g(q))^{2m-1}$. Here Δ , Δ^+ , and I are as in Lemma 1. It seems likely that a mapping satisfying the hypothesis of the theorem may be shown to be a branched immersion up to the boundary, using the result of Proposition 2. We shall be satisfied here with the

following description of the set $\pi^{-1}(\pi(\partial M))$. The proof follows immediately from Proposition 2.

COROLLARY 3. *Suppose $\pi: \bar{M} \rightarrow \bar{M}_1$ satisfies the hypotheses of the theorem. Then $\pi^{-1}(\pi(\partial M))$ consists of ∂M along with a finite union of Jordan curves and arcs in M , plus the finite set $B = B_r \cap M$. Each such Jordan arc tends at each end to a point of B_r . Each ramified point $p \in B_r$ is the endpoint of $2o(p) + 2$ arcs of $\pi^{-1}(\pi(\partial M))$, including, for $p \in B_a$, the two adjoining arcs of ∂M .*

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